Exotic Einstein metrics on S^6 and $S^3 \times S^3$, nearly Kähler 6-manifolds and G_2 cones

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What is G_2? G_2 holonomy and Ricci-flat metrics

i. the automorphism group of the octonions $\mathbb{O}$

ii. the stabilizer of a generic 3-form in $\mathbb{R}^7$

Define a vector cross-product on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$

$$u \times v = \mathrm{Im}(uv)$$

where *uv* denotes octonionic multiplication. Cross-product has an associated 3-form

$$\varphi(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle$$

φ is a generic 3-form so in fact

$$G_2 = \{A \in \mathsf{GL}(7, \mathbb{R}) | \ A^* \varphi = \varphi \subset \mathsf{SO}(7).$$

G_2 can arise as the holonomy group of an irreducible non locally symmetric Riemannian 7-manifold **(Berger 1955, Bryant 1987, Bryant-Salamon 1989, Joyce 1995).** Any such manifold is automatically *Ricci-flat*.

G_2 **cones and nearly Kähler** 6–**manifolds**

Riemannian cone over smooth compact Riemannian manifold M:

$C(M) = \mathbb{R}^+ \times M$ endowed with the Riemannian metric $g_c = dr^2 + r^2 g$

$\mathrm{Hol}(C) \subset \mathsf{G}_2 \iff$ parallel (and hence closed) 3-form φ and 4-form $*\varphi$

$$\varphi = r^2 dr \wedge \omega + r^3 \mathrm{Re}\,\Omega, \qquad *\varphi = -r^3 dr \wedge \mathrm{Im}\,\Omega + \tfrac{1}{2} r^4 \omega^2$$

$d\varphi = 0 = d*\varphi \iff$ the SU(3)–structure (ω, Ω) on M satisfies

$$\begin{cases} d\omega = 3\,\mathrm{Re}\,\Omega \\ d\mathrm{Im}\,\Omega = -2\,\omega^2 \end{cases} \tag{NK}$$

A 6–manifold M endowed with an SU(3)–structure satisfying (**??**) is called a (strict) **nearly Kähler** (nK) 6–manifold.

- every nK 6–manifold M is Einstein with $\mathrm{Scal} = 30 \implies$ if M is complete, then it is compact with $|\pi_1(M)| < \infty \implies$ wlog can assume $\pi_1(M) = 0$.
- nK 6-manifolds and real Killing spinors
- nK 2n-manifolds and Gray–Hervella classes of almost Hermitian manifolds

The 4 examples known!

- $S^6 \subset \mathrm{Im}\,\mathbb{O}$: dates back to at least **1947** (e.g. C. Ehresmann, A. Kirchoff)
- **1968**, in Gray–Wolf's classification of 3–symmetric spaces in 6d have

  $S^3 \times S^3 = \mathrm{SU}(2)^3 / \triangle \mathrm{SU}(2) \quad CP^3 = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{Sp}(1) \quad F_3 = \mathrm{SU}(3)/T^2$

  A 3-symmetric space has an automorphism σ with $\sigma^3 = 1$: define a
  homogeneous almost complex structure on $\ker(\sigma^2 + \sigma + \mathrm{Id})$ by

  $$J = \frac{1}{\sqrt{3}}\,(2\sigma + \mathrm{Id})$$

- Connection with G_2–holonomy noted only in the 1980's, e.g. Bryant's
  1987 first explicit example of a full G_2–holonomy metric is $C(F_3)$
- G_2–cones give local models for isolated singularities of G_2–spaces
- Infinitely many Calabi–Yau, hyperkähler and Spin(7)–cones. Why not G_2?
- **2005**, Butruille: the four known examples are the only homogeneous nK
  6–manifolds
- **2006**, Bryant: local generality (via Cartan-Kähler theory) of 6d nK
  structures same as for 6d Calabi–Yaus (also Reyes Carrion thesis 1993)

Main Theorem and possible proof strategies

Main Theorem (Foscolo–Haskins, 2015) *There exists a complete inhomogeneous nearly Kähler structure on S^6 and on $S^3 \times S^3$.*

Two natural strategies to find nK 6–manifolds:

- Symmetries: cohomogeneity one nK 6–manifolds.
- Desingularisation of singular nK spaces.

