Tangent cones of Kähler-Einstein metrics

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joint work with
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Calabi-Yau Theorem: *Let* X *be a compact Kähler manifold of complex dimension* n *with* $c_1(X) = 0$*. Then every Kähler class* $\mathfrak{k} \in H^2(X)$ *contains a unique Ricci-flat Kähler metric* $\omega \in \mathfrak{k}$*.*

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Example: Let f be a homogeneous complex polynomial of degree $n+2$ in $n+2$ complex variables. Let

$$X = X_f = \{[z_1 : \ldots : z_{n+2}] \in \mathbb{CP}^{n+1} : f(z_1, \ldots, z_{n+2}) = 0\}.$$

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Can take $\mathfrak{k} = 2\pi c_1(\mathcal{O}(1)|_X)$. Then $\omega_{\mathsf{FS}}|_X \in \mathfrak{k}$, so there exists a smooth function $\varphi : X \to \mathbb{R}$, unique up to constants, such that the Kähler form $\omega = \omega_{\mathsf{FS}}|_X + i\partial\bar{\partial}\varphi \in \mathfrak{k}$ is Ricci-flat.

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Today: Let $f = f_t$ move in a holomorphic family parametrized by $t \in \mathbb{C}$. Assume $X_t = X_{f_t}$ is smooth as above for all $t \neq 0$ but X_0 is singular. What happens to the Ricci-flat metric ω_t $(t \neq 0)$ representing $\mathfrak{k}_t = 2\pi c_1(\mathcal{O}(1)|_{X_t})$ as $t \to 0$?

We are lightyears away from understanding this properly. Main enemy is collapsing. In the $n = 1$ cubic example, (X_t, ω_t) is a flat 2-torus for all $t \neq 0$ that GH-converges to a line as $t \to 0$.

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• $n = 2$: canonical $\Leftrightarrow$ locally biholomorphic to $\mathbb{C}^2/\Gamma$ for a finite group $\Gamma \subset \mathrm{SU}(2)$ acting freely on $S^3 \subset \mathbb{C}^2$

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- For $n \geq 3$, a canonical singularity need not be isolated. Even if it is isolated, it is rarely (for us: never) of the form $\mathbb{C}^n/\Gamma$.

If $n = 2$ and if X_0 has only canonical singularities (i.e. isolated orbifold singularities of the form $\mathbb{C}^2/\Gamma$), then the behavior of the Ricci-flat metrics ω_t on X_t as $t \to 0$ is completely understood.

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1) orbifold version of the Calabi-Yau theorem (folklore) $\Rightarrow$ there is a unique Ricci-flat Kähler orbifold metric $\omega_0 \in \mathfrak{k}_0$ on X_0

I.e. if $\pi : \mathbb{C}^2 \to \mathbb{C}^2/\Gamma$ is the quotient map, then locally

$$\pi^*\omega_0 = \omega_{\mathbb{C}^2} + \text{smooth errors.}$$

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2) Gluing construction ($\exists$ many complete noncompact Ricci-flat Kähler manifolds asymptotic to $\mathbb{C}^2/\Gamma$ at infinity) $\Rightarrow$ ω_t converges smoothly to ω_0 away from X_0^{sing}, and globally in the GH sense. (Biquard-Rollin 2012, Spotti 2012)

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• There exists a smooth function φ_0 on X_0^{reg}, globally bounded, unique up to constants, such that $\omega_0 = \omega_{\mathrm{FS}}|_{X_0} + i\partial\bar{\partial}\varphi_0$ is Ricci-flat. (Eyssidieux-Guedj-Zeriahi 2009, Demailly-Pali 2010.) But: No information about second derivatives of φ_0 near X_0^{sing}.

• ω_t converges to ω_0 smoothly on compact subsets of X_0^{reg}, and (X_t, ω_t) GH-converges globally to the completion of (X_0^{reg}, ω_0). Volume fixed, diameter bounded above. (Rong-Zhang 2011)

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• No! (Donaldson-Sun 2014, Song 2015)

• All sequential pointed GH limits $(X_0, \lambda_j^2 \omega_0, x)$, where $x \in X_0^{sing}$ and $\lambda_j \to +\infty$, are metric cones. (Cheeger-Colding 2000)

Metric cone: a metric space of the form $C = C(Y) = [0, \infty) \times Y$ (Y is a complete geodesic metric space of diameter at most π, can be singular) with metric "$g_C = dr^2 + r^2 g_Y$".

