A NOTE ON TOPOLOGICAL METHODS FOR A CLASS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. We study a particular class of autonomous Differential-Algebraic Equations that are equivalent to Ordinary Differential Equations on manifolds. Under appropriate assumptions we determine a straightforward formula for the computation of the degree of the associated tangent vector field that does not require any explicit knowledge of the manifold. We use this formula to study the set of harmonic solutions to periodic perturbations of our equations. Two different classes of applications are provided.

1. Introduction

In this paper we apply topological methods to the study of the set of periodic solutions of periodic perturbations of a particular class of differential-algebraic equations (DAEs). Namely, we consider the following DAE in semi-explicit form:

$$\begin{cases} \dot{x} = f(x,y), \\ g(x,y) = 0, \end{cases} \tag{1.1}$$

where $g : U \to \mathbb{R}^s$ and $f : U \to \mathbb{R}^k$ are continuous maps defined on an open connected set $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$, with $g \in C^\infty$ and $\partial_2 g(p,q)$, the partial derivative of g with respect to the second variable, invertible for each $(p,q) \in U$. Given $T > 0$, we will consider T-periodic perturbations of f in (1.1) and study the set of T-periodic solution of the resulting T-periodic DAE. Namely, for $\lambda \geq 0$, we look at the T-periodic solutions of

$$\begin{cases} \dot{x} = f(x,y) + \lambda h(t,x,y), \\ g(x,y) = 0, \end{cases} \tag{1.2}$$

where $h : \mathbb{R} \times U \to \mathbb{R}^k$ is continuous and T-periodic in the first variable. Roughly speaking, we will give conditions ensuring the existence of a connected component of elements $(\lambda; x, y)$, $\lambda \geq 0$ and (x, y) a T-periodic solution to (1.2), that emanates from the set of constant solutions of (1.1) and is not compact. This kind of results is useful to study existence and multiplicity of T-periodic solutions of (1.2).

Since $\partial_2 g(p,q)$ is invertible for all $(p,q) \in U$, equations (1.1) and (1.2) are index 1 differential algebraic equation and have strangeness index 0 (see e.g. [8]). However, our argument will not require any knowledge of the theory of DAEs.

The assumption on $\partial_2 g(p,q)$ implies that $0 \in \mathbb{R}^s$ is a regular value of g, thus $M := g^{-1}(0)$ is a C^∞ submanifold of $\mathbb{R}^k \times \mathbb{R}^s$. Notice that M, locally, can be represented as graph of some map from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^s$. Thus equations (1.1) and (1.2) can be locally decoupled. However, globally, this might not be true. Observe also that even when M is a graph of some map φ, it might happen that the expression of φ is complicated (or even impossible to determine analytically), so that the decoupled version of (1.1) or (1.2) may be impractical. We have a very

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simple example of this fact if we take $k = s = 1$, $U = \mathbb{R} \times \mathbb{R}$, $g(p,q) = q^7 + q - p^2$ and $f(p,q) = q$.

It is well known (compare [8, §4.5]) and easy to see that, when $\partial_2 g(p,q)$ is invertible for all $(p,q) \in U$, equation (1.1) induces a tangent vector field Ψ on M, that is, it gives rise to an autonomous ordinary differential equation on M. Equation (1.2), then, leads to a T-periodic perturbation of this ODE. In our main result (Theorem 5.1 below), in order to get information about the set of T-periodic solutions of (1.2), we apply an argument of [4] about periodic perturbation of an autonomous ordinary differential equation on a differentiable manifold. The results of [4], however, require some knowledge of the degree of the perturbed tangent vector field. In the present setting this means the degree of the tangent vector field Ψ on M. Since M is known only implicitly, and the form of Ψ may not be very simple, a direct application of [4] is of limited interest. Thus, our first step will be to determine a formula (Theorem 4.1 below) that allows the computation of the absolute value of the degree of Ψ by means of the degree of the "morally" simpler vector field $F : U \to \mathbb{R}^k \times \mathbb{R}^s$, given by

$$(p,q) \mapsto \big(f(p,q), g(p,q)\big). \tag{1.3}$$

We stress the fact (as we shall briefly discuss below) that, since in Euclidean spaces vector fields can be regarded as maps and vice versa, the degree of the vector field F is essentially the well known Brouwer degree, with respect to 0, of F seen as a map. Hence the degree of F has a simpler nature than that of Ψ and, as a consequence, it is usually easier to compute.

Notation. Throughout this paper, $|\cdot|$ will denote the absolute value in $\mathbb{R}$ while $|\cdot|_n$ will be the norm in $\mathbb{R}^n$ given by

$$|a|_n = \sum_{i=1}^{n} |a_i| \quad \text{for all } a = (a_1, \ldots, a_n) \in \mathbb{R}^n,$$

Thus, coherently with this notation we have $\big|(p,q)\big|_{k+s} = |p|_k + |q|_s$, for $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s$.

2. Associated vector fields

In this section, we associate ordinary differential equations on the manifold $M = g^{-1}(0)$ to (1.1) and to (1.2), in quite a natural way (compare [8, §4.5]).

Let $I \subseteq \mathbb{R}$ be an interval and $W \subseteq \mathbb{R}^n$ be open. Given $r \in \mathbb{N} \cup \{0\}$, the set of all W-valued C^r functions defined on I is denoted by by $C^r(I,W)$. For simplicity, we use $C(I,W)$ as a synonym of $C^0(I,W)$.

Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, and let $g : U \to \mathbb{R}^s$, $f : U \to \mathbb{R}^k$ and $h : \mathbb{R} \times U \to \mathbb{R}^k$ be continuous maps with $g \in C^\infty$ and $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$. We also assume, throughout this paper, that h is T-periodic in the first variable for some given $T > 0$.

A solution of (1.2) for a given $\lambda \geq 0$ consists of a pair of functions $x \in C^1(I, \mathbb{R}^k)$ and $y \in C(I, \mathbb{R}^s)$, I an interval, with the property that

$$\begin{cases} \dot{x}(t) = f\big(x(t), y(t)\big) + \lambda h\big(t, x(t), y(t)\big), \\ g\big(x(t), y(t)\big) = 0, \end{cases}$$

for each $t \in I$. Notice that the assumptions on g and the Implicit Function Theorem imply that y is actually a C^1 function. In fact, in what follows, it will be convenient to consider a solution of (1.2) as a function $\zeta := (x,y) \in C^1(I, \mathbb{R}^k \times \mathbb{R}^s)$.

Let $(x,y) \in C^1(I, \mathbb{R}^k \times \mathbb{R}^s)$ be a solution of (1.2) for a given $\lambda \geq 0$, defined on some interval $I \subseteq \mathbb{R}$. Then, differentiating the identity $g\big(x(t), y(t)\big) = 0$, we get

$$\partial_1 g\big(x(t), y(t)\big)\dot{x}(t) + \partial_2 g\big(x(t), y(t)\big)\dot{y}(t) = 0,$$

which yields

$$\dot{y}(t) = -\big[\partial_2 g\big(x(t), y(t)\big)\big]^{-1} \partial_1 g\big(x(t), y(t)\big) \Big[f\big(x(t), y(t)\big) + \lambda h\big(t, x(t), y(t)\big)\Big] \tag{2.1}$$

for all $t \in I$.

As already observed, because of the assumptions on $\partial_2 g(p,q)$, $0 \in \mathbb{R}^s$ is a regular value of g. Thus, $M = g^{-1}(0)$ is a C^∞ submanifold of $\mathbb{R}^k \times \mathbb{R}^s$ and, given $(p,q) \in M$, the tangent space $T_{(p,q)}M$ to M at (p,q) is given by the kernel $\ker d_{(p,q)}g$ of the differential $d_{(p,q)}g$ of g at (p,q).

Consider $\Psi : M \to \mathbb{R}^k \times \mathbb{R}^s$ and $\Upsilon : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ given by

$$\Psi(p,q) = \big(f(p,q)\,,\, -[\partial_2 g(p,q)]^{-1} \partial_1 g(p,q) f(p,q)\big)\,, \tag{2.2a}$$

and

$$\Upsilon(t,p,q) = \big(h(t,p,q), -[\partial_2 g(p,q)]^{-1} \partial_1 g(p,q) h(t,p,q)\big)\,. \tag{2.2b}$$

Clearly, Υ is T-periodic in the first variable. Let us show that Ψ and Υ are tangent to M in the sense that, for any $(t,p,q) \in \mathbb{R} \times M$,

$$\Psi(p,q) \in T_{(p,q)}M \quad \text{and} \quad \Upsilon(t,p,q) \in T_{(p,q)}M. \tag{2.3}$$

Consider for instance Ψ. We have

$$d_{(p,q)}g[\Psi(p,q)] = \begin{pmatrix} \partial_1 g(p,q) & \partial_2 g(p,q) \end{pmatrix} \begin{pmatrix} f(p,q) \\ -[\partial_2 g(p,q)]^{-1} \partial_1 g(p,q) f(p,q) \end{pmatrix} = 0.$$

Since $T_{(p,q)}M = \ker d_{(p,q)}g$, the first relation in (2.3) is proved. The second one follows from a similar argument and is left to the reader.