Our proof uses elements from **both** viewpoints.

Simplest singular nK spaces: sine-cones (reduced holonomy $SU(3) \subset G_2$) *cross-section of a "split" G_2 cone, i.e. $\mathbb{R} \times C$ for C a Calabi–Yau cone*

(N^5, g_N) smooth Sasaki–Einstein, i.e. $C(N)$ is a Calabi–Yau (CY) cone

The **sine-cone** over N: $SC(N) = [0, \pi] \times N$ endowed with the Riemannian metric $dr^2 + \sin^2 r \, g_N$ (aka **metric suspension** of N)

$SC(N)$ is nK but has 2 isolated singularities each modelled on CY cone $C(N)$

Idea: *Try to desingularise $SC(N)$ by replacing conical singularities with smooth asymptotically conical CY 3–folds.*

A simple nK sine-cone and desingularisations

A simple example comes from the so-called **conifold**:

- $C(N)$ is the **conifold** $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$
- $N = \mathrm{SU}(2) \times \mathrm{SU}(2)/\triangle \mathrm{U}(1)$ which is diffeomorphic to $S^2 \times S^3$

$C(N)$ has 2 Calabi–Yau desingularisations (*Candelas–de la Ossa, Stenzel*)

- $Y =$ the small resolution $\simeq$ total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$
  vertex of cone replaced with a totally geodesic holomorphic $\mathbb{P}^1$
- $Y' =$ the smoothing $\simeq T^*S^3$
  vertex of cone replaced with a totally geodesic special Lagrangian S^3

The **conifold** itself and its asymptotically conical CY desingularisations are **cohomogeneity one**, i.e. $\exists$ some Lie group G acting isometrically with generic orbit of codimension one

Two examples above have only 1 singular orbit: $\mathbb{P}^1$ or S^3

Sine-cone $C(N)$, conifold and its desingularisations are cohomogeneity one. So obvious question is: **Can we desingularise this sine-cone as a cohomogeneity one space?**

Cohomogeneity one nK 6-manifolds

2010, **Podestà–Spiro**: *potential complete cohomogeneity one nK 6–mfds M*. Compact Lie group G acts with K, K_1, K_2 as its principal and singular isotropy groups. Principal orbit is G/K; 2 singular orbits G/K_i.

G	K	K_1	K_2	M
$SU(2) \times SU(2)$	$\triangle U(1)$	$\triangle SU(2)$	$\triangle SU(2)$	$S^3 \times S^3$
$SU(2) \times SU(2)$	$\triangle U(1)$	$\triangle SU(2)$	$U(1) \times SU(2)$	S^6
$SU(2) \times SU(2)$	$\triangle U(1)$	$U(1) \times SU(2)$	$SU(2) \times U(1)$	CP^3
$SU(2) \times SU(2)$	$\triangle U(1)$	$U(1) \times SU(2)$	$U(1) \times SU(2)$	$S^2 \times S^4$
$SU(3)$	$SU(2)$	$SU(3)$	$SU(3)$	S^6

$\Rightarrow N_{1,1} = SU(2) \times SU(2)/\Delta U(1)$ is only possible interesting principal orbit!

Rough outline of proof

1. Understand the local theory for cohomogeneity one nK 6–mfds in neighbourhood of principal orbit $N_{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\Delta \mathrm{U}(1)$.

  ☐ Our approach: study the geometry induced on (invariant) hypersurfaces and how it varies. Decomposes into a "static" and "dynamic" part.

  ☐ Static $=$ understand exactly what geometric structures can appear on an (invariant) hypersurface.

  Answer $=$ (invariant) *nearly hypo $SU(2)$ structures* (Fernandez et al);

  Space of invariant nearly hypo structures can be identified with a connected open subset of $SO_0(1, 2) \times S^1$. S^1 factor corresponds to obvious continuous symmetries of the equations.

  So up to symmetry *there exists a 3-dimensional family of invariant nearly hypo structures*.

  ☐ Dynamic $=$ (cohom 1) nK metrics correspond to differential equations for evolving a 1–parameter family of (invariant) nearly hypo structures.