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Open Question: Given $x \in X_0^{sing}$, how to determine the metric tangent cone to (X_0, ω_0) at x?

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Let $Z \subset \mathbb{CP}^n$ be a smooth complex hypersurface of degree $\leq n$ with a Kähler metric ω_Z with $\mathrm{Ric}(\omega_Z) = \omega_Z$. This induces a Ricci-flat Kähler cone metric ω_* on the complex affine cone $C_{\mathbb{C}}(Z)$ over Z in $\mathbb{C}^{n+1}$. (Calabi 1979)

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Example: $Z = \{z_1^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{CP}^n$
$C_{\mathbb{C}}(Z) = \{z_1^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{C}^{n+1}$
Ricci-flat Kähler cone metric $\omega_* = i\partial\bar{\partial}|z|^{2(n-1)/n}$ on $C_{\mathbb{C}}(Z)$
$C_{\mathbb{C}}(Z) = C(Y)$, $Y = T_1 S^n$, fibration $S^1 \to Y \to Z$

For $n = 2$: $Z =$ conic $\subset \mathbb{CP}^2$, $C_{\mathbb{C}}(Z) = \mathbb{C}^2/\mathbb{Z}_2$, $Y = T_1 S^2 = \mathbb{RP}^3$. But for $n \geq 3$, Y is not a spherical space form, $C_{\mathbb{C}}(Z) \ncong \mathbb{C}^n/\Gamma$.

Theorem (H-Sun): Assume the following:

- $n \geq 3$
- $X_t = \{f_t = 0\} \subset \mathbb{CP}^{n+1}$ is a family of complex hypersurfaces of degree $n+2$, smooth for $t \neq 0$, singular for $t = 0$.
- X_0 has at worst isolated canonical singularities, and ω_0 is the unique weak Ricci-flat Kähler metric cohomologous to $\omega_{\mathsf{FS}}|_{X_0}$.
- Each singularity of X_0 is of the form $C_{\mathbb{C}}(Z)$, where $Z \subset \mathbb{CP}^n$ is a hypersurface of degree $\leq n$ with a Kähler metric ω_Z such that $\mathrm{Ric}(\omega_Z) = \omega_Z$. (Here Z may vary from point to point.)
- ω_* is Calabi's Ricci-flat Kähler cone metric on $C_{\mathbb{C}}(Z)$.

Then the following conclusion holds:

Every singularity of X_0 has a small open neighborhood V such that there exists a biholomorphism $\Phi : U \to V$ with some small open neighborhood U of the apex in $C_{\mathbb{C}}(Z)$ such that

$$|\nabla^k_{\omega_*}(\Phi^*\omega_0 - \omega_*)|_{\omega_*} = O(r^{\lambda-k})$$

for some $\lambda > 0$ and all $k \in \mathbb{N}_0$.

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• There exist many admissible model cones $C_{\mathbb{C}}(Z)$ beyond the 'standard' example where $Z \subset \mathbb{CP}^n$ is a quadric. E.g. for $n = 3$, Z can be any smooth cubic; then Calabi's Ricci-flat Kähler cone metric ω_* on $C_{\mathbb{C}}(Z)$ is not explicit and has no Killing fields.

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• We do not really need X_t, X_0, Z to be hypersurfaces.

Outline of proof

Introduce a new parameter $s \in [0,1]$ and a new family of Kähler metrics ω_s on X_0^{reg}. Here ω_0 is again the unique weak Ricci-flat metric on X_0^{reg}, with as of now unknown asymptotics.

ω_1 is a 'brute force' initial metric: it is equal to Calabi's ω_* model (hence Ricci-flat and precisely conical) near each singularity, but completely arbitrary in the interior of X_0^{reg}.

For $s \in (0,1)$ define ω_s as the unique weak solution to a Monge-Ampère equation $MA(\omega_s) = f_s$, where f_s interpolates between $MA(\omega_1)$ and whatever right-hand side makes ω_0 Ricci-flat.

Easy key property: Each singularity has a fixed neighborhood V such that $\omega_s|_V$ is Ricci-flat for all $s \in [0,1]$.

Remains to prove: The set $\mathcal{S} = \{s \in [0,1] : \omega_s$ *has nice conical asymptotics at each singularity of* $X_0\}$ is open and closed.

