A SET OF AXIOMS FOR THE DEGREE OF TANGENT VECTOR FIELDS ON DIFFERENTIABLE MANIFOLDS

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Abstract. Given a tangent vector field on a finite dimensional real smooth manifold, its degree (also known as *characteristic* or *rotation*) is, in some sense, an algebraic count of its zeros and gives useful information for its associated ordinary differential equation. When, in particular, the ambient manifold is an open subset U of $\mathbb{R}^m$, a tangent vector field v on U can be identified with a map $\vec{v}\colon U \to \mathbb{R}^m$, and its degree, when defined, coincides with the Brouwer degree with respect to zero of the corresponding map $\vec{v}$.

As well known, the Brouwer degree in $\mathbb{R}^m$ is uniquely determined by three axioms, called *Normalization*, *Additivity* and *Homotopy Invariance*. Here we shall provide a simple proof that in the context of differentiable manifolds the degree of a tangent vector field is uniquely determined by suitably adapted versions of the above three axioms.

1. Introduction

The degree of a tangent vector field on a differentiable manifold is a very well known tool of nonlinear analysis used, in particular, in the theory of ordinary differential equations and dynamical systems. This notion is more often known with the names of *rotation* or of *(Euler) characteristic* of a vector field (see e.g. [2, 3, 6, 7, 8, 10]). Here, we depart from the established tradition by choosing the name "degree" because of the following consideration: In the case when the ambient manifold is an open subset U of $\mathbb{R}^m$, there is a natural identification of a vector field v on U with a map $\vec{v}\colon U \to \mathbb{R}^m$, and the degree $\deg(v, U)$ of v on U, when defined, is just the Brouwer degree $\deg_B(\vec{v}, U, 0)$ of $\vec{v}$ on U with respect to zero. Thus the degree of a vector field can be seen as a generalization to the context of differentiable manifolds of the notion of Brouwer degree in $\mathbb{R}^m$. As well known, this extension of $\deg_B$ does not require the orientability of the underlying manifold, and is therefore different from the classical extension of $\deg_B$ for maps acting between oriented differentiable manifolds.

A well known result of Amann and Weiss [1] (see also [4]) asserts that the Brouwer degree in $\mathbb{R}^m$ is uniquely determined by three axioms: Normalization, Additivity and Homotopy Invariance. A similar statement is true (e.g. as a consequence of a result of Staecker [9]) for the degree of maps between oriented differentiable manifolds of the same dimension. In this paper, that is strictly related in both spirit and demonstrative techniques to [5], we shall prove that suitably adapted versions of the above axioms are sufficient to uniquely determine the degree of a tangent vector field on a (not necessarily orientable) differentiable manifold. We will not deal with the problem of existence of such a degree, for which we refer to [2, 3, 6, 7, 8].

2. Preliminaries

Given two sets X and Y, by a *local map* with *source* X and *target* Y we mean a triple $g = (X, Y, \Gamma)$, where Γ, the *graph* of g, is a subset of $X \times Y$ such that for any $x \in X$ there exists at most one $y \in Y$ with $(x, y) \in \Gamma$. The domain $\mathcal{D}(g)$ of g is the set of all $x \in X$ for which there exists $y = g(x) \in Y$ such that $(x, y) \in \Gamma$; namely, $\mathcal{D}(g) = \pi_1(\Gamma)$, where π_1 denotes the projection of $X \times Y$ onto the first factor. The *restriction* of a local map $g = (X, Y, \Gamma)$ to a subset C of X is the triple

$$g|_C = \big(C, Y, \Gamma \cap (C \times Y)\big)$$

with domain $C \cap \mathcal{D}(g)$.

Incidentally, we point out that sets and local maps (with the obvious composition) constitute a category. Although the notation $g\colon X \to Y$ would be acceptable in the context of category theory, it will be reserved for the case when $\mathcal{D}(g) = X$.

Whenever it makes sense (e.g. when source and target spaces are differentiable manifolds), local maps are tacitly assumed to be continuous.

