Level Set Flow

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July 11, 2016

Mean curvature

- Suppose $\Sigma \subset \mathbf{R}^{n+1}$ is a hypersurface.
- $\mathbf{n}$ is the unit normal of Σ.
- $H = \text{div}_\Sigma(\mathbf{n})$ is the mean curvature.
- Here $\text{div}_\Sigma(\mathbf{n}) = \sum_{i=1}^{n} \langle \nabla_{e_i} \mathbf{n}, e_i \rangle$; where e_i is an orthonormal basis for the tangent space of Σ.

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Mean curvature flow

A one-parameter family of smooth hypersurfaces $M_t \subset \mathbf{R}^{n+1}$ flows by the MCF if

$$x_t = -H\,\mathbf{n}\,,$$

where H and $\mathbf{n}$ are the mean curvature and unit normal, respectively, of M_t at the point x.

Two key properties

- H is the gradient of area, so MCF is the negative gradient flow for volume (Vol M_t decreases most efficiently).
- Avoidance property: If M_0 and N_0 are disjoint, then M_t and N_t remain disjoint.

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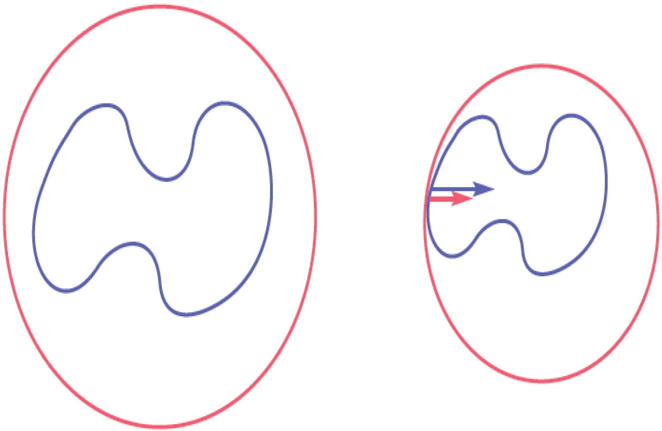

Curve shortening flow

- When $n = 1$ and the hypersurface is a curve, the flow is the curve shortening flow.
- A (round) circle shrinks through (round) circles to a point in finite time.
- Example of a snake.
- Theorem (Grayson): Any simple closed curve shrinks to a round point in finite time.

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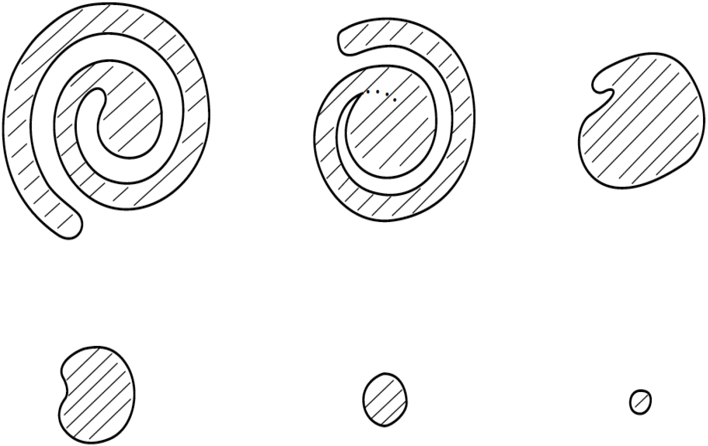
The snake



Level set flow

- Given a closed hypersurface $\Sigma \subset \mathbf{R}^{n+1}$, choose a function $u_0 : \mathbf{R}^{n+1} \to \mathbf{R}$ so that Σ is the level set $\{u_0 = 0\}$.
- If we simultaneously flow $\{u_0 = s_1\}$ and $\{u_0 = s_2\}$ for $s_1 \neq s_2$, then avoidance implies they stay disjoint.
- In the level set flow, we look for $u : \mathbf{R}^{n+1} \times \mathbf{R} \to \mathbf{R}$ so that each level set $t \to \{u(\cdot, t) = s\}$ flows by MCF.

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Level set flow II

- If $\nabla u \neq 0$ and the level sets of u flow by MCF, then

$$u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

- This is degenerate parabolic and undefined when $\nabla u = 0$. It may not have classical solutions.
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Singularities: Examples

Under MCF:

- A round sphere remains round but shrinks and eventually becomes extinct in a point.
- A round cylinder remains round and eventually becomes extinct in a line.
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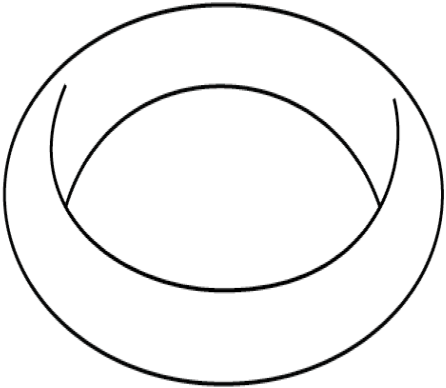
The marriage ring shrinks to a circle then disappears



Dumbbell

- The dumbbell in $\mathbf{R}^3$.
- Under the mean curvature flow the neck first pinches off and the surface disconnects into two components.
- Later each component (bell) shrinks to a round point.