1) $\mathcal{S}$ is open.

Given $s_0 \in \mathcal{S}$, then for all $s \in [0,1]$ close to s_0 we want to solve $MA(\omega_s) = f_s$ for an ω_s with nice conical asymptotics.

Ansatz: $\omega_s = \omega_{s_0} + i\partial\bar{\partial}\varphi_s$ with $\sup|\varphi_s| = O(|s - s_0|)$

Since ω_{s_0} solves $MA(\omega_{s_0}) = f_{s_0}$, we may hope to construct φ_s by an implicit function theorem. Since ω_{s_0} has nice asymptotics, the linearization of MA at ω_{s_0}, i.e. the Laplacian $\Delta_{\omega_{s_0}}$ acting on scalar functions, is invertible in weighted function spaces. But we really need $i\partial\bar{\partial} \circ \Delta_{\omega_0}^{-1}$ to be a bounded operator, and this does not follow from general elliptic theory in weighted spaces.

Theorem (H-Sun): *Let $C = C(Y)$ be a Ricci-flat Kähler cone (Y smooth) with cone metric ω_*. If $\Delta_{\omega_*} h = 0$ and if $h \sim r^\mu$ for some $\mu \in [0,2]$, then $i\partial\bar{\partial}h = \mathcal{L}_X\omega_*$ for some holomorphic vector field X on C commuting with dilations.*

Here we are not assuming that (C, ω_*) is of the form $C_{\mathbb{C}}(Z)$ with a Calabi metric, e.g. (C, ω_*) could certainly be irregular.

2) $\mathcal{S}$ is closed.

Closedness would follow from Yau's estimates if they could be applied. But this requires the model cone to have a one-sided sectional curvature bound—which holds only for flat cones.

Let $s_i \in \mathcal{S}$ with $s_i \to s_\infty \in [0,1]$. Thanks to Donaldson-Sun, the metric ω_{s_∞} has a unique tangent cone $C(Y_\infty)$. This may be different from the given model cone $C = C(Y) = C_{\mathbb{C}}(Z)$, which is by assumption the tangent cone of ω_{s_i} for all $i < \infty$.

If Y_∞ were smooth, with polynomial convergence of ω_{s_∞} to the metric of $C(Y_\infty)$, then the general openness theorem of 1) would immediately tell us that $s_\infty \in \mathcal{S}$.

In reality we need to argue as follows. First, Vol$(Y_\infty) \leq$ Vol(Y) by Bishop-Gromov, *morally* with equality if and only if $Y_\infty \cong Y$. Second, Vol$(Y_\infty) \geq$ Vol(Y) by considerations of K-stability (this is a beautiful recent result of Chi Li and Yuchen Liu), also *morally* with equality if and only if $Y_\infty \cong Y$. So Vol$(Y_\infty) =$ Vol(Y), and it appears we can go either way for the equality discussion.

Going through the equality case in Bishop-Gromov to prove that $Y_\infty \cong Y$ is technically beyond us. So we go through the equality case in Li-Liu, using that our model cone is an affine cone.

By Donaldson-Sun there is a filtration of the local ring $\mathcal{O}_x$ whose associated graded ring degenerates to the coordinate algebra of $C(Y_\infty)$, with constant Hilbert functions. This filtration is always coarser than the standard $\mathcal{O}_x \supset \mathfrak{m}_x \supset \mathfrak{m}_x^2 \supset \dots$ But $C(Y_\infty)$ and $C(Y) = C_{\mathbb{C}}(Z)$ have the same Hilbert function as well, and since $C_{\mathbb{C}}(Z)$ is an affine cone locally isomorphic to (X, x), this is equal to the Hilbert function of the standard $\mathcal{O}_x \supset \mathfrak{m}_x \supset \dots$ Thus $C(Y_\infty)$ is a degeneration of the model cone $C_{\mathbb{C}}(Z)$. Standard K-stability kicks in (Berman 2015), implying that $Y_\infty \cong Y$.

Once we know that $C(Y_\infty) \cong C(Y)$, polynomial convergence of ω_{s_∞} to the tangent cone metric (which is crucially needed as input for the openness of $\mathcal{S}$ at s_∞) follows using the method of Allard-Almgren even without assuming integrability. The reason is that the tangent cone is locally biholomorphic to the original space, and the Kähler-Ricci-flat equation never has any nonintegrable linearized solutions preserving the complex structure.