Periodic perturbations with delay of autonomous differential equations on manifolds

Massimo Furi

Dipartimento di Matematica Applicata 'G. Sansone',
Università di Firenze, Via S. Marta 3, 50139 Firenze, Italia
e-mail: massimo.furi@unifi.it

Marco Spadini

Dipartimento di Matematica Applicata 'G. Sansone',
Università di Firenze, Via S. Marta 3, 50139 Firenze, Italia
e-mail: marco.spadini@unifi.it

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Abstract

We apply topological methods to the study of the set of harmonic solutions of periodically perturbed autonomous ordinary differential equations on differentiable manifolds, allowing the perturbing term to contain a fixed delay.

In the crucial step, in order to cope with the delay, we define a suitable (infinite dimensional) notion of Poincaré T-translation operator and prove a formula that, in the unperturbed case, allows the computation of its fixed point index.

1   Introduction

In this paper we shall study the set of harmonic solutions to periodic perturbations of autonomous ODEs on (smooth) manifolds, allowing for the perturbation to contain a delay. Namely, given $T > 0$, $r \geq 0$ and a manifold $M \subseteq \mathbb{R}^k$, we will consider the T-periodic solutions to

$$\dot{x}(t) = g\big(x(t)\big) + \lambda f\big(t, x(t), x(t-r)\big), \quad \lambda \geq 0, \tag{1.1}$$

where g is a tangent vector field on M and f is T-periodic in t and tangent to M in the second variable (the meaning of these terms will be explained in due course). Roughly speaking, we will give conditions ensuring the existence of a connected component of pairs (λ, x), $\lambda \geq 0$ and x a T-periodic solution to the above equation, that emanates from the set of zeros of g and is not compact. We point out that, although this result is valid for f and g merely continuous, its proof boils down, by an approximation procedure, to the case when f and g are C^1.

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Carrying out this program requires topological tools like the fixed point index and the degree (also called the rotation or characteristic) of a tangent vector field, that shall be briefly recalled in Section 2. In fact, in the case when the perturbation f is independent of the delay, as in [FS97], the existence of such a connected component of T-periodic solutions is based on the computation of the fixed point index of the translation operator (at time T) associated to the equation (1.1) when f and g are C^1. This computation is derived from a formula (see e.g. [FS96]) that relates the degree of g with the fixed point index of the finite dimensional Poincaré T-translation operator P at time T associated to the unperturbed equation $\dot{x} = g(x)$. However, since in our case the perturbing term f contains a delay, the T-translation operator P must be replaced by its infinite dimensional version. Namely, the operator Q that to any function $\varphi \in \widetilde{M} := C([-r,0], M)$ associates the function of $\widetilde{M}$ given by $\theta \mapsto x\big(\varphi(0), \theta + T\big)$. Here $x(p, \cdot)$ denotes the unique solution to the Cauchy problem

$$\dot{x} = g(x), \quad x(0) = p.$$

Clearly P and Q are closely related, although they operate in different spaces, only one of which is finite dimensional. The relation between these operators is discussed in Section 3, where we derive a formula that deduces the fixed point index of Q from the degree of the tangent vector field g.

Let us be more precise about the above mentioned formulas for the fixed point indices of P and Q. It has been proved in [FS96] that, given $U \subseteq M$ open, one has

$$\mathrm{ind}(P, U) = \deg(-g, U), \tag{1.2}$$

provided that the left hand side member of (1.2) is defined. Formula (1.2) is a generalization of a result of [CMZ92] valid for $M = \mathbb{R}^k$, that was related to an earlier theorem by Krasnosel'skiĭ [Kr68]. This latter result holds for nonautonomous differential equations on manifolds, but requires a rather restrictive assumption called T-*irreversibility* (which, in our settings, simply means that the closure $\overline{U}$ of U in M is compact and the map $p \mapsto x(p, t)$ is fixed point free on ∂U for all $t \in (0, T]$). Equation (1.2) does away with this heavy assumption and allows, by means of the properties of Commutativity of the fixed point index and Excision of the degree, to deduce a similar formula for Q. In fact, given $W \subseteq \widetilde{M}$ open, we have

$$\mathrm{ind}(Q, W) = \deg(-g, \check{W}), \tag{1.3}$$

provided that the left hand side member is defined. Here $\check{W}$ denotes the set of points of M that, when regarded as constant functions of $\widetilde{M}$, belong to W.

The formula described above for the computation of the fixed point index of Q allows us, in Sections 4 and 5, to follow the lines of [FS97] in order to prove our main result about the connected sets of T-periodic solutions (λ, x). We point out that in Sections 3 and 4 the maps f and g are always considered C^1, while in Section 5 the merely continuous case is considered.