  Answer in cohom 1 case $=$ explicit ODEs for a curve in the space of invariant nearly hypo structures.

  ☐ **Upshot: $\exists$ 2–parameter family of cohomogeneity 1 local nK metrics.**

Rough outline of proof II

Don't know how to find explicit solutions for general solution to the ODEs.
Special explicit solutions do exist, have geometric significance and play important role in our proof.

Generic solution in 2–parameter family does NOT extend to a complete metric.

Fundamental difficulty: *recognise which local solutions extend to complete metrics.*

Proceed in two steps; separate the two singular orbits that appear and study separately.

1. *Understand the possible singular orbits (uses Lie group theory) and which solutions extend over a given singular orbit (need to solve singular IVP).*

2. *Understand how to "match" a pair of solutions from the previous step.*

Step 1 fits into a general framework for cohomogeneity 1 Einstein metrics (**Eschenburg–Wang** 2000); some care needed because of isotropy repn.

Step 2 is the most subtle part of argument. Closest to previous work of **Böhm** on Einstein metrics on spheres (**Inventiones** 1998).

Solutions extending over a singular orbit

Neighbourhood of singular orbit is a G-equivariant disc bundle over singular orbit. Use representation theory to express conditions that a G-invariant section extend smoothly over the zero section. Get a *singular initial value problem* for 1st order nonlinear ODE system. Smoothness gives constraints on the initial values permitted.

Podestà–Spiro: up to symmetries *possible singular orbits are*

$$SU(2) \times SU(2)/U(1) \times SU(2) \simeq S^2 \qquad SU(2) \times SU(2)/\triangle SU(2) \simeq S^3$$

Proposition (Nearly Kähler deformations of small resolution & smoothing)
- *There exist two 1–parameter families $\{\Psi_a\}_{a>0}$ and $\{\Psi_b\}_{b>0}$ of solutions to the fundamental ODE system which extend smoothly over a singular orbit S^2 and S^3, respectively. a and b measure size of singular orbits.*
- *As $a, b \to 0$, appropriately rescaled, the local nK structures Ψ_a and Ψ_b converge to the CY structures on the small resolution and the smoothing.*

Think of the two 1–parameter families as *local* **nearly Kähler deformations** of CY metrics on small resolution and smoothing.

Now the parameter a or b is NOT just a global rescaling (as in CY case).

Matching pairs of solns: maximal volume orbits

M complete cohom 1 nK $\implies$ orbital volume $V(t)$ has a unique maximum. But generic member of our 1-parameter families of solutions is **not** complete.

Key properties of space of invariant maximal volume orbits $\mathcal{V}$:
- $\mathcal{V} \simeq \mathbb{R}^2 \times S^1 \subset \mathbb{R}^3 \times S^1$
- $V \geq 1$ on $\mathcal{V}$ and $V = 1$ precisely for the Sasaki–Einstein structure on $N_{1,1}$
- $\mathcal{V} \cap \{V \leq C\}$ is *compact*

Key Proposition *Every member of the families $\{\Psi_a\}_{a>0}$ and $\{\Psi_b\}_{b>0}$ has a unique maximal volume orbit.*

Idea of proof: a continuity argument in the parameter a or b.

Nonempty; open; closed.

Nonempty: 3 of 4 known homogeneous examples appear in these families; these clearly have max vol orbits.

Openness: easy using nondegeneracy conditions that are satisfied.

Closedness is main point: uses compactness of $\mathcal{V} \cap \{V \leq C\}$ plus standard ODE theory and basic comparison theory.

Strategy for finding complete nK metrics: *Match pairs of solutions in the two families across their maximal volume orbits using discrete symmetries.*

- α, β continuous curves in $\mathbb{R}^2 \simeq \mathcal{V}/S^1$ parametrising the maximal volume orbits of $\{\Psi_a\}_{a>0}$ and $\{\Psi_b\}_{b>0}$
- Discrete symmetries = reflections along the axes
- **Matching** means:
  (i) curves α, β must intersect (up to a discrete symmetry), or
  (ii) self-intersect, or
  (iii) intersect either axis.
- Intersection points with axes correspond to solutions with a special *"doubling symmetry"*, i.e. $\exists$ *a reflection that exchanges the two singular orbits* (therefore are of same type and size).
- Intersection points of α curve with axes give $S^2 \times S^4$ or CP^3.
- Intersection points of β curve with either axis gives $S^3 \times S^3$.
- Intersection points of α and β curves (up to action of reflection) gives S^6.