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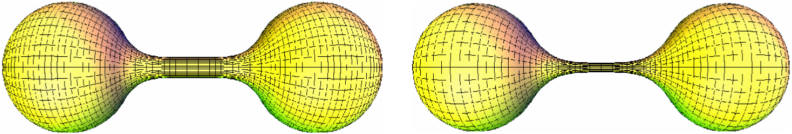

Figure: Grayson’s dumbbell; initial surface and step 1.

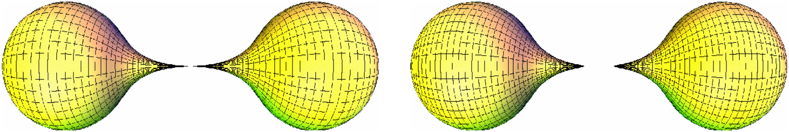

Figure: The dumbbell; steps 2 and 3.

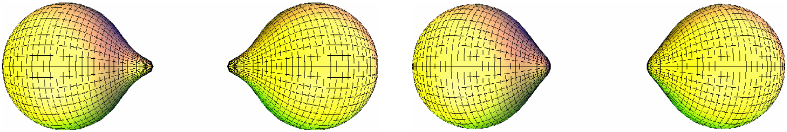

Figure: The dumbbell; steps 4 and 5.

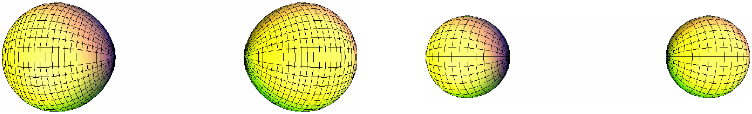

Figure: The dumbbell; steps 6 and 7.

Under MCF:

- Closed hypersurfaces contract, develop singularities and eventually become extinct.
- The **singular set** $\mathcal{S}$ is the set of points in space and time where the flow is not smooth.

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Singular set S II

- In the first 3 examples (the sphere, the cylinder and the marriage ring):
- S is a point, a line, and a closed curve, respectively.
- In each case, the singularities occur only at a single time.
- In contrast, the dumbbell has two singular times with one singular point at the first time and two at the second.

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Mean convex flows

- A hypersurface is **mean convex** if $H > 0$, i.e., if the sum of the principal curvatures is positive at every point.
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- MCF $M_t \subset \mathbf{R}^{n+1}$, M_0 closed smooth mean convex.
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Level set flow for mean convex

- When the hypersurfaces are mean convex the equation can be rewritten as degenerate elliptic.
- Write $u(x) = \{t \mid x \in M_t\}$.
- u is the **arrival time** - the time the hyper-surfaces M_t arrives at x.
- $v(x,t) = u(x) - t$ satisfies the level set flow.

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- This is degenerate elliptic and undefined when $\nabla u = 0$.
- Ex: $u = -\frac{1}{2}(x_1^2 + x_2^2)$ is the arrival time for shrinking round cylinders in $\mathbf{R}^3$.
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CM, 2015:

- u is twice differentiable everywhere and smooth away from the critical set.
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Regularity of solutions

- Huisken (90): u is C^2 for **convex** M_0.
- Ex (Ilmanen, 92): Rotationally symmetric **mean convex** M_0 where u is not C^2.
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In $\mathbf{R}^3$ with M_0 mean convex:

CM, 2016: u is C^2 **iff**:

- There is exactly one singular time T.
- The singular set $\mathcal{S}$ is either:
  1. A single point with a spherical singularity.
  2. A simple closed C^1 curve of cylindrical singularities.

Ex's: Sphere, marriage ring, dumbbell.