For what concerns the basic theory of delay differential equations we refer to the book [HL93] and to the paper [Ol69].

2 Preliminaries and notation

This section is devoted to some facts and notation that will be needed in this paper. In particular we recall the notions of fixed point index of a map and of degree of a tangent vector field.

Let us begin with the fixed point index. We recall that a metrizable space E is an *absolute neighborhood retract* (ANR) if, whenever it is homeomorphically embedded as a closed subset C of a metric space X, there exists an open neighborhood U of C in X and a retraction $r : U \to C$ (see e.g. [Bo67, GD03]). Polyhedra and differentiable manifolds are examples of ANRs. Let us also recall that a continuous map between topological spaces is called *locally compact* if it has the property that each point in its domain has a neighborhood whose image is contained in a compact set.

Let E be an ANR and let $\psi : \mathcal{D}(\psi) \to E$ be a locally compact map defined on an open subset $\mathcal{D}(\psi)$ of E. Given an open subset U of $\mathcal{D}(\psi)$, if the set Fix (ψ, U) of the fixed points of ψ in U is compact, then it is well defined an integer, $\text{ind}(\psi, U)$, called *the fixed point index of ψ in U* (see, e.g. [GD03, Gr72, Nu91]). Roughly speaking, $\text{ind}(\psi, U)$ counts algebraically the elements of Fix (ψ, U).

The fixed point index turns out to be completely determined by the following four properties that, therefore, could be used as axioms (see [Br70]). Here, E is an ANR and $U \subseteq E$ is open.

Normalization. *Let $\psi : E \to E$ be constant. Then $\text{ind}(\psi, E) = 1$.*

Additivity. *Given a locally compact map $\psi : U \to E$ with $\text{Fix}(\psi, U)$ compact, if U_1 and U_2 are disjoint open subsets of U such that $\text{Fix}(\psi, U) \subseteq U_1 \cup U_2$, then*

$$\text{ind}(\psi, U) = \text{ind}(\psi, U_1) + \text{ind}(\psi, U_2).$$

Homotopy Invariance. *Assume that $H : U \times [0,1] \to E$ is an admissible homotopy in U; that is, H is locally compact and the set $\{(x, \lambda) \in U \times [0,1] : H(x, \lambda) = x\}$ is compact. Then*

$$\text{ind}\big(H(\cdot, 0), U\big) = \text{ind}\big(H(\cdot, 1), U\big).$$

Commutativity. *Let E_1, E_2 be ANRs and let $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ be open. Suppose $\psi_1 : U_1 \to E_2$ and $\psi_2 : U_2 \to E_1$ are locally compact maps. If one of the sets*

$$\{x \in \psi_1^{-1}(U_2) : \psi_2 \circ \psi_1(x) = x\} \quad or \quad \{y \in \psi_2^{-1}(U_1) : \psi_1 \circ \psi_2(y) = y\}$$

is compact, then so is the other and

$$\text{ind}\big(\psi_2 \circ \psi_1, \psi_1^{-1}(U_2)\big) = \text{ind}\big(\psi_1 \circ \psi_2, \psi_2^{-1}(U_1)\big).$$

It is easily shown that the Additivity Property implies the following two important ones:

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Solution. Let $\psi : U \to E$ be locally compact with $\mathrm{Fix}(\psi, U) = \emptyset$. Then $\mathrm{ind}(\psi, U) = 0$.

Excision. *Given a locally compact map $\psi : U \to E$ with $\mathrm{Fix}(\psi, U)$ compact, and an open subset V of U containing $\mathrm{Fix}(\psi, U)$, one has $\mathrm{ind}(\psi, U) = \mathrm{ind}(\psi, V)$.*

From the Homotopy Invariance and Excision properties one could deduce the following property:

Generalized Homotopy Invariance. *Let $W \subseteq E \times [0, 1]$ be open. Assume that $H : W \to E$ is locally compact and such that the set $\{(x, \lambda) \in W : H(x, \lambda) = x\}$ is compact. Let W_λ denote the slice $W_\lambda := \{x \in E : (x, \lambda) \in W\}$. Then, $\mathrm{ind}\big(H(\cdot, \lambda), W_\lambda\big)$ does not depend on $\lambda \in [0, 1]$.*

In the case when E is a finite dimensional manifold, the fixed point index is uniquely determined by the first three properties (see [FPS04]). It is also worth mentioning that when $E = \mathbb{R}^n$, U is bounded, ψ is defined on $\overline{U}$ and fixed point free on ∂U, then $\mathrm{ind}(\psi, U)$ is just the Brouwer degree $\deg_B(I - \psi, U, 0)$, where I denotes the identity on $\mathbb{R}^n$.

We now recall some basic notions about tangent vector fields on manifolds.