To understand if there exist new complete cohomogeneity 1 nK metrics is equivalent to:

How many axis crossings/intersection/self-intersection points do the curves α and β (and their images under the reflections) have?

Geometry of the α and β curves

Two obvious ways to get some information about the α and β curves.

1. Standard nK metrics on CP^3, S^6 and $S^3 \times S^3$ give points on these curves.
CP^3 and $S^3 \times S^3$ give intersection points of α and β curves with the axes; S^6 gives an intersection point between α and (reflection of) β curves.
What about the sine-cone?

2. Study limits of α and β curves as the parameters a and $b \to 0$.
Desingularisation philosophy suggests: Ψ_a and Ψ_b should both converge to the *sine-cone* away from the two singular orbits. Max vol orbit in sine-cone is the origin in the plane ("rotated" SE structure). So expect that the α and β curves both limit to the origin. Need to prove:

Proposition. *As $a, b \to 0$ Ψ_a and Ψ_b converge to the sine-cone over the standard Sasaki–Einstein structure on* $N_{1,1}$.

Proof ingredients: use convergence of "bubbles" to asymptotically conical CY structures; the *Böhm functional* $\mathcal{B}$ for cohom 1 Einstein metrics; invariance of $\mathcal{B}$ under rescaling and fact that it gives a power of *Vol* on a max vol orbit; rotated SE metric is the absolute min of *Vol* on all max vol orbits.

Existence of the new metric on $S^3 \times S^3$

First look only for solutions obtained by "doubling" some Ψ_b.

Idea: *exploit the convergence of Ψ_b as $b \to 0$ to the sine-cone and the existence of the homogeneous nK metric on $S^3 \times S^3$ (this has $b = 1$). Find a new nK metric "between" these two metrics, i.e. with $0 < b < 1$.*

Observation: Can detect a doubled metric on $S^3 \times S^3$ via condition that

$$v_0 = 0$$

on a *max volume orbit* where v_0 is one component of nearly hypo structure. *ODE system $\Rightarrow$ Zeros of v_0 are nondegenerate; count the number of them that occur before a max vol orbit: we call this $\mathcal{C}(b)$.*

Key fact: $\mathcal{C}(b)$ *is locally constant in b unless we hit a* **doubled metric**. *Idea of proof:* $b = 1$ is standard $S^3 \times S^3$ and we check $C(b) = 1$. To get a new cohomogeneity 1 metric on $S^3 \times S^3$ it's enough to show $C(b) \geq 2$ for $b > 0$ sufficiently small. This implies there is another doubled metric for some $b \in (0, 1)$.

Need to prove: $C(b) \geq 2$ for $b > 0$ *sufficiently small*.

$C(b) \geq 2$ for $b > 0$ **sufficiently small.**

Want to count zeroes of v_0 before max vol orbit; by ODE system this is equivalent to counting *critical points of* u_0 (another component of the nearly hypo structure)

u_0 is a solution of the second order IVP

$$(*) \quad (\lambda u_0')' + 12\lambda u_0 = 0, \quad u_0(0) = 0, \quad u_0'(0) = 2b^2 > 0.$$

Convergence of Ψ_b to sine-cone implies $\lambda(t) \to \sin t$ as $b \to 0$.

Idea: *Compare u_0 to a solution of the limiting equation*

$$(\sin t\, \xi')' + 12 \sin t\, \xi = 0.$$

This is Legendre's equation with $k = 3$. There are explicit solutions:

$$\xi_0(t) = C_1(5\cos^3 t - 3\cos t) +$$

$$C_2 \left(\frac{5}{2}\cos^2 t + \frac{1}{8}\cos t(4\cos^2 t - 6\sin^2 t) \log \frac{1 - \cos t}{1 + \cos t} - \frac{2}{3} \right)$$

1st solution is **regular** at endpoints 0 and π while 2nd is **singular**.