Throughout the paper all the differentiable manifolds will be assumed to be finite dimensional, smooth, real, Hausdorff and second countable. Thus, they can be embedded in some $\mathbb{R}^k$. Moreover, M and N will always denote arbitrary differentiable manifolds. Given any $x \in M$, T_xM will denote the tangent space of M at x. Furthermore TM will be the tangent bundle of M, namely

$$TM = \big\{(x, v) : x \in M, v \in T_xM\big\}.$$

The map $\pi\colon TM \to M$ given by $\pi(x, v) = x$ will be the *bundle projection* of TM. It will be also convenient, given any $x \in M$, to denote by 0_x the zero element of T_xM.

Let $f\colon M \to N$ be smooth. Then, it is defined a map $Tf\colon TM \to TN$ that to each $(x, v) \in TM$ associates $\big(f(x), df_x(v)\big) \in TN$. Here $df_x\colon T_xM \to T_{f(x)}N$ denotes the differential of f at x. Notice that if $f\colon M \to N$ is a diffeomorphism, then so is $Tf\colon TM \to TN$ and one has $T(f^{-1}) = (Tf)^{-1}$.

By a *local tangent vector field on* M we mean a local map v having M as source and TM as target, with the property that the composition $\pi \circ v$ is the identity on $\mathcal{D}(v)$. Therefore, given a local tangent vector field v, for all $x \in \mathcal{D}(v)$ there exists $\vec{v}(x) \in T_xM$ such that $v(x) = \big(x, \vec{v}(x)\big)$.

Let V and W be differentiable manifolds and let $\psi\colon V \to W$ be a diffeomorphism. Recall that two tangent vector fields $v\colon V \to TV$ and $w\colon W \to TW$ *correspond under* ψ if the following diagram commutes:

$$\begin{tikzcd} TV \arrow[r, "T\psi"] & TW \\ V \arrow[u, "v"] \arrow[r, "\psi"] & W \arrow[u, "w"'] \end{tikzcd}$$

Let V be an open subset of M and suppose that v is a local tangent vector field on M with $V \subseteq \mathcal{D}(v)$. We say that v is *identity-like* on V if there exists a diffeomorphism ψ of V onto $\mathbb{R}^m$ such that $v|_V$ and the identity in $\mathbb{R}^m$ correspond under ψ. Notice that any diffeomorphism ψ from an open subset V of M onto $\mathbb{R}^m$ induces an identity-like vector field on V.

Let v be a local tangent vector field on M and let $p \in M$ be a *zero* of v; that is, $\vec{v}(p) = 0_p$. Consider a diffeomorphism φ of a neighborhood $U \subseteq M$ of p onto $\mathbb{R}^m$ and let $w\colon \mathbb{R}^m \to T\mathbb{R}^m$ be the tangent vector field on $\mathbb{R}^m$ that corresponds to v under φ. Since $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$, then the map $\vec{w}$ associated to w sends $\mathbb{R}^m$ into itself. Assuming that v is smooth about p, the function $\vec{w}$ is Fréchet differentiable at $q = \varphi(p)$. Denote by $D\vec{w}(q)\colon \mathbb{R}^m \to \mathbb{R}^m$ its Fréchet derivative and let $v'(p)\colon T_pM \to T_pM$ be the endomorphism of T_pM which makes the following diagram commutative:

$$\begin{array}{ccc} T_pM & \xrightarrow{v'(p)} & T_pM \\ {\scriptstyle d\varphi_p}\downarrow & & \downarrow{\scriptstyle d\varphi_p} \\ \mathbb{R}^m & \xrightarrow{D\vec{w}(q)} & \mathbb{R}^m \end{array} \tag{2.1}$$

Using the fact that p is a zero of v, it is not difficult to prove that $v'(p)$ does not depend on the choice of φ. This endomorphism of T_pM is called the *linearization* of v at p. Observe that when $M = \mathbb{R}^m$, the linearization $v'(p)$ of a tangent vector field v at a zero p is just the Fréchet derivative $D\vec{v}(p)$ at p of the map $\vec{v}$ associated to v.

The following fact will play an important rôle in the proof of our main result.