Taking (2.1) into account, one can see that (1.2) is equivalent to the following ODE on M:

$$\dot{\zeta} = \Psi(\zeta) + \lambda \Upsilon(t,\zeta), \qquad \lambda \geq 0, \tag{2.4}$$

where, we recall, $\zeta = (x,y)$. By the same argument one can see that (1.1) is equivalent to

$$\dot{\zeta} = \Psi(\zeta). \tag{2.5}$$

Remark 2.1. *Let g and f and h be as above. When f is C^1, so is the vector field Ψ. Thus, by virtue of the equivalence of* (1.1) *with* (2.5)*, the local results on existence, uniqueness and continuous dependence of local solutions of the initial value problems translate to* (1.1) *from the theory of ordinary differential equations on manifolds. Of course, if also h is C^1, a similar statement holds for* (1.2).

Notice that the importance of the hypotheses on g goes beyond ensuring the smoothness of M. In fact, even when M is a differentiable manifold and g is C^∞, if we drop our assumption on $\partial_2 g$, (1.1) may fail to induce a (continuous) tangent vector field Ψ on M and, even if this happens, (1.1) might not be equivalent to (2.5). The following simple examples illustrates these possibilities.

Example 2.2. *Take $k = s = 1$ and let $U = \mathbb{R} \times \mathbb{R}$. Consider the following DAE:*

$$\dot{x} = 1, \qquad x - y^3 = 0. \tag{2.6}$$

Clearly, $M = \{(p,q) \in \mathbb{R} \times \mathbb{R} : p = q^3\}$ is a C^∞ submanifold of $\mathbb{R} \times \mathbb{R}$. Equation (2.6) *induces the vector field*

$$(p,q) \mapsto \left(1, \frac{1}{3q^2}\right)$$

on all points of M, with the exception of $(0,0)$. Clearly, this vector field cannot be extended to a continuous tangent vector field on M.

Example 2.3. *Let $k = s = 1$ and let $U = \mathbb{R} \times \mathbb{R}$, as in the previous example. Consider the following DAE:*

$$\dot{x} = y, \qquad x^2 + y^2 = 1. \tag{2.7}$$

In this case, the manifold M is the unit circle S^1 of $\mathbb{R} \times \mathbb{R}$ centered at the origin. Clearly, (2.7) *induces on $S^1 \setminus \{(\pm 1, 0)\}$ the vector field $(p, q) \mapsto (q, -p)$ that can be extended uniquely to a vector field Ψ defined on the whole S^1. Notice, however, that* (2.7) *is not equivalent to* (2.5) *on S^1. In fact, the maps $t \mapsto (\pm 1, 0)$ are solutions of* (2.7)*, but not of* (2.5)*.*

Observe that taking $U = (\mathbb{R} \times \mathbb{R}) \setminus \{(0, 0)\}$ in Example 2.2, the manifold $M = \{(p, q) \in U : p = q^3\}$ consists of two connected sets which the vector field $\Psi(p, q) = \big(1, 1/(3q^2)\big)$ is tangent to. Now, (2.6) turns out to be equivalent to (2.5) on M.

Similarly, taking $U = (\mathbb{R}\times\mathbb{R}) \setminus \{(\pm 1, 0)\}$ in Example 2.3, one has that M consists of two connected components and the above construction of Ψ can be carried out on M.

In order to investigate the T-periodic solutions of (1.2) we will study the set of T-periodic solutions of the equivalent equation (2.4). Our first step will be to consider the case $\lambda = 0$ and determine a formula for the computation of the degree (sometimes called characteristic or rotation) of the tangent vector field Ψ on U. Before doing that, however, we will recall some basic facts about the notion of the degree of a tangent vector field.

3. The degree of a tangent vector field

We now recall some basic notions about tangent vector fields on manifolds.

Let $M \subseteq \mathbb{R}^n$ be a manifold. Given any $p \in M$, $T_pM \subseteq \mathbb{R}^n$ denotes the tangent space of M at p. Let w be a tangent vector field on M, that is, a continuous map $w : M \to \mathbb{R}^n$ with the property that $w(p) \in T_pM$ for any $p \in M$. If w is (Fréchet) differentiable at $p \in M$ and $w(p) = 0$, then the differential $d_pw : T_pM \to \mathbb{R}^k$ maps T_pM into itself (see e.g. [9]), so that the determinant $\det d_pw$ of d_pw is defined. If, in addition, p is a nondegenerate zero (i.e. $d_pw : T_pM \to \mathbb{R}^n$ is injective) then p is an isolated zero and $\det d_pw \neq 0$.

Let W be an open subset of M in which we assume w admissible for the degree; that is, the set $w^{-1}(0) \cap W$ is compact. Then, one can associate to the pair (w, W) an integer, $\deg(w, W)$, called the *degree (or characteristic) of the vector field w in W*, which, roughly speaking, counts (algebraically) the zeros of w in W (see e.g. [3, 7, 9] and references therein). For instance, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap W$ is finite and

$$\deg(w, W) = \sum_{q \in w^{-1}(0) \cap W} \operatorname{sign} \det d_q w. \tag{3.1}$$

When $M = \mathbb{R}^n$, $\deg(w, W)$ is just the classical Brouwer degree, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap W$ whose closure is contained in W.

For the purpose of future reference, we mention a few of the properties of the degree of a tangent vector field that shall be useful in the sequel. Here W is an open subset of a manifold $M \subseteq \mathbb{R}^n$ and $w : M \to \mathbb{R}^n$ is a tangent vector field.

Solution: *If (w, W) is admissible and $\deg(w, W) \neq 0$, then w has a zero in W.*

Additivity: *Let (w, W) be admissible. If W_1 and W_2 are two disjoint open subsets of W whose union contains $w^{-1}(0) \cap W$, then*

$$\deg(w, W) = \deg(w, W_1) + \deg(w, W_2).$$

Homotopy Invariance: *Let $h : M \times [0,1] \to \mathbb{R}^n$ be a homotopy of tangent vector fields admissible in W; that is, $h(p,\lambda) \in T_pM$ for all $(p,\lambda) \in M \times [0,1]$ and $h^{-1}(0) \cap W \times [0,1]$ is compact. Then $\deg\big(h(\cdot,\lambda), W\big)$ is independent of λ.*

Invariance under diffeomorphisms: *Let $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ be differentiable manifolds and let $v : N \to \mathbb{R}^n$ and $w : M \to \mathbb{R}^m$ be tangent vector fields. Let also $V \subseteq N$ and $W \subseteq M$ be open, and let $\varphi : W \to V$ be a diffeomorphism. If*

$$v(q) = d_{\varphi^{-1}(q)}\varphi\big[w\big(\varphi^{-1}(q)\big)\big] \quad \forall q \in V,$$

we say that $v|_V$ and $w|_W$ correspond under the diffeomorphism φ. In this case, if either v is admissible in V or or w is admissible in W, then so is the other and

$$\deg(v,V) = \deg(w,W).$$

Remark 3.1. *Let $M \subseteq \mathbb{R}^n$ be a differentiable manifold and let $W \subseteq M$ be open and relatively compact. If $w : M \to \mathbb{R}^n$ is such that $w(p) \neq 0$ on the boundary $\mathrm{Fr}\,(W)$ of W, then (w,W) is admissible. Let $\varepsilon = \min_{p\in \mathrm{Fr}\,(W)} |w(p)|_n$. Then, for any $v : M \to \mathbb{R}^n$ such that $\max_{p\in \mathrm{Fr}\,(W)} |w(p) - v(p)|_n < \varepsilon$, we have that (v,W) is admissible and that the homotopy $h : M \times [0,1] \to \mathbb{R}^n$ given by*

$$h(p,\lambda) = \lambda w(p) + (1-\lambda)v(p)$$

is admissible in W. Hence, by the Homotopy Invariance Property,

$$\deg(w,W) = \deg(v,W).$$

The Additivity Property implies the following important one:

Excision: *Let (w,W) be admissible. If $V \subseteq W$ is open and contains $w^{-1}(0) \cap V$, then $\deg(w,W) = \deg(w,V)$.*

The Excision Property allows the introduction of the notion of index of an isolated zero of a tangent vector field. Let $w : M \to \mathbb{R}^n$ be a vector field tangent to the differentiable manifold $M \subseteq \mathbb{R}^n$, and let $q \in M$ be an isolated zero of w. Clearly, $\deg(w,V)$ is well defined for each open $V \subseteq M$ such that $V \cap w^{-1}(0) = \{q\}$. By the Excision Property $\deg(w,V)$ is constant with respect to such V's. This common value of $\deg(w,V)$ is, by definition, the *index of w at q*, and is denoted by $\mathrm{i}\,(w,q)$. Using this notation, if (w,W) is admissible, by the Additivity Property we get that if all the zeros in W of w are isolated, then

$$\deg(w,W) = \sum_{q\in w^{-1}(0)\cap W} \mathrm{i}\,(w,q). \tag{3.2}$$

By formula (3.1) we have that if q is a nondegenerate zero of w, then

$$\mathrm{i}\,(w,q) = \operatorname{sign}\det d_q w.$$

Notice that (3.1) and (3.2) differ in the fact that, in the latter, the zeros of w are not necessarily nondegenerate as they have to be in the former. In fact, in (3.2), w need not be differentiable at its zeros.

4. The degree of Ψ

In this section we shall obtain a simple formula for the computation of the degree of Ψ that does not require an "explicit" expression of the manifold $M = g^{-1}(0)$. Namely, let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be be open and connected. Define $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ by (1.3) and $\Psi : M \to \mathbb{R}^k \times \mathbb{R}^s$ by (2.2a). We shall prove a formula that allows the computation of $\deg(\Psi, M)$ from the degree of F in U (notice that $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ being an open set is a differentiable manifold, so that $\deg(F,U)$ makes sense).

Theorem 4.1. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, and let $g : U \to \mathbb{R}^s$ and $f : U \to \mathbb{R}^k$ be such that f is continuous and g is C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$. Let also $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ and $\Psi : M \to \mathbb{R}^k \times \mathbb{R}^s$ be given by* (1.3) *and* (2.2a)*, respectively. If either* $\deg(\Psi, M)$ *or* $\deg(F, U)$ *is well defined, so is the other, and*

$$\big|\deg(\Psi, M)\big| = \big|\deg(F, U)\big|. \tag{4.1}$$

The proof of Theorem 4.1 makes use of the following technical lemma.

Lemma 4.2. *Let U, f and g be as in Theorem 4.1. Given $\varepsilon > 0$, there exists a C^1 map $f_\varepsilon : U \to \mathbb{R}^k$ such that*

$$\sup_{(p,q)\in U} \big|f_\varepsilon(p,q) - f(p,q)\big|_k < \varepsilon$$

and such that $(0,0) \in \mathbb{R}^k \times \mathbb{R}^s$ is a regular value of the map $F_\varepsilon : U \to \mathbb{R}^k \times \mathbb{R}^s$ given by $F_\varepsilon(p,q) = \big(f_\varepsilon(p,q), g(p,q)\big)$.

Proof. What follows is a fairly usual argument in transversality theory (see e.g. [6]). For the sake of completeness, though, we will provide a complete proof.

By standard approximation results in Euclidean spaces, there exists a C^1 map $\tilde f : U \to \mathbb{R}^k$ such that

$$\sup_{(p,q)\in U} \big|\tilde f(p,q) - f(p,q)\big|_k < \frac{\varepsilon}{2}.$$

Denote by B the $\frac{\varepsilon}{2}$-ball of $\mathbb{R}^k$ centered at the origin, and define $\mathcal{F} : \mathbb{R}^k \times \mathbb{R}^s \times B \to \mathbb{R}^k \times \mathbb{R}^s$ by $\mathcal{F}(p,q,b) = \big(\tilde f(p,q) + b, g(p,q)\big)$. Since the origin of $\mathbb{R}^s$ is a regular value for g, the origin $(0,0) \in \mathbb{R}^k \times \mathbb{R}^s$ is a regular value for $\mathcal{F}$. Thus, $X = \mathcal{F}^{-1}(0,0)$ is a k-dimensional C^1 submanifold of $\mathbb{R}^k \times \mathbb{R}^s \times \mathbb{R}^k$.

Denote by π the projection of $\mathbb{R}^k \times \mathbb{R}^s \times B$ onto its third factor. Clearly, the restriction $\pi|_X$ of π to X is C^1. By the well known Morse–Sard Theorem (see e.g. [7]) it is possible to choose an element $\bar b \in B$ which is a regular value for $\pi|_X$. Let us show that, with such a choice of $\bar b$, $(0,0) \in \mathbb{R}^k \times \mathbb{R}^s$ is a regular value of the map $\mathcal{F}_{\bar b} = \mathcal{F}(\cdot,\cdot,\bar b)$.

To see that, we need to show that for any $(p,q) \in \mathcal{F}_{\bar b}^{-1}(0,0)$, the differential $d_{(p,q)}\mathcal{F}_{\bar b} : T_{(p,q)}(\mathbb{R}^k \times \mathbb{R}^s) \to T_{(0,0)}(\mathbb{R}^k \times \mathbb{R}^s)$ of $\mathcal{F}_{\bar b}$ at (p,q) is surjective. We will prove that, given any $\alpha = (\alpha_1, \alpha_2) \in T_{(p,q)}(\mathbb{R}^k \times \mathbb{R}^s) = \mathbb{R}^k \times \mathbb{R}^s$, there exists $v \in T_{(0,0)}(\mathbb{R}^k \times \mathbb{R}^s) = \mathbb{R}^k \times \mathbb{R}^s$ such that $d_{(p,q)}\mathcal{F}_{\bar b} v = \alpha$.

Denote by $d_{(p,q,\bar b)}\mathcal{F}$ the differential of $\mathcal{F}$ at $(p,q,\bar b)$. Since $(0,0) \in \mathbb{R}^k \times \mathbb{R}^s$ is a regular value for $\mathcal{F}$, there exists an element (w_1, w_2, e) of $\mathbb{R}^k \times \mathbb{R}^s \times \mathbb{R}^k$ (i.e. of the tangent space to $\mathbb{R}^k \times \mathbb{R}^s \times B$ at $(p,q,\bar b)$) such that $d_{(p,q,\bar b)}\mathcal{F}(w_1,w_2,e) = \alpha$. Moreover, since $\bar b$ is a regular value for $\pi|_X$, the differential of $\pi|_X$ at $(p,q,\bar b)$

$$d_{(p,q,\bar b)}\pi|_X : T_{(p,q,\bar b)}X \to T_{\bar b}B = \mathbb{R}^k$$

is surjective. Thus, there exists an element $(u_1,u_2,e) \in T_{(p,q,\bar b)}X$ such that

$$d_{(p,q,\bar b)}\pi|_X(u_1,u_2,e) = e.$$

Observe that $d_{(p,q,\bar b)}\mathcal{F}(u_1,u_2,e) = (0,0)$ because, as it is well known, $T_{(p,q,\bar b)}X = \ker d_{(p,q,\bar b)}\mathcal{F}$. Thus, taking $v = (w_1 - u_1, w_2 - u_2)$, and $\bar v = (w_1 - u_1, w_2 - u_2, 0)$ we have

$$\begin{aligned} d_{(p,q)}\mathcal{F}_{\bar b} v = d_{(p,q,\bar b)}\mathcal{F}\bar v &= d_{(p,q,\bar b)}\mathcal{F}(w_1 - u_1, w_2 - u_2, 0) \\ &= d_{(p,q,\bar b)}\mathcal{F}\big((w_1,w_2,e) - (u_1,u_2,e)\big) \\ &= d_{(p,q,\bar b)}\mathcal{F}(w_1,w_2,e) = \alpha. \end{aligned}$$

Thus, $(0,0) \in \mathbb{R}^k \times \mathbb{R}^s$ is a regular value of $\mathcal{F}_{\bar{b}}$ as claimed.

To conclude the proof it is now sufficient to define $f_\varepsilon(p,q) = \tilde{f}(p,q) + \bar{b}$ for all $(p,q) \in U$. $\square$

Proof of Theorem 4.1. The first part of the assertion is an obvious consequence of the fact that $F^{-1}(0,0)$ coincides with the set $\{(p,q) \in M : \Psi(p,q) = (0,0)\}$.

We now proceed to prove (4.1). Let $\mathfrak{s}$ be the constant sign of $\det \partial_2 g(p,q)$ in the connected set U. The following formula:

$$\deg(\Psi, M) = \mathfrak{s} \deg(F, U), \tag{4.2}$$

obviously imply (4.1). Let us prove (4.2). Let V be an open and bounded subset of U with the property that the closure $\overline{V}$ of V is contained in U. Assume that $F^{-1}(0,0) \subseteq V$, clearly one has that $\Psi^{-1}(0,0)$ is contained in V as well and, by the excision property of the degree of a vector field, we get

$$\deg(F, U) = \deg(F, V), \qquad \deg(\Psi, M) = \deg(\Psi, V \cap M).$$

Therefore it is sufficient to prove that $\deg(\Psi, V \cap M) = \mathfrak{s} \deg(F, V)$.

Let $\varepsilon = \min\{|F(p,q)|_{k+s} : (p,q) \in \operatorname{Fr}(V)\}$. By Lemma 4.2, one can find a C^1 map $f_\varepsilon : U \to \mathbb{R}^k$, with

$$\sup_{(p,q)\in U} \big| f_\varepsilon(p,q) - f(p,q) \big|_k < \varepsilon, \tag{4.3}$$

and such that $(0,0)$ is a regular value of $F_\varepsilon : U \to \mathbb{R}^k \times \mathbb{R}^s$ given by $F_\varepsilon(p,q) = \big(f_\varepsilon(p,q), g(p,q)\big)$. Consider $\Psi_\varepsilon : M \to \mathbb{R}^k \times \mathbb{R}^s$ given by

$$\Psi_\varepsilon(p,q) = \big(f_\varepsilon(p,q), -[\partial_2 g(p,q)]^{-1} \partial_1 g(p,q) f_\varepsilon(p,q)\big)$$

for any $(p,q) \in M$. Clearly Ψ_ε is tangent to M. By (4.3) we have

$$\max_{(p,q)\in \operatorname{Fr}(V)} \big| F_\varepsilon(p,q) - F(p,q) \big|_{k+s} < \varepsilon,$$

so that, as in Remark 3.1, we have that $\deg(F_\varepsilon, V) = \deg(F, V)$. Also, the homotopy $(p,q;\lambda) \mapsto \lambda \Psi_\varepsilon(p,q) + (1-\lambda)\Psi(p,q)$ is admissible on $V \cap M$ since its $\mathbb{R}^k$-component never vanishes for $(p,q;\lambda) \in \operatorname{Fr}(V \cap M) \times [0,1]$. Thus, $\deg(\Psi_\varepsilon, V \cap M) = \deg(\Psi, V \cap M)$. Therefore, it is sufficient to show that $\deg(\Psi_\varepsilon, V \cap M) = \mathfrak{s} \deg(F_\varepsilon, V)$.

As with F and Ψ, one has that $F_\varepsilon(p,q) = (0,0)$ if and only if $(p,q) \in M$ and $\Psi_\varepsilon(p,q) = (0,0)$. Since $(0,0)$ is a regular value of F_ε, all the zeros of F_ε are nondegenerate, thus isolated. Since $\overline{V}$ is compact, $F_\varepsilon^{-1}(0,0)$ is finite. Let $F_\varepsilon^{-1}(0,0) = \{(p_i,q_i)\}_{i=1,\dots,n}$. Clearly, for each $i = 1, \dots, n$, $(p_i,q_i) \in M$ and (p_i,q_i) is an isolated zero of Ψ_ε. From (3.1) and (3.2) we have

$$\deg(F_\varepsilon, V) = \sum_{i=1}^n \operatorname{sign} \det d_{(p_i,q_i)} F_\varepsilon$$

$$\deg(\Psi_\varepsilon, V \cap M) = \sum_{i=1}^n \mathrm{i}\big(\Psi_\varepsilon, (p_i,q_i)\big)$$

The assertion follows if we prove that

$$\mathrm{i}\big(\Psi_\varepsilon, (p_i,q_i)\big) = \big(\operatorname{sign} \det \partial_2 g(p_i,q_i)\big) \big(\operatorname{sign} \det d_{(p_i,q_i)} F_\varepsilon\big) \tag{4.4}$$

for $i = 1, \dots, n$.

Let $i \in \{1, \dots, n\}$ be fixed. In order to compute $\operatorname{sign} \det d_{(p_i,q_i)} F_\varepsilon$ we write $d_{(p_i,q_i)} F_\varepsilon$ in block-matrix form:

$$d_{(p_i,q_i)} F_\varepsilon = \begin{pmatrix} \partial_1 f_\varepsilon(p_i,q_i) & \partial_2 f_\varepsilon(p_i,q_i) \\ \partial_1 g(p_i,q_i) & \partial_2 g(p_i,q_i) \end{pmatrix}.$$

Being $\det \partial_2 g(p_i, q_i) \neq 0$, the so-called generalized Gauss algorithm (see e.g. [5]) yields

$$\begin{aligned} \det d_{(p_i,q_i)} F_\varepsilon &= \det \partial_2 g(p_i, q_i)\cdot \\ &\cdot \det\Big(\partial_1 f_\varepsilon(p_i,q_i) - \partial_2 f_\varepsilon(p_i,q_i)\big(\partial_2 g(p_i,q_i)\big)^{-1}\partial_1 g(p_i,q_i)\Big). \end{aligned} \tag{4.5}$$

Let W_i be a neighborhood of (p_i, q_i) in $\mathbb{R}^k \times \mathbb{R}^s$ such that $F_\varepsilon(p,q) \neq (0,0)$ for any $(p,q) \in W_i \setminus \{(p_i,q_i)\}$. Clearly, $\Psi_\varepsilon(p,q) \neq (0,0)$ for any $(p,q) \in W_i \cap M \setminus \{(p_i,q_i)\}$. Without loss of generality we can assume that $W_i = U_i \times V_i$ for appropriate open sets $U_i \subseteq \mathbb{R}^k$ and $V_i \subseteq \mathbb{R}^s$.

Since $\partial_2 g(p_i,q_i)$ is invertible, the implicit function theorem implies that, taking a smaller U_i if necessary, we can assume that there exists a C^1 function $\gamma_i : U_i \to \mathbb{R}^s$ such that $g\big(p, \gamma_i(p)\big) = 0$ for any $p \in U_i$. The continuity of γ_i imply that, taking again a smaller U_i if necessary, we can assume $\gamma_i(U_i) \subseteq V_i$. Thus the map $G_i : p \mapsto \big(p, \gamma_i(p)\big)$ is a diffeomorphism of U_i onto $W_i \cap M$, its inverse being the projection $\pi : W_i \cap M \to U_i$ given by $\pi(p,q) = p$.

The property of invariance under diffeomorphisms of the degree of tangent vector fields implies that

$$\deg(\Psi_\varepsilon, W_i \cap M) = \deg\big(\pi \circ \Psi_\varepsilon \circ G_i, U_i).$$

Notice that p_i is an isolated zero of $\pi \circ \Psi_\varepsilon \circ G_i$. The differential of this map at p_i is

$$\partial_1 f_\varepsilon(p_i,q_i) - \partial_2 f_\varepsilon(p_i,q_i)\big(\partial_2 g(p_i,q_i)\big)^{-1}\partial_1 g(p_i,q_i)$$

(recall that $q_i = \gamma_i(p_i)$). By (4.5) and the fact that $(0,0)$ is a regular value for F_ε, it follows that this differential is invertible. Therefore we have

$$\begin{aligned} &\mathrm{i}\big(\Psi_\varepsilon,(p_i,q_i)\big) = \\ &= \operatorname{sign}\det\Big(\partial_1 f_\varepsilon(p_i,q_i) - \partial_2 f_\varepsilon(p_i,q_i)\big(\partial_2 g(p_i,q_i)\big)^{-1}\partial_1 g(p_i,q_i)\Big). \end{aligned} \tag{4.6}$$

Equations (4.5) and (4.6) clearly imply (4.4). The assertion follows. □

Remark 4.3. *Let U, f, g, M, Ψ and F be as in Theorem 4.1. In fact, an inspection of its proof reveals that we have proved a slightly more precise, albeit less elegant, formula concerning* $\deg(\Psi, M)$*. Namely,*

$$\deg(\Psi, M) = \operatorname{sign}\big(\det \partial_2 g(p,q)\big)\deg(F, U).$$

(Recall that $\det \partial_2 g(p,q)$ *has constant sign for* (p,q) *in the connected set* U*.)*

Let us illustrate Theorem 4.1 with two examples.

Example 4.4. *Consider the following second order DAE in* $\mathbb{R} \times \mathbb{R}$*:*

$$\begin{cases} \ddot{x} = -x + y - \dot{x}, \\ y^3 + y - x^2 = 0. \end{cases} \tag{4.7}$$

We rewrite (4.7) *as the following equivalent first order system in* $U = \mathbb{R}^2 \times \mathbb{R}$*:*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + y - x_2, \\ y^3 + y - x_1^2 = 0. \end{cases} \tag{4.8}$$

Let $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ *be given by* $g(p_1, p_2; q) = q^3 + q - p_1^2$*. As in Section 2, Equation* (4.8) *is equivalent to the ordinary differential equation* $\dot{\zeta} = \Psi(\zeta)$ *on* $M = g^{-1}(0) \subseteq \mathbb{R}^2 \times \mathbb{R}$ *where*

$$\Psi(p_1, p_2; q) = \left(p_2, -p_1 + q - p_2\,;\, \frac{2p_1p_2}{1 + 3q^2}\right)$$

for all $(p_1, p_2; q) \in M$*. Computing* $|\deg(\Psi, M)|$ *directly from the expression of* Ψ *is possible, of course. However, an easier way is to observe that by Theorem 4.1 we have* $\big|\deg(\Psi, M)\big| = \big|\deg(F, U)\big|$*, where* $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ *is given by*

$$F(p_1, p_2; q) = \big(p_2, -p_1 + q - p_2\,;\, q^3 + q - p_1^2\big)\,.$$

A simple computation shows that the unique zero of F *is* $(0, 0; 0)$ *and that we have* $\deg(F, U) = 1$*. Hence,* $\big|\deg(\Psi, M)\big| = 1$*. Actually, according to Remark 4.3,*

$$\deg(\Psi, M) = \operatorname{sign}\big(\det \partial_2 g(p_1, p_2; q)\big) \deg(F, U) = \deg(F, U) = 1.$$

Example 4.5. (Index 2 DAE in Hessenberg form.) *Let* U_1 *and* U_2 *be open subsets of* $\mathbb{R}^k$*, and* $\mathbb{R}^s$*, respectively; and let* $U = U_1 \times U_2$*. Consider the following DAE*

$$\begin{cases} \dot{x} = f(x, y), \\ \gamma(x) = 0, \end{cases} \tag{4.9}$$

where $f : U \to \mathbb{R}^k$ *and* $\gamma : U_1 \to \mathbb{R}^s$ *are* C^∞*. Assume that* $d_p\gamma[\partial_2 f(p, q)] : \mathbb{R}^s \to \mathbb{R}^s$ *is an invertible linear operator for all* $(p, q) \in U$*. In this case* (4.9) *is an index 2 differential-algebraic equation in Hessenberg form (see e.g.* [8]*). With a simple index reduction, we see that* (4.9) *is equivalent to the following DAE:*

$$\begin{cases} \dot{x} = f(x, y), \\ d_x\gamma\big(f(x, y)\big) = 0, \end{cases} \tag{4.10}$$

Let us set $g(p, q) = d_p\gamma\big(f(p, q)\big)$ *for all* $(p, q) \in U$*. Then,*

$$\partial_2 g(p, q) = d_p\gamma\big(\partial_2 f(p, q)\big) : \mathbb{R}^s \to \mathbb{R}^s$$

is invertible. The vector field Ψ*, constructed as in Section 2 and tangent to* $M = g^{-1}(0)$*, has the following expression*

$$\Psi(p, q) = \Big(f(p, q)\,,\, -\big[d_p\gamma\big(\partial_2 f(p, q)\big)\big]^{-1} \\ \big\{d_p^2\gamma\big(f(p, q), f(p, q)\big) + d_p\gamma\big[\partial_1 f(p, q)\big] f(p, q)\big\}\Big).$$

where the bilinear form $d_p^2\gamma(\cdot, \cdot)$ *is the second differential of* γ *at* p*. Theorem 4.1 shows that the rather unappealing task of computing* $|\deg(\Psi, M)|$ *reduces to the computation of the comparatively simpler* $|\deg(F, U)|$ *where,*

$$F(p, q) = \Big(f(p, q)\,,\, d_p\gamma\big[f(p, q)\big]\Big).$$

5. The set of T-periodic solutions of (1.2)

This section is devoted to the study of the set of T-periodic solutions of equation (1.2). Recall that $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is open and connected, $f : U \to \mathbb{R}^k$, $g : U \to \mathbb{R}^s$ and $h : \mathbb{R} \times U \to \mathbb{R}^k$ are continuous, and we assume that h is T-periodic in the first variable for a given $T > 0$, and g is C^∞ with the property that $\det \partial_2 g(p, q) \neq 0$ for all $(p, q) \in U$.

We need to introduce some further notation: denote by $C_T(U)$ the metric subspace of the Banach space $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ of all the continuous T-periodic functions taking values in U. We say that $(\mu; x, y) \in [0, \infty) \times C_T(U)$ is a *solution pair of* (1.2) if (x, y) satisfies (1.2) for $\lambda = \mu$; here the pair (x, y) is thought of as a single element of $C_T(U)$. It is convenient, given any $(p, q) \in \mathbb{R}^k \times \mathbb{R}^s$, to denote by $(\hat{p}, \hat{q})$ the map in $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ that is constantly equal to (p, q). A solution pair of the form $(0; \hat{p}, \hat{q})$ is called *trivial.*

As mentioned in the Introduction, the main result of this section, Theorem 5.1 below, follows from a combination of Theorem 4.1 and an argument of [4], where most of the technical difficulties that arise when working with branches of solution pairs are solved (this fact explains the simplicity of the proof of Theorem 5.1).

Let $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ be given by (1.3). As one immediately checks, $(\hat{p}, \hat{q})$ is a constant solution of (1.2) corresponding to $\lambda = 0$ if and only if $F(p,q) = (0,0)$. Thus, with this notation, the set of trivial solution pairs can be written as

$$\{(0; \hat{p}, \hat{q}) \in [0,\infty) \times C_T(U) : F(p,q) = (0,0)\}.$$

Given $\Omega \subseteq [0,\infty) \times C_T(U)$, with $U \cap \Omega$ we denote the set of points of U that, regarded as constant functions, lie in Ω. Namely,

$$U \cap \Omega = \{(p,q) \in U : (0; \hat{p}, \hat{q}) \in \Omega\}.$$

We are now ready to state and prove our main result concerning the T-periodic solutions of (1.2).

Theorem 5.1. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected. Let $g : U \to \mathbb{R}^s$, $f : U \to \mathbb{R}^k$, $h : \mathbb{R} \times U \to \mathbb{R}^k$ and $T > 0$ be such that f and h are continuous, h is T-periodic in the first variable, and g is C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$. Let also $F(p,q) = \big(f(p,q), g(p,q)\big)$. Given $\Omega \subseteq [0,\infty) \times C_T(U)$ open, assume $\deg(F, U \cap \Omega)$ is well-defined and nonzero. Then, there exists a connected set Γ of nontrivial solution pairs of (1.2) whose closure in $[0,\infty) \times C_T(U)$ meets the set $\{(0, \hat{p}, \hat{q}) \in \Omega : F(p,q) = (0,0)\}$ and is not contained in any compact subset of Ω.*

Proof. Denote by $C_T(M)$ metric subspace of the Banach space $C_T(\mathbb{R}^k \times \mathbb{R}^s)$, of all the continuous T-periodic functions taking values in $M = g^{-1}(0)$. Let Ψ and Υ be as in (2.2). Then (1.2) is equivalent to (2.4) on M.

By Theorem 4.1 we have that $\deg(\Psi, M \cap \Omega) \neq 0$, here by $M \cap \Omega$ we mean the set $\{(p,q) \in M : (0; \hat{p}, \hat{q}) \in \Omega\}$. Theorem 3.3 of [4] implies the existence of a connected subset Γ of

$$\big\{(\lambda; x, y) \in \big([0,\infty) \times C_T(M)\big) \cap \Omega : (x,y) \text{ is a nonconstant solution of (2.4)}\big\}$$

whose closure $\overline{\Gamma}$ in $[0,\infty) \times C_T(M)$ is not contained in any compact subset of $\big([0,\infty) \times C_T(M)\big) \cap \Omega$ and meets the set

$$\big\{(0; \hat{p}, \hat{q}) \in \Omega : \Psi(p,q) = (0,0)\big\},$$

that coincides with $\big\{(0, \hat{p}, \hat{q}) \in \Omega : F(p,q) = (0,0)\big\}$.