Let $M \subseteq \mathbb{R}^k$ be a manifold. Given any $p \in M$, $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p. Let w be a tangent vector field on M, that is, a (continuous) map $w : M \to \mathbb{R}^k$ with the property that $w(p) \in T_pM$ for any $p \in M$. If $p \in M$ is such that $w(p) = 0$, then the Fréchet derivative $w'(p) : T_pM \to \mathbb{R}^k$ maps T_pM into itself (see e.g. [Mi65]), so that the determinant $\det w'(p)$ of $w'(p)$ is defined. If, in addition, p is a nondegenerate zero (i.e. $w'(p) : T_pM \to \mathbb{R}^k$ is injective) then p is an isolated zero and $\det w'(p) \neq 0$.

Let U be an open subset of M in which we assume w admissible for the degree; that is, the set $w^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (w, U) an integer, $\deg(w, U)$, called the *degree (or characteristic) of the vector field w in U*, which, roughly speaking, counts (algebraically) the zeros of w in U (see e.g. [Hi76, Mi65, FPS05] and references therein). For instance, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap U$ is finite and

$$\deg(w, U) = \sum_{q \in w^{-1}(0) \cap U} \mathrm{sign}\, \det w'(q).$$

When $M = \mathbb{R}^k$, $\deg(w, U)$ is just the classical Brouwer degree, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap U$ whose closure is in U. Moreover, when M is a compact manifold, the celebrated Poincaré-Hopf Theorem states that $\deg(w, M)$ coincides with the Euler-Poincaré characteristic $\chi(M)$ of M and, therefore, is independent of w.

For the pourpose of future reference, we mention a few of the properties of the degree of a tangent vector field that shall be useful in the sequel. Here U is an open subset of a manifold $M \subseteq \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ is a tangent vector field.

Solution. *If (g, U) is admissible and $\deg(g, U) \neq 0$, then g has a zero in U.*

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Additivity. *Let (g, U) be admissible. If U_1 and U_2 are two disjoint open subsets of U whose union contains $g^{-1}(0) \cap U$, then*

$$\deg(g, U) = \deg(g, U_1) + \deg(g, U_2).$$

Homotopy Invariance. *Let $h \colon U \times [0, 1] \to \mathbb{R}^k$ be an admissible homotopy of tangent vector fields; that is, $h(x, \lambda) \in T_x M$ for all $(x, \lambda) \in U \times [0, 1]$ and $h^{-1}(0)$ is compact. Then $\deg\big(h(\cdot, \lambda), U\big)$ is independent of λ.*

As in the case of the fixed point index, the Additivity Property implies the following important one:

Excision *Let (g, U) be admissible. If $V \subseteq U$ is open and contains $g^{-1}(0) \cap U$, then $\deg(g, U) = \deg(g, V)$.*

3 Poincaré-type translation operators

Let $M \subseteq \mathbb{R}^k$ be a manifold, and $g \colon M \to \mathbb{R}^k$ a tangent vector field on M. Let $f \colon \mathbb{R} \times M \times M \to \mathbb{R}^k$ be (continuous and) tangent to M in the second variable (i.e. such that $f(t, p, q) \in T_p M$ for all $(t, p, q) \in \mathbb{R} \times M \times M$). Given $T > 0$, assume also that f is T-periodic in t.

Given $r > 0$, consider the following delay differential equation depending on a parameter $\lambda \geq 0$:

$$\dot{x}(t) = g\big(x(t)\big) + \lambda f\big(t, x(t), x(t - r)\big). \tag{3.4}$$

We are interested in the T-periodic solutions of (3.4). Without loss of generality we will assume that $T \geq r$ ([Fr07]). In fact, for $n \in \mathbb{N}$, equations (3.4) and

$$\dot{x}(t) = g\big(x(t)\big) + \lambda f\big(t, x(t), x(t - (r - nT))\big)$$

have the same T-periodic solutions. Thus, if necessary, one can replace r with $r - nT$, where $n \in \mathbb{N}$ is such that $0 < r - nT \leq T$.

Let us introduce some notation.

Given any $X \subseteq \mathbb{R}^k$, $\widetilde{X}$ denotes the metric space $C\big([-r, 0], X\big)$ with the distance inherited from the Banach space $\widetilde{\mathbb{R}}^k = C([-r, 0], \mathbb{R}^k)$ with the usual supremum norm. Notice that $\widetilde{X}$ is complete if and only if X is closed in $\mathbb{R}^k$. Given any $p \in M$, denote by $\widehat{p} \in \widetilde{M}$ the constant function $\widehat{p}(t) \equiv p$ and, for any $U \subseteq M$, define $\widehat{U} = \big\{\widehat{p} \in \widetilde{M} : p \in U\big\}$. