# Spinor structures on compact homogeneous manifolds

Dmitri Alekseevsky
Institute for Information Transmission Problems RAS

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# Abstract

We study spin structures on compact simply-connected homogeneous pseudo-Riemannian manifolds $(M = G/H, g)$ of a compact semisimple Lie group $G$.
We classify flag manifolds $F = G/H$ of a compact simple Lie group which are spin. This yields also the classification of all flag manifolds carrying an invariant metaplectic structure. Then we give a description of spin structures on C-spaces, i.e. homogeneous principal torus bundles over flag manifolds.

# Outline

# Spinor group

Let $V := \mathbb{R}^{p,q}$ $(n = p + q)$ be the pseudo-Euclidean vector space with the pseudo-Euclidean metric $g = \langle\ ,\ \rangle$ of signature $(p, q) = (-\cdots-, +\cdots+)$. We denote by $\mathrm{SO}(V) = \mathrm{SO}_{p,q}$ the pseudo-orthogonal group and by $\mathrm{S}pin(V) = \mathrm{S}pin_{p,q}$ the spinor group.
Recall that $Spin(V) = Spin_{p,q}$ is a $\mathbb{Z}_2$ covering of $SO_{p,q}$. It is defined as the subgroup of the group $\mathrm{Cl}(V)^*$ of invertible even elements $a$ of the Clifford algebra $\mathrm{Cl}(V)$ which normalizes $V \subset \mathrm{Cl}(V)$ ; $aVa^{-1} = V$. The natural representation

$$\mathrm{Spin}(V) \ni a \mapsto \mathrm{Ad}_a,\ \mathrm{Ad}_a v := ava^{-1}$$

is a $\mathbb{Z}_2$-covering $\mathrm{Ad} : Spin(V) \to SO(V) = Spin(V)/\{\pm 1\}$.

# Spinor structure on a pseudo-Riemannian manifold $(M, g)$

Let $\pi : SO(M) \to M$ be the $SO(V)$-principal bundle of orthonormal frames of a pseudo-Roemannian manifold $M$.
A $\mathrm{S}pin_{p,q}$-structure (shortly *spin structure*) on $M$ is a $\mathrm{S}pin_{p,q}$-principal bundle $\tilde{\pi} : Q = \mathrm{Spin}(M) \to M$, which is a $\mathbb{Z}_2$ covering $\Lambda : \mathrm{Spin}(M) \to \mathrm{SO}(M)$ such that the following diagram is commutative:

$$
\begin{array}{ccccc}
\mathrm{S}pin(M) \times \mathrm{S}pin_{p,q} & \longrightarrow & \mathrm{S}pin(M) & & \\
\downarrow{\scriptstyle \Lambda \times \mathrm{Ad}} & & \downarrow{\scriptstyle \Lambda} & \searrow^{\tilde{\pi}} & \\
\mathrm{SO}(M) \times \mathrm{SO}_{p,q} & \longrightarrow & \mathrm{SO}(M) & \xrightarrow{\pi} & M
\end{array}
$$

Two spin structures $(Q_1, \Lambda_1)$ and $(Q_2, \Lambda_2)$ are said *equivalent* if there is an ($\mathrm{S}pin_{p,q}$-equivariant) isomorphism $\mathcal{U} : Q_1 \to Q_2$ which commutes with the projections on $\mathrm{SO}(M)$.

# Existence of spin structure

The tangent bundle of an oriented $n$-dimensional pseudo-Riemannian manifold $(M, g)$ admits an orthogonal decomposition $TM = \eta_- \oplus \eta_+$ into timelike subbundle $\eta_-$ s.t. $g|_{\eta_-} < 0$ and spacelike subbundle $\eta_+$ s.t. $g|_{\eta_+} > 0$.
Conversely, any decomposition $TM = \eta_- \oplus \eta_+$ is an orthogonal decomposition into timelike and spacelike subbundles for some metric.
$(M^n, g)$ is *time-oriented* (resp. *space-oriented, oriented*) if $w_1(\eta_-) = 0$ (resp. $w_1(\eta_+) = 0$, $w_1(M) = w_1(\eta_-) + w_1(\eta_+) = 0$).
Proposition An oriented pseudo-Riemannian manifold $(M, g)$ admits a spin structure if and only is $w_2(\eta_-) + w_2(\eta_+) = 0$ or equivalently, $w_2(M) = w_1(\eta_-) \smile w_1(\eta_+)$, where $\eta_-$, $\eta_+$ are timelike and spacelike subbundles. If $M$ is timelike or spacelike orientable, then this is equivalent to $w_2(M) = 0$.

# Case of almost complex manifold

Assume that $M$ is a compact manifold with a complex structure $J$. Then $M$ is oriented and $w_2(M) = c_1(M, J)(mod 2)$. Moreover , the following result holds

Theorem (Lawson-Michelsohn; Atiyah) Let $M^{2n}$ be a compact complex manifold with complex structure $J$. Then $M$ admits a spin structure if and only if the first Chern class $c_1(M) \in H^2(M, \mathbb{Z})$ is even, i.e. it is divisible by 2 in $H^2(M, \mathbb{Z})$. Moreover, spin structures are in 1-1 correspondence with isomorphism classes of holomorphic line bundles $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} = K_M$ where $K_M$ is the canonical line bundle.

# Invariant spin structures

A spin structure $\tilde{\pi} : Q \to M$ on a oriented homogeneous pseudo-Riemannian manifold $(M = G/L, g)$ is called $G$-invariant if the natural action of $G$ on the bundle $SO(M)$ of oriented orthonormal frames of $M$, can be extended to an action on the $Spin_{p,q}$-principal bundle $\tilde{\pi} : Q = \mathrm{S}pin(M) \to M$.
Invariant spin structures on reductive homogeneous spaces can be described in terms of lifts of the isotropy representation into the spin group.

# Theorem(Cahen)

Let $(M = G/L, g)$ be an oriented homogeneous pseudo-Riemannian manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$. A lift $\widetilde{\vartheta} : L \to Spin(\mathfrak{q})$ of the isotropy representation $\vartheta : L \to SO(\mathfrak{m})$ onto the spinor group $Spin(\mathfrak{m})$ defines a $G$-invariant spin structure

$$Q = G \times_{\widetilde{\vartheta}} Spin(\mathfrak{q}).$$

Conversely, if $G$ is simply-connected then any spin structure on $(M = G/L, g)$ is invariant and is defined by a such lift. A spin structure (if it exists) is unique if $M$ is simply connected.

Let $(V = \mathbb{R}^{2n}, \omega)$ be the symplectic vector space and $\mathrm{Sp}(V) = \mathrm{Sp}_n(\mathbb{R}) := \mathrm{Aut}(V, \omega)$ the symplectic group. The metaplectic group $\mathrm{Mp}_n(\mathbb{R})$ is the unique connected (double) cover of $\mathrm{Sp}_n(\mathbb{R})$ . For a symplectic manifold $(M^{2n}, \omega)$, we denote by $\mathrm{Sp}(M)$ the $\mathrm{Sp}_n(\mathbb{R})$-principal bundle of symplectic frames, i.e. frames $e_1, \cdots, e_n, f_1, \cdots, f_n$ such that
$\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, $\omega(e_i, f_j) = \delta_{ij}$.
A metaplectic structure on a symplectic manifold $(M^{2n}, \omega)$ is a $\mathrm{Mp}_n(\mathbb{R})$-equivariant lift of the symplectic frame bundle $\mathrm{Sp}(M) \to M$ to a principal $(\mathrm{Mp})_\mathrm{n}(\mathbb{R})$-bundle $\mathrm{Mp}(M) \to M$ ( with respect to the double covering $\rho : \mathrm{Mp}_n(\mathbb{R}) \to \mathrm{Sp}_n\mathbb{R})$ .

# Existence of metaplectic structure

The first Chern class of $(M^{2n}, \omega)$ is defined as the first Chern class of $(TM, J)$, where $J$ is a $\omega$-compatible almost complex structure. Since the space of $\omega$-compatible almost complex structures is contractible, $c_1(M, \omega) := c_1(TM, J)$ is independent of $J$.
Theorem(K. Habermann) A symplectic manifold $(M^{2n}, \omega)$ admits a metaplectic structure if and only if $w_2(M) = 0$ or equivalently, the first Chern class $c_1(M, \omega)$ is even. In this case, the set of metaplectic structures on $(M^{2n}, \omega)$ stands in a bijective correspondence with $H^1(M; \mathbb{Z}_2)$.
Invariant metaplectic structures A simply connected compact homogeneous symplectic manifold $(M = G/H, \omega)$ (flag manifold) admits a metaplectic structure if and only if $w_2(M) = 0$. Such structure is unique and invariant.

A flag manifold is an adjoint orbit $F = G/H = \mathrm{Ad}_G w \subset \mathfrak{g} = T_e G$ of a compact connected semisimple Lie group $G$. A flag manifold $F$ admits invariant complex, symplectic and Kähler structures.

Infinitesimal description:

The $B$-orthogonal reductive decomposition of $F$ can be written as

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (Z(\mathfrak{h}) + \mathfrak{h}') + \mathfrak{m}$$

We fix a Cartan subalgebra $\mathfrak{a} = Z(\mathfrak{h}) + \mathfrak{a}' \subset \mathfrak{h}$ and denote by $R_H$, (resp., $R = R_H \cup R_F$) the root system of $(\mathfrak{h}^{\mathbb{C},\mathfrak{a}^{\mathbb{C}}})$ ( resp., $(\mathfrak{g}^{\mathbb{C},\mathfrak{a}^{\mathbb{C}}})$). Recall that roots take real values on the real form $\mathfrak{a}_0 := i\mathfrak{a} = \mathfrak{t} + \mathfrak{a}'_0$, $\mathfrak{t} := iZ(\mathfrak{h}), \mathfrak{a}'_0 = i\mathfrak{a}'$ of the Cartan subalgebra.

# Reductive decomposition of $\mathfrak{g}^{\mathbb{C}}$

$$\mathfrak{g}^{C} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}} = (\mathfrak{a}^{C} + \mathfrak{g}(R_H)) + \mathfrak{g}(R_F)$$

where for $Q \subset R$, we denote $\mathfrak{g}(Q) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha$.
Note that $R_H|_{\mathfrak{t}} = 0$ where $\mathfrak{t} := iZ(\mathfrak{h}) \subset \mathfrak{a}_0$ is the real form of the center $Z(\mathfrak{h})$.
The isotropy representation $\vartheta : H \to \mathrm{Aut}(\mathfrak{m})$ in $\mathfrak{m} = T_o F$ is the restriction $\mathrm{Ad_H}|_{\mathfrak{m}}$ of the adjoint representation.

# Black and white simple roots and weights and painted Dynkin diagram (PDD)

Let $\Pi_W = \{\beta_1, \cdots, \beta_v\}$ be the fundamental system of $R_H$ and $\Pi = \Pi_W \cup \Pi_B$ its extension to the fundamental system of $R$.
Graphically, the decomposition $\Pi = \Pi_W \cup \Pi_B$ is described by painted Dynkin diagram (PDD) where roots from $\Pi_B$ are painted in black.
"Black"fundamental weights $\Lambda_1, \cdots, \Lambda_v$ associated to the black simple roots $\beta_1, \cdots \beta_v \in \Pi_B$ are linear forms on $\mathfrak{a}_0$ which vanish on $\mathfrak{a}_0'$. They defined by

$$(\Lambda_i|\beta_j) := \frac{2(\Lambda_i, \beta_j)}{(\beta_j, \beta_j)} = \delta_{ij}, \quad (\Lambda_i|\Pi_W) = 0.$$

and span a latice $\mathcal{P}_T \subset \mathfrak{t}^*$ which we call $T$-weight lattice.

# Bijective correspondence between $(F = G/H, J_F)$ and PDD

i) A PDD determines the flag manifold $F = G/H$ together with an invariant complex structure $J$. The stability subalgebra is $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h})$ where $\mathfrak{h}'$ is the subalgebra associated with $\Pi_W$ and $\mathfrak{z}(\mathfrak{h}) = i\mathfrak{t} = \{h \in \mathfrak{a} : (h, \alpha_i) = 0, \forall\, \alpha_i \in \Pi_W\}$.
A basis of $\mathfrak{t}^* \cong \mathfrak{t}$ is given by $\Lambda_i$, $i = 1, \ldots, v$. The complex structure $J$ is defined by its $\pm i$-eigenspace decomposition $\mathfrak{m}^{\pm} = \mathfrak{g}(\pm([\Pi_B]))$ As a complex manifold, $F$ is identified with the homogeneous space $F = G^{\mathbb{C}}/P$ where $P$ is the parabolic subgroup generated by the subalgebra

$$\mathfrak{p} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{+}$$

ii) Invariant complex structures on $F = G/H$ bijectively correspond to extensions of a fixed fundamental system $\Pi_W$ of the subalgebra $\mathfrak{h}^{\mathbb{C}}$ to a fundamental system $\Pi = \Pi_W \cup \Pi_B$ of $\mathfrak{g}^{\mathbb{C}}$, or equivalently, to parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$ with reductive part $\mathfrak{h}^{\mathbb{C}}$.

# T-roots , decomposition of isotropy representation and invariant metrics

A *T*-root is the restriction $\kappa(\alpha) = \alpha|_{\mathfrak{t}}$ of a complementary root $\alpha \in R_F$ to $\mathfrak{t} := iZ(\mathfrak{h})$ . We denote by $R_T := \kappa(R_F) \subset \mathfrak{t}^*$ the set of $T$-roots.

$T$-roots $\xi$ bijectively correspond to irreducible $H$-submodules $\mathfrak{f}(\xi) := \mathfrak{g}(\kappa^{-1}(\xi))$ of $\mathfrak{m}^{\mathbb{C}}$ . Note that $\mathfrak{m}(\xi) = (\mathfrak{f}(\xi) + \mathfrak{f}(-\xi)) \cap \mathfrak{g}$ is irreducible submodule of $\mathfrak{m}$ .

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_T} \mathfrak{f}(\xi), \qquad \mathfrak{m} = \sum_{\xi \in R_T^+} \mathfrak{m}(\xi).$$

By Schur lemma, any $G$-invariant pseudo-Riemannian metric $g$ on a flag manifold $F = G/H$ is given by

$$g_o := \sum_{i=1}^{d} x_{\xi_i} B_{\xi_i},$$

where $B_{\xi_i} := -B|_{\mathfrak{m}_{\xi_i}}$ $\xi_i \in R_T^+$, and $x_{\xi_i} \neq 0$ are real numbers, for any $i = 1, \ldots, d := \sharp(R_T^+)$.

# Invariant closed 2-form on a flag manifold

Let $(F = G/H, J)$ be a flag manifold with reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{h}' + i\mathfrak{t}) + \mathfrak{m}.$$

The complex $\Omega(F)^G$ of invariant differential forms is identified with subcomplex

$$\Lambda(\mathfrak{m}^*)^H = \{\omega \in \Lambda(\mathfrak{g}^*)^H,\ X \lrcorner \omega = 0\ \forall X \in \mathfrak{h}\} \subset \Lambda(\mathfrak{g}^*)^H$$

of the complex $\Lambda(\mathfrak{g}^*)^H$. Denote by $\omega^\beta$ the basis of 1-forms $(\mathfrak{m}^{\mathbb{C}})^*$ dual to the basis $E_\beta$, $\beta \in R_F$ of $\mathfrak{m}^{\mathbb{C}}$.

Propossition There is a natural isomorphism (transgression)

$$\tau : \mathfrak{t}^* \to \Lambda^2_{cl}(\mathfrak{m}^*)^H \cong H^2(\mathfrak{m}^*)^H \simeq H^2(F, \mathbb{R})$$

given by

$$\mathfrak{a}_0^* \supset t^* \ni \xi \mapsto \tau(\xi) \equiv \omega_\xi := \frac{i}{2\pi} d\xi = \frac{i}{2\pi} \sum_{\alpha \in R_F^+} \frac{2(\xi, \alpha)}{(\alpha, \alpha)} \omega^\alpha \wedge \omega^{-\alpha}.$$

In particular, $\tau(\mathcal{P}_T) \cong H^2(F, \mathbb{Z})$ .

# Correspondence between $T$ -weights $\lambda \in \mathcal{P}_T$, characters $\chi \in \mathrm{Hom}(H, T^1)$ and holomorphic line bundles

A $T$-weight $\lambda \in \mathcal{P}_T$ defines a character
$\chi_\lambda \in \chi(H) = \mathrm{Hom}(H, T^1) = \mathrm{Hom}(Z(H), T^1)$

$$\chi_\lambda : H \to T^1, \ \exp(2\pi i X) \mapsto e^{2\pi i \lambda(X)}, \ X \in \mathfrak{t}.$$

It admits a holomorphic extension to the holomorphic character $\chi_\lambda^{\mathbb{C}} \in \chi(P) = \mathrm{Hom}(P, \mathbb{C}^*)$ of the parabolic subgroup $P = H^{\mathbb{C}} \cdot N^+ \subset G^{\mathbb{C}}$. This defines an isomorphism $\mathcal{P}_T \simeq \chi(H) \simeq \chi(P)$ of the vector group of $T$-weights $\mathcal{P}_T$, the group of real characters $\chi(H)$ of $H$ and the group $\chi(P)$ of holomorphic characters of $P$.
Denote by $\mathcal{L}_\lambda$ the holomorphic line bundle on $F = G/H = G^{\mathbb{C}}/P$ associated with the holomorphic character $\chi^{\mathbb{C}_\lambda}$. Then the Chern class $c_1(\mathcal{L}_\lambda) = [\omega_\lambda]$.

# The Koszul form

We set $\sigma_G := \frac{1}{2}\sum_{\alpha \in R^+} \alpha, \;\; \sigma_H := \frac{1}{2}\sum_{\alpha \in R_H^+} \alpha$. Recall that $\sigma_G$ is the sum of fundamental weights.

Definition The Koszul 1-form associated with an invariant complex structure $J$ on flag manifold $F = G/H$ is define by

$$\sigma^J = 2(\sigma_G - \sigma_H) = \sum_{\alpha \in R_F^+} \alpha$$

# The Koszul numbers

The Koszul form is a linear combination of black fundamental weights with positive integers coefficients, given by:

$$\sigma^J = \sum_{j=1}^{v} k_j \Lambda_j = \sum_{j=1}^{v} (2 + b_j)\Lambda_j \in \mathcal{P}_T, \quad \text{where } b_j = -\frac{4(\sigma_H, \beta_j)}{(\beta_j, \beta_j)} \geq 0.$$

The integers $k_j \in \mathbb{Z}_+$ are called Koszul numbers and the vector $\vec{k} := (k_1, \ldots, k_v) \in \mathbb{Z}_+^v$ is called Koszul vector.

# Chern form of an invariant complex structure

The first Chern class $c_1(J) \in H^2(F;\mathbb{Z})$ of the invariant complex structure $J$ in $F$, is represented by the closed invariant 2-form $\gamma_J := \omega_{\sigma^J}$, ( Chern form ).
A flag manifold $F = G/H$ admits a ( $G$-invariant) spin (or metaplectic) structure, if and only is the first Chern class $c_1(J)$ of an invariant complex structure $J$ on $F$ is even, that is all Koszul numbers are even. In this case, such spin (or metablectic) structure is unique.
Corollary The divisibility by two of the Koszul numbers on a flag manifold $F = G/H = G^{\mathbb{C}}/P$ does not depend on the complex structure.

# Invariant spin and metaplectic structures on classical flag manifolds

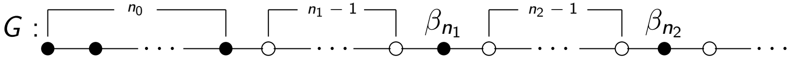
Flag manifolds of classical groups:

$$\begin{aligned}
A(\vec{n}) &= SU_{n+1}/U_1^{n_0} \times S(U_{n_1} \times \cdots \times U_{n_s}), \quad \vec{n} = (n_0, n_1, \cdots, n_s), \\
B(\vec{n}) &= SO_{2n+1}/U_1^{n_0} \times U_{n_1} \times \cdots \times U_{n_s} \times SO_{2r+1}, \quad n = \sum n_j + r, \\
C(\vec{n}) &= Sp_n/U_1^{n_0} \times U_{n_1} \times \cdots \times U_{n_s} \times Sp_r, \quad n = \sum n_j + r, \\
D(\vec{n}) &= SO_{2n}/U_1^{n_0} \times U_{n_1} \times \cdots \times U_{n_s} \times SO_{2r}, \quad n = \sum n_j + r,
\end{aligned}$$

where $\vec{n} = (n_0, n_1, \cdots, n_s, r)$ for the group $B_n, C_n, D_n$.

# Standard complex structures

We chose the standard complex structure on a flag manifold $G(\vec{n})$ for one of the groups $G = \mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ as complex structure associated the following standard painted Dynkin diagram:



# Koszul numbers for classical groups

For a flag manifold $(F = G/H, J)$ of a classical group , the Koszul number $k_j$ associated with the black simple root $\beta_j \in \Pi_B$ equals to $2 + b_j$, where $b_j$ is the number of white roots connected with the black root by a string of white roots with the following exceptions:
(a) For $B_n$, each long white root of the last white string which corresponds to the simple factor $SO_{2r+1}$ is counted with multiplicity two, and the last short white root is counted with multiplicity one. If the last simple root is painted black, i.e. $\beta_{s_n} = \alpha_n$, then the coefficient $b_{n_s}$ is twice the number of white roots which are connected with this root.
(b) For $C_n$, each root of the last white string which corresponds to the factor $Sp_r$ is counted with multiplicity two.
(c) For $D_n$, the last white chain which defines the root system of $D_r = \mathrm{SO}_{2r}$ is considered as a chain of length $2(r-1)$. If $r = 0$ and one of the last right roots is white and other is black, then the Koszul number $k_{n_s}$ associated to this black end root $\beta_{n_s}$, is $2(n_s - 1)$.

## The Koszul vector $\vec{k} := (k_1, \cdots, k_v) \in \mathbb{Z}_+^v$ associated to the standard complex structure $J_0$ on a flag manifold $G(\vec{n})$ of classical type

$$\begin{aligned} A(\vec{n}): \quad \vec{k} = &\ (2, \cdots, 2, 1+n_1, n_1+n_2, \cdots, n_{s-1}+n_s), \\ B(\vec{n}): \quad \vec{k} = &\ (2, \cdots, 2, 1+n_1, n_1+n_2, \cdots, n_{s-1}+n_s, n_s+2r), \\ C(\vec{n}): \quad \vec{k} = &\ (2, \cdots, 2, 1+n_1, n_1+n_2, \cdots, n_{s-1}+n_s, n_s+2r+1), \\ D(\vec{n}): \quad \vec{k} = &\ (2, \cdots, 2, 1+n_1, n_1+n_2, \cdots, n_{s-1}+n_s, n_s+2r-1). \end{aligned}$$

If $r = 0$, then the last Koszul number (over the end black root) is $2n_s$ for B($\vec{n}$), $n_s + 1$ for C($\vec{n}$) and $2(n_s - 1)$ for D($\vec{n}$).

# Spin (= metaplectic) flag manifolds of classical groups

(a) The flag manifold $A(\vec{n})$ is spin if and only if $n_0 > 0$ and all the numbers $n_1, \ldots, n_s$ are odd or $n_0 = 0$, and the numbers $n_1, \ldots, n_s$ have the same parity.
(b) The flag manifold $B(\vec{n})$ is spin if and only if either $n_0 > 0$ and $r = 0$, and all the numbers $n_1, \ldots, n_s$ are odd or $n_0 = 0$ and $r > 0$ and all the numbers $n_1, \ldots, n_s$ are even. Finally, or $n_0 = r = 0$, and all the numbers $n_1, \ldots, n_s$ have the same parity.
(c) The flag manifold $C(\vec{n})$ is spin if $n_0 \geq 0$ and all the numbers $n_1, \ldots, n_s$ are odd.
(d) The flag manifold $D(\vec{n})$ is spin if $n_0 > 0$ and all the numbers $n_1, \ldots, n_s$ are odd, or $n_0 = 0$ and $r > 0$, and all the numbers $n_1, \ldots, n_s$ are odd or $n_0 = r = 0$ and the numbers $n_1, \ldots, n_s$ have the same parity.

# Flag manifolds of exceptional Lie groups $G_2$ and $F_4$

| $G$ | $F = G/H$ | $b_2(F)$ | $d$ | $\sigma^J$ |
|---|---|---|---|---|
| $\mathrm{G}_2$ | $\mathrm{G}_2(0) = \mathrm{G}_2/\mathrm{T}^2$ | 2 | 6 | $2(\Lambda_1 + \Lambda_2)$ |
| | $\mathrm{G}_2(1) = \mathrm{G}_2/U_2^l$ | 1 | 3 | $5\Lambda_2$ |
| | $\mathrm{G}_2(2) = \mathrm{G}_2/U_2^s$ | 1 | 2 | $3\Lambda_1$ |
| $\mathrm{F}_4$ | $\mathrm{F}_4(0) = \mathrm{F}_4/\mathrm{T}^4$ | 4 | 24 | $2(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)$ |
| | $\mathrm{F}_4(1) = \mathrm{F}_4/\mathrm{A}_1^l \times \mathrm{T}^3$ | 3 | 16 | $3\Lambda_2 + 2(\Lambda_3 + \Lambda_4)$ |
| | $\mathrm{F}_4(4) = \mathrm{F}_4/\mathrm{A}_1^s \times \mathrm{T}^3$ | 3 | 13 | $2(\Lambda_1 + \Lambda_2) + \Lambda_3$ |
| | $\mathrm{F}_4(1,2) = \mathrm{F}_4/\mathrm{A}_2^l \times \mathrm{T}^2$ | 2 | 9 | $6\Lambda_3 + 2\Lambda_4$ |
| | $\mathrm{F}_4(1,4) = \mathrm{F}_4/\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^2$ | 2 | 8 | $3\Lambda_2 + 3\Lambda_3$ |
| | $\mathrm{F}_4(2,3) = \mathrm{F}_4/\mathrm{B}_2 \times \mathrm{T}^2$ | 2 | 6 | $5\Lambda_1 + 6\Lambda_4$ |
| | $\mathrm{F}_4(3,4) = \mathrm{F}_4/\mathrm{A}_2^s \times \mathrm{T}^2$ | 2 | 6 | $2\Lambda_1 + 4\Lambda_2$ |
| | $\mathrm{F}_4(1,2,4) = \mathrm{F}_4/\mathrm{A}_2^l \times \mathrm{A}_1^s \times \mathrm{T}$ | 1 | 4 | $7\Lambda_3$ |
| | $\mathrm{F}_4(1,3,4) = \mathrm{F}_4/\mathrm{A}_2^s \times \mathrm{A}_1^l \times \mathrm{T}$ | 1 | 3 | $5\Lambda_2$ |
| | $\mathrm{F}_4(1,2,3) = \mathrm{F}_4/\mathrm{B}_3 \times \mathrm{T}$ | 1 | 2 | $11\Lambda_4$ |
| | $\mathrm{F}_4(2,3,4) = \mathrm{F}_4/\mathrm{C}_3 \times \mathrm{T}$ | 1 | 2 | $8\Lambda_1$ |

# Flag manifolds of group $E_6$

| $F = G/H$ | $b_2(F)$ | $d$ | $\sigma^J$ |
|---|---|---|---|
| $E_6(0) = E_6 / T^6$ | 6 | 36 | $2(\Lambda_1 + \cdots + \Lambda_6)$ |
| $E_6(1) = E_6 / A_1 \times T^5$ | 5 | 25 | $3\Lambda_2 + 2(\Lambda_3 + \cdots + \Lambda_6)$ |
| $E_6(3,5) = E_6 / A_1^2 \times T^4$ | 4 | 17 | $2\Lambda_1 + 3\Lambda_2 + 4\Lambda_4 + 3\Lambda_6$ |
| $E_6(4,5) = E_6 / A_2 \times T^4$ | 4 | 15 | $2(\Lambda_1 + \Lambda_2 + 2\Lambda_3 + \Lambda_6)$ |
| $E_6(1,3,5) = E_6 / A_1^3 \times T^3$ | 3 | | $4(\Lambda_2 + \Lambda_4) + 3\Lambda_6$ |
| $E_6(2,4,5) = E_6 / A_2 \times A_1 \times T^3$ | 3 | 10 | $3\Lambda_1 + 5\Lambda_3 + 2\Lambda_6$ |
| $E_6(3,4,5) = E_6 / A_3 \times T^3$ | 3 | 8 | $2\Lambda_1 + 5(\Lambda_2 + \Lambda_6)$ |
| $E_6(2,3,4,5) = E_6 / A_4 \times T^2$ | 2 | 4 | $6\Lambda_1 + 8\Lambda_6$ |
| $E_6(1,3,4,5) = E_6 / A_3 \times A_1 \times T^2$ | 2 | 5 | $6\Lambda_2 + 5\Lambda_6$ |
| $E_6(1,2,4,5) = E_6 / A_2^2 \times T^2$ | 2 | 6 | $6\Lambda_3 + 2\Lambda_6$ |
| $E_6(2,4,5,6) = E_6 / A_2 \times A_1^2 \times T^2$ | 2 | 6 | $3\Lambda_1 + 6\Lambda_3$ |
| $E_6(2,3,4,6) = E_6 / D_4 \times T^2$ | 2 | 3 | $8(\Lambda_1 + \Lambda_5)$ |
| $E_6(1,2,4,5,6) = E_6 / A_2^2 \times A_1 \times T$ | 1 | 3 | $7\Lambda_3$ |
| $E_6(1,2,3,4,5) = E_6 / A_5 \times T$ | 1 | 2 | $11\Lambda_6$ |
| $E_6(1,3,4,5,6) = E_6 / A_1^2 \times T$ | 1 | 2 | $9\Lambda_2$ |
| $E_6(2,3,4,5,6) = E_6 / D_5 \times T$ | 1 | 1 | $12\Lambda_1$ |

# Flag manifolds of group $E_7$

| $F = G/H$ | $b_2$ | $d$ | $\sigma^J$ |
|---|---|---|---|
| $E_7(0) = E_7 / T^7$ | 7 | 63 | $2(\Lambda_1 + \cdots + \Lambda_7)$ |
| $E_7(1) = E_7 / A_1 \times T^6$ | 6 | 46 | $3\Lambda_2 + 2(\Lambda_3 + \cdots + \Lambda_7)$ |
| $E_7(4,6) = E_7 / A_1^2 \times T^5$ | 5 | 33 | $2(\Lambda_1 + \Lambda_2) + 3(\Lambda_3 + \Lambda_7) + 4\Lambda_5$ |
| $E_7(5,6) = E_7 / A_2 \times T^5$ | 5 | 30 | $2(\Lambda_1 + \cdots + \Lambda_4 + \Lambda_7)$ |
| $E_7(1,3,5) = E_7 / A_1^3 \times T^4$ | 4 | 23 | $4\Lambda_2 + 4\Lambda_4 + 3\Lambda_6 + 2\Lambda_7$ |
| $E_7(1,3,7) = E_7 / A_1^3 \times T^4$ | 4 | 24 | $4\Lambda_2 + 4\Lambda_4 + 2\Lambda_5 + 2\Lambda_6$ |
| $E_7(3,5,6) = E_7 / A_2 \times A_1 \times T^4$ | 4 | 21 | $2\Lambda_1 + 3\Lambda_2 + 5\Lambda_4 + 2\Lambda_7$ |
| $E_7(4,5,6) = E_7 / A_3 \times T^4$ | 4 | 18 | $2\Lambda_1 + 2\Lambda_2 + 5\Lambda_3 + 5\Lambda_7$ |
| $E_7(1,2,3,4) = E_7 / A_4 \times T^3$ | 3 | 10 | $6\Lambda_5 + 2\Lambda_6 + 6\Lambda_7$ |
| $E_7(1,2,3,5) = E_7 / A_3 \times A_1 \times T^3$ | 3 | 12 | $6\Lambda_4 + 3\Lambda_6 + 2\Lambda_7$ |
| $E_7(1,2,3,7) = E_7 / A_3 \times A_1 \times T^3$ | 3 | 13 | $6\Lambda_4 + 2\Lambda_5 + 2\Lambda_6$ |
| $E_7(1,2,4,5) = E_7 / A_2^2 \times T^3$ | 3 | 13 | $6\Lambda_3 + 4\Lambda_6 + 4\Lambda_7$ |
| $E_7(1,2,4,6) = E_7 / A_2 \times A_1^2 \times T^3$ | 3 | 14 | $5\Lambda_3 + 4\Lambda_5 + 3\Lambda_7$ |
| $E_7(1,3,5,7) = E_7 / (A_1)^4 \times T^3$ | 3 | 16 | $4\Lambda_2 + 5\Lambda_4 + 3\Lambda_6$ |
| $E_7(3,4,5,7) = E_7 / D_4 \times T^3$ | 3 | 9 | $2\Lambda_1 + 8\Lambda_2 + 8\Lambda_6$ |

| $F = G/H$ | $b_2$ | $d$ | $\sigma^J$ |
|---|---|---|---|
| $E_7(1,2,3,4,5) = E_7 / A_5 \times T^2$ | 2 | 5 | $7\Lambda_6 + 10\Lambda_7$ |
| $E_7(1,2,3,4,7) = E_7 / A_5 \times T^2$ | 2 | 6 | $10\Lambda_5 + 2\Lambda_6$ |
| $E_7(1,2,3,4,6) = E_7 / A_4 \times A_1 \times T^2$ | 2 | 6 | $7\Lambda_5 + 6\Lambda_7$ |
| $E_7(1,2,3,5,6) = E_7 / A_3 \times A_2 \times T^2$ | 2 | 7 | $7\Lambda_4 + 2\Lambda_7$ |
| $E_7(1,2,3,5,7) = E_7 / A_3 \times A_1^2 \times T^2$ | 2 | 8 | $7\Lambda_4 + 3\Lambda_6$ |
| $E_7(1,3,4,5,7) = E_7 / D_4 \times A_1 \times T^2$ | 2 | 6 | $9\Lambda_2 + 4\Lambda_6$ |
| $E_7(1,2,5,6,7) = E_7 / A_2 \times A_1^2 \times T^2$ | 2 | 8 | $4\Lambda_3 + 5\Lambda_4$ |
| $E_7(1,3,5,6,7) = E_7 / A_2 \times (A_1)^3 \times T^2$ | 2 | 9 | $4\Lambda_2 + 6\Lambda_4$ |
| $E_7(3,4,5,6,7) = E_7 / D_5 \times T^2$ | 2 | 4 | $2\Lambda_1 + 12\Lambda_2$ |
| $E_7(1,2,3,4,5,6) = E_7 / A_6 \times T$ | 1 | 2 | $14\Lambda_7$ |
| $E_7(2,3,4,5,6,7) = E_7 / E_6 \times T$ | 1 | 1 | $18\Lambda_1$ |
| $E_7(1,3,4,5,6,7) = E_7 / D_5 \times A_1 \times T$ | 1 | 2 | $13\Lambda_2$ |
| $E_7(1,2,4,5,6,7) = E_7 / A_4 \times A_2 \times T$ | 1 | 3 | $10\Lambda_3$ |
| $E_7(1,2,3,5,6,7) = E_7 / A_3 \times A_2 \times A_1 \times T$ | 1 | 4 | $8\Lambda_4$ |
| $E_7(1,2,3,4,6,7) = E_7 / A_5 \times A_1 \times T$ | 1 | 2 | $12\Lambda_5$ |
| $E_7(1,2,3,4,5,7) = E_7 / D_6 \times T$ | 1 | 2 | $17\Lambda_6$ |

# Flag manifolds of group $E_8$

| $F = G/H$ | $b_2$ | $d$ | $\sigma^J$ |
|---|---|---|---|
| $E_8(0) = E_8/T^8$ | 8 | 120 | $2(\Lambda_1 + \cdots + \Lambda_8)$ |
| $E_8(1) = E_8/A_1 \times T^7$ | 7 | 91 | $3\Lambda_2 + 2(\Lambda_3 + \cdots + \Lambda_8)$ |
| $E_8(1,2) = E_8/A_2 \times T^6$ | 6 | 63 | $4\Lambda_3 + 2(\Lambda_4 + \cdots + \Lambda_8)$ |
| $E_8(1,3) = E_8/A_1 \times A_1 \times T^6$ | 6 | 68 | $4\Lambda_2 + 3\Lambda_4 + 2(\Lambda_5 + \cdots + \Lambda_8)$ |
| $E_8(1,2,3) = E_8/A_3 \times T^5$ | 5 | 41 | $5\Lambda_4 + 2(\Lambda_5 + \cdots + \Lambda_8)$ |
| $E_8(1,2,4) = E_8/A_2 \times A_1 \times T^5$ | 5 | 46 | $5\Lambda_3 + 3\Lambda_5 + 2\Lambda_6 + 2\Lambda_7 + 2\Lambda_8$ |
| $E_8(1,3,5) = E_8/(A_1)^3 \times T^5$ | 5 | 50 | $4\Lambda_2 + 4\Lambda_4 + 3\Lambda_6 + 2\Lambda_7 + 3\Lambda_8$ |
| $E_8(1,2,3,4) = E_8/A_4 \times T^4$ | 4 | 25 | $6\Lambda_5 + 2(\Lambda_6 + \Lambda_7 + \Lambda_8)$ |
| $E_8(1,2,3,5) = E_8/A_3 \times A_1 \times T^4$ | 4 | 29 | $6\Lambda_4 + 3\Lambda_6 + 2\Lambda_7 + 3\Lambda_8$ |
| $E_8(1,2,4,5) = E_8/A_2 \times A_2 \times T^4$ | 4 | 30 | $6\Lambda_3 + 4\Lambda_6 + 2\Lambda_7 + 4\Lambda_8$ |
| $E_8(1,2,4,6) = E_8/A_2 \times A_1^2 \times T^4$ | 4 | 33 | $5\Lambda_3 + 4\Lambda_5 + 3\Lambda_7 + 2\Lambda_8$ |
| $E_8(1,3,5,7) = E_8/(A_1)^4 \times T^4$ | 4 | 36 | $2(\Lambda_2 + \Lambda_4 + \Lambda_6) + 3\Lambda_8$ |
| $E_8(4,5,6,8) = E_8/D_4 \times T^4$ | 4 | 24 | $2(\Lambda_1 + \Lambda_2) + 8(\Lambda_3 + \Lambda_7)$ |

| $F = G/H$ | $b_2$ | $d$ | $\sigma^J$ |
|---|---|---|---|
| $E_8(1,2,3,4,5) = E_8 / A_5 \times T^3$ | 3 | 14 | $7\Lambda_6 + 2\Lambda_7 + 7\Lambda_8$ |
| $E_8(1,2,3,4,6) = E_8 / A_4 \times A_1 \times T^3$ | 3 | 17 | $7\Lambda_5 + 3\Lambda_7 + 2\Lambda_8$ |
| $E_8(1,2,3,5,6) = E_8 / A_3 \times A_2 \times T^3$ | 3 | 18 | $7\Lambda_4 + 4\Lambda_7 + 4\Lambda_8$ |
| $E_8(1,2,3,5,7) = E_8 / A_3 \times A_1 \times A_1 \times T^3$ | 3 | 20 | $6\Lambda_4 + 2\Lambda_6 + 3\Lambda_8$ |
| $E_8(1,2,4,5,7) = E_8 / A_2 \times A_2 \times A_1 \times T^3$ | 3 | 21 | $6\Lambda_3 + 5\Lambda_6 + 4\Lambda_8$ |
| $E_8(1,2,4,6,8) = E_8 / A_2 \times (A_1)^3 \times T^3$ | 3 | 23 | $5\Lambda_3 + 5\Lambda_5 + 3\Lambda_7$ |
| $E_8(1,4,5,6,8) = E_8 / D_4 \times A_1 \times T^3$ | 3 | 16 | $3\Lambda_2 + 8\Lambda_3 + 8\Lambda_7$ |
| $E_8(4,5,6,7,8) = E_8 / D_5 \times T^3$ | 3 | 13 | $2(\Lambda_1 + \Lambda_2 + \Lambda_3)$ |
| $E_8(1,2,3,4,5,6) = E_8 / A_6 \times T^2$ | 2 | 7 | $8\Lambda_7 + 12\Lambda_8$ |
| $E_8(1,2,3,4,5,7) = E_8 / A_5 \times A_1 \times T^2$ | 2 | 9 | $8\Lambda_6 + 7\Lambda_8$ |
| $E_8(1,2,3,4,6,7) = E_8 / A_4 \times A_2 \times T^2$ | 2 | 10 | $8\Lambda_5 + 2\Lambda_8$ |
| $E_8(1,2,3,5,6,7) = E_8 / A_3 \times A_3 \times T^2$ | 2 | 10 | $8\Lambda_4 + 5\Lambda_8$ |
| $E_8(1,2,3,4,6,8) = E_8 / A_4 \times A_1 \times A_1 \times T^2$ | 2 | 11 | $8\Lambda_5 + 3\Lambda_7$ |

| | | | |
|---|---|---|---|
| $E_8(1,2,4,5,6,8) = E_8 / D_4 \times A_2 \times T^2$ | 2 | 9 | $10\Lambda_3 + 8\Lambda_7$ |
| $E_8(1,4,5,6,7,8) = E_8 / D_5 \times A_1 \times T^2$ | 2 | 8 | $3\Lambda_2 + 12\Lambda_3$ |
| $E_8(2,3,4,5,6,8) = E_8 / D_6 \times T^2$ | 2 | 6 | $12\Lambda_1 + 17\Lambda_7$ |
| $E_8(1,2,3,6,7,8) = E_8 / A_3 \times A_2 \times A_1 \times T^2$ | 2 | 12 | $5\Lambda_4 + 5\Lambda_5$ |
| $E_8(1,2,4,6,7,8) = E_8 / A_2 \times A_2 \times A_1 \times A_1 \times T^2$ | 2 | 14 | $5\Lambda_3 + 6\Lambda_5$ |
| $E_8(3,4,5,6,7,8) = E_8 / E_6 \times T^2$ | 2 | 6 | $2\Lambda_1 + 18\Lambda_2$ |
| $E_8(1,2,3,4,5,6,7) = E_8 / A_7 \times T$ | 1 | 3 | $17\Lambda_8$ |
| $E_8(2,3,4,5,6,7,8) = E_8 / E_7 \times T$ | 1 | 2 | $29\Lambda_1$ |
| $E_8(1,3,4,5,6,7,8) = E_8 / E_6 \times A_1 \times T$ | 1 | 3 | $19\Lambda_2$ |
| $E_8(1,2,4,5,6,7,8) = E_8 / D_5 \times A_2 \times T$ | 1 | 4 | $14\Lambda_3$ |
| $E_8(1,2,3,5,6,7,8) = E_8 / A_4 \times A_3 \times T$ | 1 | 5 | $11\Lambda_4$ |
| $E_8(1,2,3,4,6,7,8) = E_8 / A_4 \times A_2 \times A_1 \times T$ | 1 | 6 | $9\Lambda_5$ |
| $E_8(1,2,3,4,5,7,8) = E_8 / A_6 \times A_1 \times T$ | 1 | 4 | $13\Lambda_6$ |
| $E_8(1,2,3,4,5,6,8) = E_8 / D_7 \times T$ | 1 | 2 | $23\Lambda_7$ |

Let $F = G/H = G/H' \cdot T^k$ be a flag manifold with reductive decomposition $\mathfrak{g} = (\mathfrak{h}' + i\mathfrak{t}) + \mathfrak{m}$ and $T^k = T_0^\ell \times T_1^{2m}$ a direct product decomposition. Then
$\pi : M = G/L = G/H' \cdot T^\ell \to F = G/H$ is a principal $T_1^{2m}$-bundle.
The manifold $M = G/L$ with reductive decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{p} = (\mathfrak{h}' + i\mathfrak{t}_0) + (i\mathfrak{t}_1 + \mathfrak{m})$$

is called a C-space. An invariant complex structure $J_F$ defined by $J_\mathfrak{m}$ is extended to an invariant complex structure $J_M$ defined by $J_\mathfrak{p} = J_{i\mathfrak{t}_1} \oplus J_\mathfrak{m}$. The Chern form $\gamma_{J_M} = \pi^* \gamma_{J_F}$.
Theorem The C-space $M = G/L$ is spin if $\mathfrak{t}_1^*$ contains all black fundamental weights from $\Pi_B$ with odd Koszul numbers.