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- A tangent flow is the limit of a sequence of rescalings at a singularity, where the convergence is on compact subsets.
- A tangent flow to M_t at the origin in space-time is the limit of a sequence of rescaled flows $\frac{1}{\delta_i} M_{\delta_i^2 t}$ where $\delta_i \to 0$.
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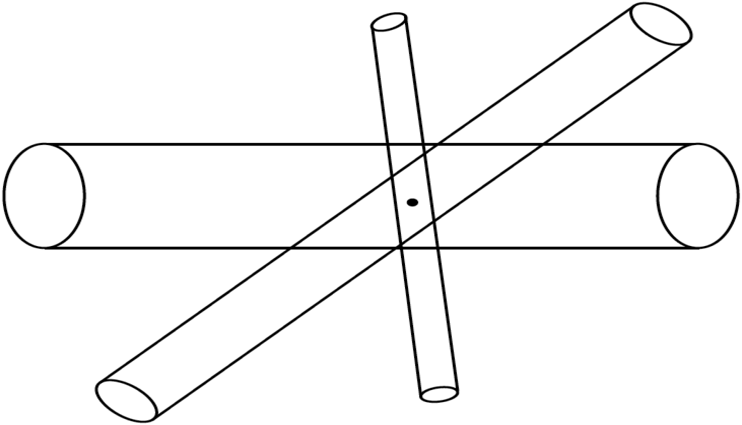
Blow up analysis and tangent flows

- A tangent flow is the limit of a sequence of rescalings at a singularity, where the convergence is on compact subsets.
- A tangent flow to M_t at the origin in space-time is the limit of a sequence of rescaled flows $\frac{1}{\delta_i} M_{\delta_i^2 t}$ where $\delta_i \to 0$.
- Non-uniqueness: Different sequences δ_i could give different tangent flows.



Snapshots of the flow at 3 times near one singular time. The axis of one cylinder could potentially rotate slowly in time.

Tangent flows = shrinkers

- Monotonicity formula of Huisken + Ilmanen and White:
- Tangent flows are shrinkers, i.e., self-similar solutions of MCF that evolve by rescaling.

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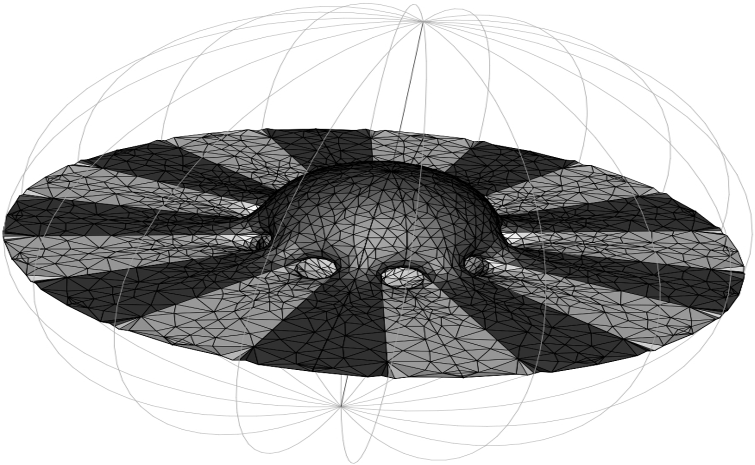

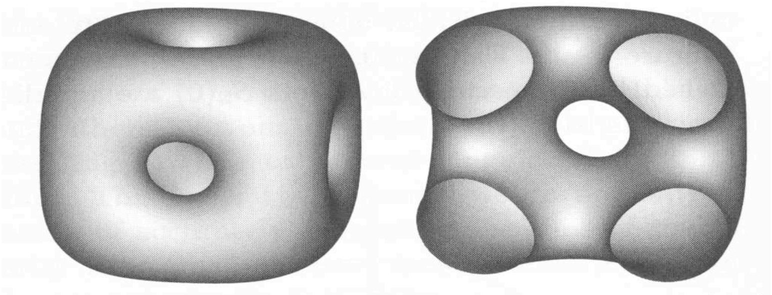

The shrinking cube. Half of the shrinking cube.

A numerical example of Chopp, Exper. Math. 1994.

Most important shrinkers

- Generalized round cylinders $\mathcal{C} := \mathbf{S}^{\mathbf{k}} \times \mathbf{R}^{\mathbf{n}-\mathbf{k}}$.
- Here the $\mathbf{S}^{\mathbf{k}}$ is centered at 0 with radius $\sqrt{2k}$ and we allow all possible rotations by $SO(n+1)$.

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Importance of cylindrical singularities

From now on: Consider flows with cylindrical singularities.

- They are the only singularities for mean convex MCF (White; Huisken-Sinestrari, Andrews, Haslhofer-Kleiner).
- They are the generic singularities in general (CM, 2012).
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Mean convex flows

- **Corollary**: [CM] Tangent flows of mean convex MCF are unique.

Strong rectifiability I

- Suppose a MCF in $\mathbf{R}^{n+1}$ has cylindrical sings (e.g., mean cvx).
- $\mathcal{S} \subset \mathbf{R}^{n+1} \times \mathbf{R}$ is the space-time singular set.
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