Clearly, each $(\lambda; x, y) \in \Gamma$ is a nontrivial solution pair of (1.2). Since M is closed in U, it is not difficult to prove that $[0,\infty) \times C_T(M)$ is closed in $[0,\infty) \times C_T(U)$. Thus, $\overline{\Gamma}$ coincides with the closure of Γ in $[0,\infty) \times C_T(U)$. Consequently Γ satisfies the assertion. □

Example 5.2. *Consider the 2π-periodically perturbed DAE in $U = \mathbb{R} \times (-1,1)$*

$$\begin{cases} \dot{x} = -y - \lambda \sin t, \\ x - \frac{1}{3}y^3 + y = 0, \end{cases} \tag{5.1}$$

that is obtained by writing in the Liénard plane the following forced van der Pol differential equation and taking the limit as $\varepsilon \to 0$:

$$\varepsilon\ddot{y} + (y^2 - 1)\dot{y} + y + \lambda \sin t = 0, \quad y \in (-1,1).$$

Let $F(p,q) = \big(-q, p - \frac{1}{3}q^3 + q\big)$, and put $\Omega = [0,\infty] \times C_{2\pi}(U)$. Clearly, one has $U = U \cap \Omega$, $F^{-1}(0,0) = \{(0,0)\}$ and $\deg(F,U) = 1$. Theorem 5.1, yields a connected set Γ of nontrivial solution pairs of (5.1) whose closure in $[0,\infty) \times C_T(U)$ meets the trivial solution pair $\{(0; \hat{0}, \hat{0})\}$ and is not compact. (Here $\hat{0}$ denotes the identically zero function in $\mathbb{R}$.)

As one immediately checks, the only 2π-periodic solution of (5.1), for $-1 < y < 1$ and $\lambda = 0$, is $\{(\hat{0}, \hat{0})\}$. Thus, by the connectedness of Γ one can deduce that (5.1) admits a 2π-periodic solution for sufficiently small values of $\lambda > 0$.

The result of Theorem 5.1 is slightly more intuitive when $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$ (as, for instance, if $U = \mathbb{R}^k \times \mathbb{R}^s$). In fact, in this case, the metric subspace $C_T(M) \subseteq C_T(\mathbb{R}^k \times \mathbb{R}^s)$, that consists of all continuous T-periodic and M-valued functions, is complete. In this situation, we deduce the following *Continuation Principle* from Theorem 5.1.

Corollary 5.3. *Let f, g, h, U, M, T, F and Ω be as in Theorem 5.1. Assume also that $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $\deg(F, U \cap \Omega)$ be nonzero. Then there exists a connected component of the set of solution pairs of* (1.2) *that meets $\{(0, \hat{p}, \hat{q}) \in \Omega : F(p, q) = (0, 0)\}$ and cannot be both bounded and contained in Ω.*

If, in particular, $\Omega = [0, \infty) \times A$, with $A \subseteq C_T(U)$ open, bounded, and such that there are no T-periodic solutions of (1.2) *on the boundary* $\operatorname{Fr}(A)$ *of A for $\lambda \in [0, 1]$, then equation*

$$\begin{cases} \dot{x} = f(x, y) + h(t, x, y), \\ g(x, y) = 0. \end{cases} \tag{5.2}$$

admits a T-periodic solution in A.

Proof. By Theorem 5.1, there exists a connected set Γ of nontrivial solution pairs of (1.2) whose closure $\overline{\Gamma}$ in $[0, \infty) \times C_T(U)$ meets the set $\{(0, \hat{p}, \hat{q}) \in \Omega : F(p, q) = (0, 0)\}$ and is not contained in any compact subset of Ω. Let Σ be the connected component of the set of all solution pairs that contains $\overline{\Gamma}$.

Since $M \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is closed, the metric space $[0, \infty) \times C_T(M)$ is complete. Moreover, the Ascoli-Arzelà Theorem implies that any bounded set of T-periodic solutions of (2.4) is totally bounded. Thus, if Σ is bounded, then it is also compact. If, in addition, Σ is contained in Ω then so is $\Gamma \subseteq \Sigma$, which is impossible. This contradiction proves that Σ cannot be both bounded and contained in Ω.

To prove the last part of the assertion observe that Σ is connected and that $\emptyset \neq \Sigma \cap \Omega \neq \Sigma$. Thus, Σ necessarily meets the boundary of $\Omega = [0, \infty) \times A$. Since there are no solution pairs of (1.2) in $[0, 1] \times \operatorname{Fr} A$, one has that Σ intersects $\{1\} \times A$. □

Remark 5.4. *A practical method of applying Corollary* (5.3) *is to consider a relatively compact open subset V of U with the following properties:*

- *the set $F^{-1}(0, 0) \cap V$ is compact and* $\deg(F, V) \neq 0$*;*
- *there is no T-periodic solution of* (1.2) *whose image intersects the boundary of V. (This last point might be difficult to verify and is usually proved by the means of a priori bounds.)*

In this situation, taking $A = C_T(V)$ and $\Omega = [0, \infty) \times A$, we have $U \cap \Omega = V$ and

$$\deg(F, U \cap \Omega) = \deg(F, V) \neq 0.$$

Hence, Corollary 5.3 yields a T-periodic solution of (5.2).

The next two subsections, that are meant mainly as illustrations of Theorem 5.1 and of its main consequence Corollary 5.3, are each devoted to a quite different application.

5.1. **Example of application to multiplicity results.** This subsection is devoted to some multiplicity results that can be deduced from Theorem 5.1 and from its Corollary 5.3. Throughout this subsection f, g, h, U, T and F will be as in Theorem 5.1 and, in addition, we will assume that f is C^1.

In order to obtain multiplicity results, we combine the global approach of Theorem 5.1 with a local analysis of the set of T-periodic solutions. Let (p_0, q_0) be an isolated zero of F. Since $\partial_2 g(p_0, q_0)$ is invertible, we can locally "decouple" (1.2). Namely, by the Implicit Function Theorem, there exist neighborhoods $V \subseteq \mathbb{R}^k$ of

p_0 and $W \subseteq \mathbb{R}^s$ of q_0, and a function $\gamma : V \to \mathbb{R}^s$ such that $g^{-1}(0) \cap V \times W$ is the graph of γ. Thus, in $V \times W$, equation (1.2) can be written as

$$\dot{x} = f\big(x, \gamma(x)\big) + \lambda h\big(t, x, \gamma(x)\big), \tag{5.3a}$$

$$y = \gamma(x). \tag{5.3b}$$

We will say that (p_0, q_0) is a T-resonant zero of F if the following linearization, for $\lambda = 0$, of (5.3a) at (p_0, q_0):

$$\dot{\xi} = \big[\partial_1 f(p_0, q_0) - \partial_2 f(p_0, q_0) d_{(p_0,q_0)}\gamma\big]\xi \tag{5.4}$$

admits nonzero T-periodic solutions (note that (5.4) is an ordinary differential equation in $\mathbb{R}^k$).

A simple computation shows that (p_0, q_0) is T-resonant if and only if the following linear endomorphism of $\mathbb{R}^k$:

$$\partial_1 f(p_0, q_0) - \partial_2 f(p_0, q_0)[\partial_2 g(p_0, q_0)]^{-1}\partial_1 g(p_0, q_0) \tag{5.5}$$

has eigenvalues of the form $2n\pi i/T$, where $n \in \mathbb{N} \cup \{0\}$, and i denotes the imaginary unit. Also, the generalized Gauss algorithm, as in the proof of Theorem 4.1, yields

$$\begin{aligned} \det d_{(p_0,q_0)}F = \det\big(\partial_2 g(p_0, q_0)\big)\cdot& \\ \cdot \det\Big(\partial_1 f(p_0, q_0) - \partial_2 f(p_0, q_0)&[\partial_2 g(p_0, q_0)]^{-1}\partial_1 g(p_0, q_0)\Big). \end{aligned}$$

Thus, if (p_0, q_0) is non-T-resonant, then it is a nondegenerate zero of F. Hence, $\mathrm{i}\big(F, (p_0, q_0)\big) \neq 0$.