Remark 2.1. Let v, w, p and q be as above. Then, the commutativity of diagram (2.1) implies

$$\det v'(p) = \det D\vec{w}(q).$$

3. Degree of a tangent vector field

Given an open subset U of M and a local tangent vector field v on M, the pair (v, U) is said to be *admissible on* U if $U \subseteq \mathcal{D}(v)$ and the set

$$\mathcal{Z}(v, U) := \{x \in U : \vec{v}(x) = 0_x\}$$

of the zeros of v in U is compact. In particular, (v, U) is admissible if the closure $\overline{U}$ of U is a compact subset of $\mathcal{D}(v)$ and $\vec{v}$ is nonzero on the boundary ∂U of U.

Given an open subset U of M and a (continuous) local map H with source $M \times [0,1]$ and target TM, we say that H is a *homotopy of tangent vector fields on* U if $U \times [0,1] \subseteq \mathcal{D}(H)$, and if $H(\cdot, \lambda)$ is a local tangent vector field for all $\lambda \in [0,1]$. If, in addition, the set

$$\{(x, \lambda) \in U \times [0,1] : \vec{H}(x, \lambda) = 0_x\}$$

is compact, the homotopy H is said to be *admissible*. Thus, if $\overline{U}$ is compact and $\overline{U} \times [0,1] \subseteq \mathcal{D}(H)$, a sufficient condition for H to be admissible on U is the following:

$$\vec{H}(x, \lambda) \neq 0_x, \quad \forall\, (x, \lambda) \in \partial U \times [0,1],$$

which, by abuse of terminology, will be referred to as "H is nonzero on ∂U".

We shall show that there exists at most one function that to any admissible pair (v, U) assigns a real number $\deg(v, U)$, called the *degree (or characteristic or rotation) of the tangent vector field* v *on* U, which satisfies the following three properties that will be regarded as axioms. Moreover, this function (assuming its existence) must be integer valued.

Normalization. *Let v be identity-like on an open subset U of M. Then,*

$$\deg(v, U) = 1.$$

Additivity. *Given an admissible pair (v, U), if U_1 and U_2 are two disjoint open subsets of U such that $\mathcal{Z}(v, U) \subseteq U_1 \cup U_2$, then*

$$\deg(v, U) = \deg(v|_{U_1}, U_1) + \deg(v|_{U_2}, U_2).$$

Homotopy Invariance. *If H is an admissible homotopy on U, then*

$$\deg\big(H(\cdot, 0), U\big) = \deg\big(H(\cdot, 1), U\big).$$

For now on we shall assume the existence of a function deg defined on the family of all admissible pairs and satisfying the above three properties, that we shall regard as axioms.

Remark 3.1. The pair $(v, \emptyset)$ is admissible. This includes the case when $\mathcal{D}(v)$ is the empty set ($\mathcal{D}(v) = \emptyset$ is coherent with the notion of local tangent vector field). A simple application of the Additivity Property shows that $\deg(v|_\emptyset, \emptyset) = 0$ and $\deg(v, \emptyset) = 0$.

As a consequence of the Additivity Property and Remark 3.1, one easily gets the following (often neglected) property, which shows that the degree of an admissible pair (v, U) does not depend on the behavior of v outside U. To prove it, take $U_1 = U$ and $U_2 = \emptyset$ in the Additivity Property.

Localization. *If (v, U) is admissible, then* $\deg(v, U) = \deg(v|_U, U)$.

A further important property of the degree of a tangent vector field is the following.

Excision. *Given an admissible pair (v, U) and an open subset U_1 of U containing $\mathcal{Z}(v, U)$, one has* $\deg(v, U) = \deg(v, U_1)$.

To prove this property observe that by Additivity, Remark 3.1, and Localization, one gets

$$\deg(v, U) = \deg(v|_{U_1}, U_1) + \deg(v|_\emptyset, \emptyset) = \deg(v, U_1).$$

As a consequence, we finally obtain the following property.

Solution. *If* $\deg(v, U) \neq 0$*, then* $\mathcal{Z}(v, U) \neq \emptyset$.

To obtain it, observe that if $\mathcal{Z}(v, U) = \emptyset$, taking $U_1 = \emptyset$, we get

$$\deg(v, U) = \deg(v, \emptyset) = 0.$$

4. The degree for linear vector fields

By $\mathrm{L}(\mathbb{R}^m)$ we shall mean the normed space of linear endomorphisms of $\mathbb{R}^m$, and by $\mathrm{GL}(\mathbb{R}^m)$ we shall distinguish the group of invertible ones. In this section we shall consider *linear vector fields* on $\mathbb{R}^m$. Namely, vector fields $L\colon \mathbb{R}^m \to T\mathbb{R}^m$ with the property that $\vec{L} \in \mathrm{L}(\mathbb{R}^m)$. Notice that $(L, \mathbb{R}^m)$, with L a linear vector field, is an admissible pair if and only if $\vec{L} \in \mathrm{GL}(\mathbb{R}^m)$.

The following consequence of the axioms asserts that the degree of an admissible pair $(L, \mathbb{R}^m)$, with $\vec{L} \in \mathrm{GL}(\mathbb{R}^m)$, is either 1 or -1.

Lemma 4.1. *Let $\vec{L}$ be a nonsingular linear operator in $\mathbb{R}^m$. Then*

$$\deg(L, \mathbb{R}^m) = \operatorname{sign} \det \vec{L}.$$

Proof. It is well known (see e.g. [11]) that $\mathrm{GL}(\mathbb{R}^m)$ has exactly two connected components. Equivalently, the following two subsets of $\mathrm{L}(\mathbb{R}^m)$ are connected:

$$\mathrm{GL}^+(\mathbb{R}^m) = \{A \in \mathrm{L}(\mathbb{R}^m) : \det A > 0\},$$
$$\mathrm{GL}^-(\mathbb{R}^m) = \{A \in \mathrm{L}(\mathbb{R}^m) : \det A < 0\}.$$

Since the connected sets $\mathrm{GL}^+(\mathbb{R}^m)$ and $\mathrm{GL}^-(\mathbb{R}^m)$ are open in $\mathrm{L}(\mathbb{R}^m)$, they are actually path connected. Consequently, given a linear tangent vector field L on $\mathbb{R}^m$ with $\vec{L} \in \mathrm{GL}(\mathbb{R}^m)$, the Homotopy Invariance implies that $\deg(L, \mathbb{R}^m)$ depends only on the component of $\mathrm{GL}(\mathbb{R}^m)$ containing $\vec{L}$. Therefore, if $\vec{L} \in \mathrm{GL}^+(\mathbb{R}^m)$, one has $\deg(L, \mathbb{R}^m) = \deg(I, \mathbb{R}^m)$, where $\vec{I}$ is the identity on $\mathbb{R}^m$. Thus, by Normalization, we get

$$\deg(L, \mathbb{R}^m) = 1. \tag{4.1}$$

It remains to prove that $\deg(L, \mathbb{R}^m) = -1$ when $\vec{L} \in \mathrm{GL}^-(\mathbb{R}^m)$. For this purpose consider the vector field $f\colon \mathbb{R}^m \to T\mathbb{R}^m$ determined by

$$\vec{f}(\xi_1, \dots, \xi_{m-1}, \xi_m) = (\xi_1, \dots, \xi_{m-1}, |\xi_m| - 1).$$

Notice that $\deg(f, \mathbb{R}^m)$ is well defined, as $\vec{f}^{-1}(0)$ is compact. Observe also that $\deg(f, \mathbb{R}^m)$ is zero, because f is admissibly homotopic in $\mathbb{R}^m$ to the never-vanishing vector field $g\colon \mathbb{R}^m \to T\mathbb{R}^m$ given by $\vec{g}(\xi_1, \dots, \xi_m) = (\xi_1, \dots, |\xi_m| + 1)$.

Let U_- and U_+ denote, respectively, the open half-spaces of the points in $\mathbb{R}^m$ with negative and positive last coordinate. Consider the two solutions

$$x_- = (0, \dots, 0, -1) \quad \text{and} \quad x_+ = (0, \dots, 0, 1)$$

of the equation $\vec{f}(x) = 0$ and observe that $x_- \in U_-$, $x_+ \in U_+$.

By the Additivity Property (and taking into account the Localization) we get

$$0 = \deg(f, \mathbb{R}^m) = \deg(f, U_-) + \deg(f, U_+). \tag{4.2}$$

Now, observe that f in U_+ coincides with the vector field $f_+\colon \mathbb{R}^m \to T\mathbb{R}^m$ determined by

$$\vec{f}_+(\xi_1, \dots, \xi_{m-1}, \xi_m) = (\xi_1, \dots, \xi_{m-1}, \xi_m - 1),$$

that is admissibly homotopic (in $\mathbb{R}^m$) to the tangent vector field $I\colon \mathbb{R}^m \to T\mathbb{R}^m$, given by $I(x) = (x, x)$. Therefore, because of the properties of Localization, Excision, Homotopy Invariance and Normalization, one has

$$\deg(f, U_+) = \deg(f_+, U_+) = \deg(f_+, \mathbb{R}^m) = \deg(I, \mathbb{R}^m) = 1,$$

which, by (4.2), implies

$$\deg(f, U_-) = -1. \tag{4.3}$$

Notice that f in U_- coincides with the vector field $f_-\colon \mathbb{R}^m \to T\mathbb{R}^m$ defined by

$$\vec{f}_-(\xi_1, \dots, \xi_{m-1}, \xi_m) = (\xi_1, \dots, \xi_{m-1}, -\xi_m - 1),$$

which is admissibly homotopic (in $\mathbb{R}^m$) to the linear vector field L_- defined by $\vec{L}_- \in \mathrm{GL}_-(\mathbb{R}^m)$ with

$$\vec{L}_-(\xi_1, \dots, \xi_{m-1}, \xi_m) = (\xi_1, \dots, \xi_{m-1}, -\xi_m).$$

Thus, by Homotopy Invariance, Excision, Localization and formula (4.3)

$$\deg(L_-, \mathbb{R}^m) = \deg(f_-, \mathbb{R}^m) = \deg(f_-, U_-) = \deg(f, U_-) = -1.$$

Hence, $\mathrm{GL}_-(\mathbb{R}^m)$ being path connected, we finally get $\deg(L, \mathbb{R}^m) = -1$ for all linear tangent vector fields L on $\mathbb{R}^m$ such that $\vec{L} \in \mathrm{GL}_-(\mathbb{R}^m)$, and the proof is complete. □

We conclude this section with a consequence as well as an extension of Lemma 4.1. The Euclidean norm of an element $x \in \mathbb{R}^m$ will be denoted by $|x|$.

Lemma 4.2. *Let v be a local vector field on $\mathbb{R}^m$ and let $U \subseteq \mathcal{D}(v)$ be open and such that the equation $\vec{v}(x) = 0$ has a unique solution $x_0 \in U$. If $\vec{v}$ is smooth about x_0 and the linearization $v'(x_0)$ of v at x_0 is invertible, then* $\deg(v, U) = \operatorname{sign}\det v'(x_0)$.

Proof. Since $\vec{v}$ is Fréchet differentiable at x_0 and $D\vec{v}(x_0) = v'(x_0)$, we have

$$\vec{v}(x) = v'(x_0)(x - x_0) + |x - x_0|\epsilon(x - x_0), \quad \forall x \in U,$$

where $\epsilon(h)$ is defined for $h \in -x_0 + U$, is continuous, and such that $\epsilon(0) = 0$. Consider the vector field $g\colon \mathbb{R}^m \to T\mathbb{R}^m$ determined by $\vec{g}(x) = v'(x_0)(x - x_0)$, and let H be the homotopy on U, joining g with v, defined by

$$\vec{H}(x, \lambda) = v'(x_0)(x - x_0) + \lambda|x - x_0|\epsilon(x - x_0).$$

For all x in U we have

$$|\vec{H}(x,\lambda)| \geq \big(m - |\epsilon(x-x_0)|\big)|x-x_0|,$$

where $m = \inf\{|v'(x_0)y| : |y|=1\}$ is positive, $v'(x_0)$ being invertible. This shows that there exists a neighborhood V of x_0 such that $\big(V\times[0,1]\big)\cap \vec{H}^{-1}(0)$ coincides with the compact set $\{x_0\}\times[0,1]$. Thus, by Excision and Homotopy Invariance,

$$\deg(v,U) = \deg(v,V) = \deg(g,V). \tag{4.4}$$

Let $L\colon \mathbb{R}^m \to T\mathbb{R}^m$ be the linear tangent vector field given by $\xi \mapsto \big(\xi, v'(x_0)\xi\big)$. Clearly, L is admissibly homotopic to g in $\mathbb{R}^m$. By Excision, Homotopy Invariance and Lemma 4.1, we get

$$\deg(g,V) = \deg(g,\mathbb{R}^m) = \deg(L,\mathbb{R}^m) = \operatorname{sign}\det \vec{L}. \tag{4.5}$$