Existence of the new metric on $S^3 \times S^3$

Lemma: *There exists a solution ξ_0 of this Legendre eqn with the following properties: there exists $0 < t_1 < t_2 < t_3 < \frac{\pi}{2}$ such that $\xi_0(t_1) = \xi_0(t_2) = 0$, $\xi_0 \geq 0$ on $[t_1, t_2]$ and ξ_0 has a negative minimum at t_3.*

Proof:
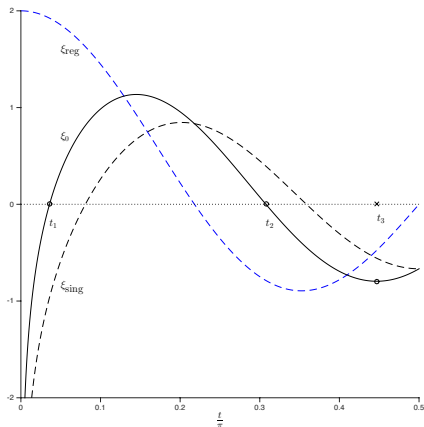
Existence of the new metric on $S^3 \times S^3$

Recall that u_0 solves

$$(*) \quad (\lambda u_0')' + 12\lambda u_0 = 0, \quad u_0(0) = 0, \quad u_0'(0) = 2b^2 > 0.$$

and $\lambda \to \sin t$ on $(0, \pi)$ as $b \to 0$.

Theorem: *There exists $\epsilon > 0$ such that for all $b < \epsilon$, u_0 the solution of (*) has a strict negative minimum before the maximal volume orbit.*

Proof sketch: Apply a (generalised) Sturm-Picone comparison argument to prove the same conclusion about the minimum holds for solution of $(*)$, using uniform convergence of $\lambda(t)$ to $\sin t$ on compact subsets of $(0, \pi)$.

Finally: initial conditions for u_0 also force a *maximum* before the minimum.

Existence of the new metric on S^6

Need to force planar curves α and β to intersect in another point (2 intersection points already exist: standard nK S^6 and sine-cone).

Idea: *use the new and old solutions on $S^3 \times S^3$ to find a closed bounded region D in the plane encircling the origin.* The α curve starts at the origin (as $a \to 0$); we want to show that eventually the α curve leaves D passing through its boundary; this point gives the new intersection point of the α and β curves.

Proposition. *The curve α exits any compact subset of $\mathbb{R}^2$ as $a \to \infty$.*

Idea of proof: based on explicit Taylor series for solutions Ψ_a and their dependence on a we consider a very particular (but non geometric) rescaling of the solution to the ODE system. Show that the rescaled solutions are well behaved as $a \to \infty$ and converge smoothly to some limiting object. Scaling used shows $V_{max} \sim ca^4$.

Heuristically: making the size of the singular orbit 2-sphere large ($a \to \infty$) forces the size of the maximal volume orbit to be large.

The intersection points of α and β
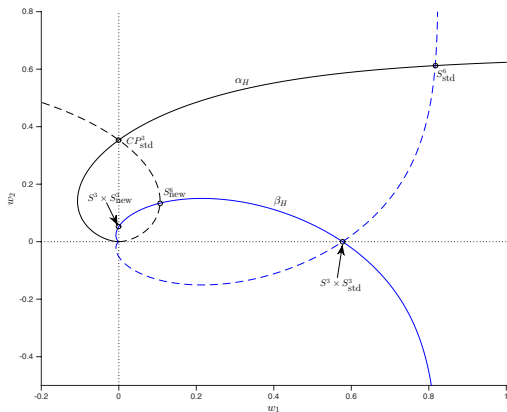
Figure: α and β curves and the locations of the 5 complete cohomogeneity one nK structures computed numerically

Nearly hypo structures

$N_{1,1} = SU(2) \times SU(2)/\triangle U(1)$
On $M^* = (a, b) \times N_{1,1}$ we write

$$\omega = \eta \wedge dt + \omega_1 \qquad \Omega = (\omega_2 + i\omega_3) \wedge (\eta + idt),$$

where $(\eta, \omega_1, \omega_2, \omega_3)$ defines an invariant $SU(2)$–structure on $N_{1,1}$.