From Theorem 5.1 we get the following lemma:

Lemma 5.5. *Assume that f, g, h, U, T and F be as in Theorem 5.1. Assume also that f is C^1 and let (p_0, q_0) be a non-T-resonant zero of F. Then*

1. *the trivial T-pair $(0; \hat{p}_0, \hat{q}_0)$ is isolated in the set of T-pairs corresponding to $\lambda = 0$;*
2. *there exists a connected set of nontrivial T-pairs of* (1.2) *whose closure in $[0, \infty) \times C_T(U)$ contains $(0; \hat{p}_0, \hat{q}_0)$ and is noncompact or intersects the set*

$$\big\{(0; \hat{p}, \hat{q}) \in [0, \infty) \times C_T(U) : F(p, q) = (0, 0)\big\} \setminus \{(0; \hat{p}_0, \hat{q}_0)\}$$

Proof. Let us prove the first part of the assertion. Assume by contradiction that there exists a sequence $\{(0; x_n, y_n)\}$, $n = 1, 2, \ldots$, of T-pairs of (1.2) with $(x_n, y_n) \to (\hat{p}_0, \hat{q}_0)$ uniformly. If we put $p_n = x_n(0)$, we clearly have $p_n \to p_0$. We claim that this is not possible. Let V and γ be as in (5.3). For any $p' \in V$, denote by $x(\cdot, p')$ the maximal solution of the Cauchy problem

$$\begin{cases} \dot{x} = f\big(x, \gamma(x)\big), \\ x(0) = p' \end{cases}$$

Well known results in the theory of ordinary differential equations imply that there exists an open neighborhood $W \subseteq V$ of p_0 such that the map P, that to $p' \in W$ associates $x(T, p') \in \mathbb{R}^k$, is defined. Also, since $p \mapsto f\big(p, \gamma(p)\big)$ is continuous in $W \subseteq V$, we know that P is C^1 in W and that its differential $d_{p_0}P$ is given by (5.5). Thus, since (p_0, q_0) is a nondegenerate zero of F, the linear operator $d_{p_0}P$ is invertible. The claim now follows from the Inverse Function Theorem.

Let us prove the second part of the assertion. Since $(0; \hat{p}_0, \hat{q}_0)$ is isolated in the set of T-pairs corresponding to $\lambda = 0$, the set

$$\big\{(0; \hat{p}, \hat{q}) \in [0, \infty) \times C_T(U) : F(p, q) = (0, 0)\big\} \setminus \{(0; \hat{p}_0, \hat{q}_0)\}$$

is closed. Thus, the set

$$\Omega = \Big([0,\infty)\times C_T(U)\Big)\setminus \Big(\big\{(0;\hat p,\hat q)\in[0,\infty)\times C_T(U) : F(p,q)=(0,0)\big\}\setminus\{(0;\hat p_0,\hat q_0)\}\Big)$$

is open. As in Theorem 5.1, we use the symbol $U\cap\Omega$ as a shorthand notation for the set $\{(p,q)\in U : (0,\hat p,\hat q)\in\Omega\}$. Since (p_0,q_0) is non-T-resonant and since $U\cap\Omega$ is just the singleton $\{(p_0,q_0)\}$, from (3.2) we have

$$\deg(F,U\cap\Omega)=\mathrm{i}\big(F,(p_0,q_0)\big)\neq 0.$$

By Theorem 5.1 there exists a connected set Γ of T-pairs that meets

$$\big\{(0;\hat p,\hat q)\in\Omega : F(p,q)=(0,0)\big\}=\{(0;\hat p_0,\hat q_0)\},$$

and is such that its closure $\overline{\Gamma}$ in $[0,\infty)\times C_T(U)$ is not contained in any compact subset of Ω. Hence Γ satisfies the second part of the assertion. □

Let us introduce some notation. Let Y be a metric space and X a subset of $[0,\infty)\times Y$. Given $\lambda\geq 0$, we denote by X_λ the slice $\big\{y\in Y : (\lambda,y)\in X\big\}$. Recall the following notion from [2]: We say that $A\subseteq X_0$ is an *ejecting set* (for X) if it is relatively open in X_0 and there exists a connected subset of X which meets A and is not contained in X_0. For example, any non-T-resonant point of (1.2) is an ejecting set (or, rather, *ejecting point*). In fact, as a consequence of Lemma 5.5, if X denotes the set of T-pairs of (1.2) and $Y=C_T(U)$, any non-T-resonant zero of F turns out to be an isolated point of X_0 which is ejecting.

Let us recall the following abstract result from [2]:

Theorem 5.6. *Let Y be a metric space and let X be a locally compact subset of $[0,\infty)\times Y$. Assume that X_0 contains $r+1$ pairwise disjoint ejecting subsets, r of which are compact. Then there exists $\lambda_*>0$ such that the cardinality of X_λ is at least $r+1$ for any $\lambda\in[0,\lambda_*)$.*

We are now in a position to state and prove the following multiplicity result:

Proposition 5.7. *Let f, g, h, U, M, T, F and Ω be as in Theorem 5.1. Assume also that f is C^1 and that $M=g^{-1}(0)$ is closed in $\mathbb{R}^k\times\mathbb{R}^s$. Let $(p_1,q_1),\ldots,(p_r,q_r)$ be non-T-resonant zeros of F such that*

$$\deg(F,U)\neq\sum_{j=1}^{r}\mathrm{i}\big(F,(p_j,q_j)\big).$$

Suppose that (1.1) *does not admit an unbounded connected set of T-periodic solutions in $C_T(U)$. Then, there are at least $r+1$ different T-periodic solution of* (1.2) *when $\lambda>0$ is sufficiently small.*

The assumption on the nonexistence of an unbounded connected set of T-periodic solutions (in $C_T(U)$) of the unperturbed equation (1.1) is often the most difficult to verify and usually shown to hold with the help of a priori bounds.

Proof of Proposition 5.7. Let

$$\Omega=\Big([0,\infty)\times C_T(U)\Big)\setminus\bigcup_{j=1}^{r}\{(0;\hat p_j,\hat q_j)\}.$$

By the additivity property of the degree and formula (3.2)

$$\deg(F,U\cap\Omega)=\deg(F,U)-\sum_{j=1}^{r}\mathrm{i}\big(F,(p_j,q_j)\big)\neq 0,$$

where, as in Theorem 5.1, we use the notation $U \cap \Omega = \{(p,q) \in U : (0,\hat{p},\hat{q}) \in \Omega\}$. Let X be the set of all T-pairs of (1.2). By Corollary 5.3, $M = g^{-1}(0)$ being closed in $\mathbb{R}^k \times \mathbb{R}^s$, there exists a connected component Γ of X that cannot be both bounded and contained in Ω. Since by assumption (1.1) does not admit an unbounded connected set of T-periodic solutions in $C_T(U)$, it is not difficult to show that the set

$$X_0 \setminus \bigcup_{j=1}^{r} \{(0;\hat{p}_j,\hat{q}_j)\}$$

is ejecting. The assertion now follows from Lemma 5.5 and Theorem 5.6. □

As an illustration of Proposition 5.7 we consider the following elementary example even though, in that case, the situation is sufficiently simple to be treatable without the help of our multiplicity result.

Example 5.8. *Consider the following DAE in $U = \mathbb{R} \times \mathbb{R}$:*

$$\left\{ \begin{array}{l} \dot{x} = y^2 - xy + \lambda h(t,x,y), \\ y - x^2 = 0, \end{array} \right. \tag{5.6}$$

where $h : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is any continuous function T-periodic in the first variable.

One immediately sees that $F(p,q) = (q^2 - pq, q - p^2)$ has only the two zeros $(0,0)$ and $(1,1)$ of which the former is T-resonant, whereas the latter is not so. Also, it not difficult to see that for $\lambda = 0$ the only possible periodic solutions of (5.6) correspond to the zeros of F. Thus, for $\lambda = 0$, equation (5.6) does not admit an unbounded connected set of T-periodic solutions.

By inspection, we see that the homotopy $H : U \times [0,1] \to \mathbb{R} \times \mathbb{R}$, given by $H(p,q;\lambda) = (q^2 - pq, q - p^2 - \lambda)$, is admissible. Since $H(p,q;1) \neq 0$ for any $(p,q) \in U$, we have $\deg\big(H(\cdot,\cdot;1),U\big) = 0$. Thus, by Homotopy Invariance, $\deg(F,U) = 0$.

Since a non-T-resonant zero of F is nondegenerate, we have $\mathrm{i}\big(F,(1,1)\big) \neq 0$. Hence, by Proposition 5.7, for sufficiently small $\lambda > 0$ there are at least two T-periodic solutions of (5.6)

5.2. **Example of application to a class of implicit differential equations.** In this subsection we will describe an application to periodic perturbations of ordinary differential equations of a particular implicit form. What follows is mostly intended as an illustration of Theorem 5.1 and of its Corollary 5.3. For this reason we do not seek generality but confine ourselves to a fairly simple situation. Namely, we consider the following equation:

$$\varphi\big(x, \dot{x} + \lambda h(t,x)\big) = 0, \tag{5.7}$$

where $\varphi : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is C^∞ with the property that $\partial_2\varphi(p,q)$ is invertible for all $(p,q) \in \mathbb{R}^k \times \mathbb{R}^k$; and $h : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous and T-periodic in the first variable, with given $T > 0$.

We will also need the following *"no blow up"* assumption on φ. Namely, we suppose that φ is such that:

(5.8) The set $\big\{q \in \mathbb{R}^k : \varphi(p,q) = 0, p \in K\big\}$ is bounded for any bounded $K \subseteq \mathbb{R}^k$.

Before we proceed we need a technical result on the degree of a special class of vector fields. Its proof is inspired by the one of Theorem 6.1 in [1] regarding the Brouwer degree.

Lemma 5.9. *Let $V \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be open, $\omega : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be continuous and such that $[\omega(\cdot,0)]^{-1}(0) \cap V$ is compact. Define $v : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k$ by $v(p,q) = \big(q, \omega(p,q)\big)$. Then, (v,V) is admissible and*

$$\deg(v,V) = -\deg\big(\omega(\cdot,0),V_0\big),$$

where $V_0 = \{p \in \mathbb{R}^k : (p,0) \in V\}$.

Proof. By the Excision Property and the compactness of $[\omega(\cdot,0)]^{-1}(0)\cap V$, taking a smaller V if necessary, we can assume that V is bounded and such that $\omega(p,0) \neq 0$ for all $(p,0)$ in the boundary $\mathrm{Fr}\,(V)$ of V.

Observe that the homotopy $H : \mathbb{R}^k \times \mathbb{R}^k \times [0,1] \to \mathbb{R}^k \times \mathbb{R}^k$ given by $H(p,q,\lambda) = \big(q, \omega(p,\lambda q)\big)$ is admissible in V, define $G(p,q) = H(p,q,0)$. By the Homotopy Invariance Property it is sufficient to show that $\deg(G,V) = -\deg\big(\omega(\cdot,0),V_0\big)$.

By known approximation results (see e.g. [6] or [7]) there exists a C^1 map $\eta : \mathbb{R}^k \to \mathbb{R}^k$, such that all its zeros contained in V_0 are nondegenerate, and with the property that

$$\max_{p\in \mathrm{Fr}\,(V_0)} |\eta(p) - \omega(p,0)|_k < \min_{p\in \mathrm{Fr}\,(V_0)} |\omega(p,0)|_k.$$

As in Remark 3.1 we have that

$$\deg(\eta, V_0) = \deg\big(\omega(\cdot,0),V_0\big). \tag{5.9}$$

Define $Q : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k$ by $Q(p,q) = \big(q, \eta(p)\big)$. By Remark 3.1 again,

$$\deg(G,V) = \deg(Q,V), \tag{5.10}$$

since

$$\begin{aligned}\max_{(p,q)\in \mathrm{Fr}\,(V)} |Q(p,q) - G(p,q)|_{2k} &= \max_{p\in \mathrm{Fr}\,(V_0)} |\eta(p) - \omega(p,0)|_k \\ &< \min_{p\in \mathrm{Fr}\,(V_0)} |\omega(p,0)|_k \leq \min_{(p,q)\in \mathrm{Fr}\,(V)} |G(p,q)|_{2k}.\end{aligned}$$

Observe also that, since all the zeros of η in V_0 are nondegenerate, so are those of Q in V. In fact, $Q^{-1}(0,0) = \{(p,0) \in \mathbb{R}^k \times \mathbb{R}^k : \eta(p) = 0\}$ and, writing the differential $d_{(p,q)}Q$ of Q at any $(p,q) \in Q^{-1}(0,0)$ in block matrix form, we get

$$\det d_{(p,q)}Q = \det \begin{pmatrix} 0 & \mathrm{Id}_{\mathbb{R}^k} \\ d_p\eta & 0 \end{pmatrix} = -\det d_p\eta\,,$$

where $\mathrm{Id}_{\mathbb{R}^k}$ denotes the identity on $\mathbb{R}^k$ and $d_p\eta$ is the differential of η at p. Thus, taking (5.10), (3.1) and (5.9) into account, we get

$$\begin{aligned}\deg(G,V) = \deg(Q,V) &= \sum_{(p,q)\in Q^{-1}(0,0)\cap V} \operatorname{sign}\det d_{(p,q)}Q = \\ &= -\sum_{p\in \eta^{-1}(0)\cap V_0} \operatorname{sign}\det d_p\eta = -\deg\big(\eta, V_0\big) = -\deg\big(\omega(\cdot,0),V_0\big),\end{aligned}$$

that implies the assertion. □

Recall that, given $p \in \mathbb{R}^k$, we denote by $\hat{p}$ the function in $C_T(\mathbb{R}^k)$ that is constantly equal to p. Given an open set $W \subseteq [0,\infty) \times C_T(\mathbb{R}^k)$, it is convenient to denote by $\mathbb{R}^k \cap W$ the open subset of $\mathbb{R}^k$ given by $\big\{p \in \mathbb{R}^k : (0,\hat{p}) \in W\big\}$.

Proposition 5.10. *Let h and φ be as above, and let $W \subseteq [0,\infty) \times C_T(\mathbb{R}^k)$ be open and such that* $\deg\big(\varphi(\cdot,0),\mathbb{R}^k\cap W\big)$ *is well defined and nonzero. Then the subspace of $[0,\infty) \times C_T(\mathbb{R}^k)$ consisting of all pairs (λ,x), with x a (clearly T-periodic) solution of* (5.7)*, contains a connected component Ξ that intersects the set*

$$\big\{(0,\hat{p}) \in W : \varphi(p,0) = 0\big\}$$

and is unbounded or meets the boundary of W.

Proof. Clearly, (5.7) is equivalent to the following DAE:

$$\left\{\begin{array}{l} \dot{x} = y - \lambda h(t,x), \\ \varphi(x,y) = 0. \end{array}\right. \tag{5.11}$$

Let $F:\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}^k\times\mathbb{R}^k$ be given by $F(p,q)=\big(q,\varphi(p,q)\big)$. Denote by Ω the set

$$\big\{(\lambda;x,y)\in[0,\infty)\times C_T(\mathbb{R}^k\times\mathbb{R}^k):(\lambda,x)\in W,\, y\in C_T(\mathbb{R}^k)\big\},$$

and by V the set $\{(p,q)\in\mathbb{R}^k\times\mathbb{R}^k:(0;\hat{p},\hat{q})\in\Omega\}$. Let $V_0=\{p\in\mathbb{R}^k:(p,0)\in V\}$. Since $V_0=\mathbb{R}^k\cap W$, Lemma 5.9 implies that

$$\deg\big(F,V\big)=\deg\big(\varphi(\cdot,0),\mathbb{R}^k\cap W\big)\neq 0. \tag{5.12}$$

Using the notation $(\mathbb{R}^k\times\mathbb{R}^k)\cap\Omega=\{(p,q)\in\mathbb{R}^k\times\mathbb{R}^k:(0;\hat{p},\hat{q})\in\Omega\}$ as in Theorem 5.1, we write $V=(\mathbb{R}^k\times\mathbb{R}^k)\cap\Omega$. By (5.12), $\deg\big(F,(\mathbb{R}^k\times\mathbb{R}^k)\cap\Omega\big)=\deg\big(F,V\big)\neq 0$. Thus, Corollary 5.3 yields the existence of a connected component Γ of the set of solutions pairs of (5.11) that meets

$$\{(0;\hat{p},\hat{q})\in\Omega:F(p,q)=0\}=\{(0;\hat{p},\hat{0}):\varphi(p,0)=0\}$$

(here $\hat{0}\in C_T(\mathbb{R}^k)$ denotes the map $\hat{0}(t)\equiv 0\in\mathbb{R}^k$) and is unbounded or intersects the boundary of Ω. Let $\Xi\subseteq[0,\infty)\times C_T(\mathbb{R}^k)$ be the connected set defined by

$$\Xi=\Big\{(\lambda,x)\in[0,\infty)\times C_T(\mathbb{R}^k):\exists y\in C_T(\mathbb{R}^k)\text{ s.t. }\big(\lambda;x,y\big)\in\Gamma\Big\}.$$

Notice that if Ξ is contained in W, then $\Gamma\subseteq\Omega$. Also, using assumption (5.8), it is not difficult to prove that if Ξ is bounded then so is Γ. Hence, Ξ cannot be both bounded and contained in W. The assertion follows from the connectedness of Ξ. $\square$

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