The assertion now follows from (4.4), (4.5) and the fact that $\vec{L}$ coincides with $v'(x_0)$. □

5. The uniqueness result

Given a local tangent vector field v on M, a zero p of v is called *nondegenerate* if v is smooth about p and its linearization $v'(p)$ at p is an automorphism of T_pM. It is known that this is equivalent to the assumption that v is transversal at p to the zero section $M_0 = \big\{(x,0_x)\in TM : x\in M\big\}$ of TM (for the theory of transversality see e.g. [6, 7]). We recall that a nondegenerate zero is, in particular, an isolated zero.

Let v be a local tangent vector field on M. A pair (v,U) will be called *nondegenerate* if U is a relatively compact open subset of M, v is smooth on a neighborhood of the closure $\overline{U}$ of U, is nonzero on ∂U, and all its zeros in U are nondegenerate. Note that, in this case, (v,U) is an admissible pair and $\mathcal{Z}(v,U)$ is a discrete set, therefore finite, being closed in the compact set $\overline{U}$.

The following result, which is an easy consequence of transversality theory, shows that the computation of the degree of any admissible pair can be reduced to that of a nondegenerate pair.

Lemma 5.1. *Let v be a local tangent vector field on M and let (v,U) be admissible. Let V be a relatively compact open subset of M containing $\mathcal{Z}(v,U)$ and such that $\overline{V}\subseteq U$. Then, there exists a local tangent vector field w on M which is admissibly homotopic to v in V and such that (w,V) is a nondegenerate pair. Consequently,* $\deg(v,U)=\deg(w,V)$.

Proof. Without loss of generality we can assume $M\subseteq\mathbb{R}^k$. Let

$$\delta = \min_{x\in\partial V}\big|\vec{v}(x)\big| > 0.$$

From the Transversality Theorem (see e.g. [6, 7]) it follows that one can find a smooth tangent vector field $w\colon U\to TU\subseteq TM$ that is transversal to the zero section M_0 of TM and such that

$$\max_{x\in\partial V}\big|\vec{v}(x)-\vec{w}(x)\big| < \delta.$$

Since M_0 is closed in TM, the set $\mathcal{Z}(w,V) = w^{-1}(M_0)\cap\overline{V}$ is a compact subset of $\overline{V}$. Thus, this inequality shows that (w,V) is admissible. Moreover, at any zero $x\in\mathcal{Z}(w,U) = w^{-1}(M_0)\cap U$ the endomorphism $w'(x)\colon T_xM\to T_xM$ is invertible. This implies that (w,V) is nondegenerate.

The conclusion follows by observing that the homotopy H on U of tangent vector fields given by

$$\vec{H}(x,\lambda) = \lambda\vec{v}(x) + (1-\lambda)\vec{w}(x)$$

is nonzero on $\partial V \times [0,1]$ and therefore it is admissible on V. The last assertion follows from Excision and Homotopy Invariance. □

We will now show that the properties of Normalization, Additivity and Homotopy Invariance imply a formula for the computation of the degree of a tangent vector field that is valid for any nondegenerate pair. Therefore, Lemma 5.1 and the properties of Excision and Homotopy Invariance imply the existence of at most one real function on the family of admissible pairs that satisfies the axioms for the degree of a tangent vector field.

Theorem 5.2 (Uniqueness of the degree)**.** *Let* deg *be a real function on the family of admissible pairs satisfying the properties of Normalization, Additivity and Homotopy Invariance. If (v,U) is a nondegenerate pair, then*

$$\deg(v,U) = \sum_{x\in\mathcal{Z}(v,U)} \operatorname{sign}\det v'(x).$$

Consequently, there exists at most one function on the family of admissible pairs satisfying the axioms for the degree of a tangent vector field, and this function, assuming its existence, must be integer-valued.