The $SU(2)$–structure induced on a hypersurface in a nK 6–manifold is called a **nearly hypo** structure. It is defined by the following equations:

$$d\omega_1 = 3\eta \wedge \omega_2 \qquad d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1$$

The evolution equations to obtain a nK manifold by flowing a nearly hypo structure are:

$$\partial_t \omega_1 = -d\eta - 3\omega_3 \qquad \partial_t(\eta \wedge \omega_2) = -d\omega_3 \qquad \partial_t(\eta \wedge \omega_3) = d\omega_2 + 4\eta \wedge \omega_1$$

Lemma. *The space of invariant nearly hypo structures on $N_{1,1}$ is a smooth manifold diffeomorphic to $\mathbb{R}^3 \times S^1$. (The S^1–factor is generated by the action of the Reeb vector field.)*

The fundamental ODE system

We parametrise invariant nearly hypo structures modulo the Reeb action by tuples $(\lambda, u, v) \in \mathbb{R}^+ \times \mathbb{R}^{1,2} \times \mathbb{R}^{1,2}$ subject to the constraints

$$\lambda^2 |u|^2 = |v|^2 > 0 \qquad \langle u, v \rangle = 0 \qquad v_1 = |u|^2 \qquad u_2 = -\lambda |u|$$

The basic equations then are:

$$\begin{aligned}
\lambda \dot{u}_0 + 3v_0 &= 0, & \dot{v}_0 - 4\lambda u_0 &= 0, \\
\lambda \dot{u}_1 + 3v_1 - 2\lambda^2 &= 0, & \dot{v}_1 - 4\lambda u_1 &= 0, \\
\lambda \dot{u}_2 + 3v_2 &= 0, & \lambda \dot{v}_2 - 4\lambda^2 u_2 + 3u_2 &= 0,
\end{aligned}$$

$$\lambda^2 |u|^2 \dot{\lambda} + 2\lambda^4 u_1 + 3u_2 v_2 = 0.$$

Proposition. *Up to symmetries, there exists a 2–parameter family of local cohomogeneity one nK stuctures on* $(a, b) \times N_{1,1}$.

The homogeneous nK structure on $S^3 \times S^3$ is

$$\lambda = 1, \quad u_0 = u_1 = \frac{1}{\sqrt{3}} \sin\left(2\sqrt{3}t\right), \quad u_2 = -\frac{2}{\sqrt{3}} \sin\left(\sqrt{3}t\right),$$

$$v_0 = -\frac{2}{3} \cos\left(2\sqrt{3}t\right), \quad v_1 = \frac{2}{3}\left(1 - \cos\left(2\sqrt{3}t\right)\right), \quad v_2 = \frac{2}{3} \cos\left(\sqrt{3}t\right),$$

for $t \in [0, \frac{\pi}{\sqrt{3}}]$.

The sine-cone is:

$$\lambda = \sin t, \quad u_0 = 0, \quad u_1 = \sin^2 t \cos t, \quad u_2 = -\sin^3 t,$$

$$v_0 = 0, \quad v_1 = \mu^2 = \sin^4 t, \quad v_2 = \sin^3 t \cos t,$$

for $t \in [0, \pi]$.

The first few terms of the Taylor series of Ψ_a at $t = 0$ are:

$$\lambda(t) = \tfrac{3}{2}t - \frac{2a^2 + 3}{12a^2}t^3 + \frac{116a^4 - 381a^2 + 261}{1440a^4}t^5 + \cdots$$

$$u_0(t) = a^2 - 3a^2t^2 + \frac{52a^2 - 3}{24}t^4 + \cdots$$

$$u_1(t) = a^2 - \tfrac{3}{2}(2a^2 - 1)t^2 + \frac{52a^4 - 32a^2 - 3}{24a^2}t^4 + \cdots$$

$$u_2(t) = \tfrac{-3\sqrt{3}}{2}at^2 + \frac{\sqrt{3}(16a^2 - 3)}{12a}t^4 + \cdots$$

Conclusion

Conjecture The Main Theorem yields all (inhomogeneous) cohomogeneity one nK structures on simply connected 6–manifolds.