Proof. Consider first the case $M=\mathbb{R}^m$. Let (v,U) be a nondegenerate pair in $\mathbb{R}^m$ and, for any $x\in\mathcal{Z}(v,U)$, let V_x be an isolating neighborhood of x. We may assume that the neighborhoods V_x's are pairwise disjoint. The Additivity and Localization properties together with Lemma 4.2 yield

$$\deg(v,U) = \sum_{x\in\mathcal{Z}(v,U)} \deg(v,V_x) = \sum_{x\in\mathcal{Z}(v,U)} \operatorname{sign}\det v'(x).$$

Now the uniqueness of the degree of a tangent vector field on $\mathbb{R}^m$ follows immediately from Lemma 5.1.

Let us now consider the general case and denote by m the dimension of M. Let W be any open subset of M which is diffeomorphic to the whole space $\mathbb{R}^m$ and let $\psi\colon W\to\mathbb{R}^m$ be any diffeomorphism onto $\mathbb{R}^m$. Denote by $\mathcal{U}$ the set of all pairs (v,U) which are admissible and such that $U\subseteq W$. We claim that for any $(v,U)\in\mathcal{U}$ one necessarily has

$$\deg(v,U) = \deg\big(T\psi\circ v\circ\psi^{-1},\psi(U)\big).$$

To show this, denote by $\mathcal{V}$ the set of admissible pairs (w,V) with $V\subseteq\mathbb{R}^m$ and consider the map $\alpha\colon\mathcal{U}\to\mathcal{V}$ defined by

$$\alpha(v,U) = \big(T\psi\circ v\circ\psi^{-1},\psi(U)\big).$$

Our claim means that the restriction $\deg|_{\mathcal{U}}$ of deg to $\mathcal{U}$ coincides with $\deg\circ\alpha$. Observe that α is invertible and

$$\alpha^{-1}(w,V) = \big(T\psi^{-1}\circ w\circ\psi,\psi^{-1}(V)\big),$$

and if two pairs $(v,U)\in\mathcal{U}$ and $(w,V)\in\mathcal{V}$ correspond under α, then the sets $\mathcal{Z}(v,U)$ and $\mathcal{Z}(w,V)$ correspond under ψ. It is also evident that the function $\deg\circ\alpha^{-1}\colon\mathcal{V}\to\mathbb{R}$ satisfies the axioms. Thus, by the first part of the proof, it coincides with the restriction $\deg|_{\mathcal{V}}$, and this implies our as claim.

Let now (v,U) be a given nondegenerate pair in M. Let $\mathcal{Z}(v,U)=\{x_1,\dots,x_n\}$ and let $W_1,\dots,W_n$ be n pairwise disjoint open subsets of U such that $x_j\in W_j$, for $j=1,\dots,n$. Since any point of M has a fundamental system of neighborhoods which are diffeomorphic to the whole space $\mathbb{R}^m$, we may assume that each W_j is diffeomorphic to $\mathbb{R}^m$ under a diffeomorphism ψ_j. The Additivity Property yields

$$\deg(v,U) = \sum_{j=1}^{n} \deg(v,W_j),$$

and, by the above claim, we get

$$\sum_{j=1}^{n} \deg(v, W_j) = \sum_{j=1}^{n} \deg\big(T\psi_j \circ v \circ \psi_j^{-1}, \psi_j(W_j)\big).$$

By Lemma 4.2, and Remark 2.1 one has

$$\begin{aligned} \deg\big(T\psi_j \circ v \circ \psi_j^{-1}, \psi_j(W_j)\big) &= \operatorname{sign}\det \Big(T\psi_j \circ v \circ \psi_j\Big)'\big(\psi_j(x_j)\big) \\ &= \operatorname{sign}\det v'(x_j), \end{aligned}$$

for $j = 1, \dots, n$. Thus

$$\deg(v, U) = \sum_{j=1}^{n} \operatorname{sign}\det v'(x_j).$$

As in the case when $M = \mathbb{R}^m$, the uniqueness of the degree of a tangent vector field is now a consequence of Lemma 5.1. □

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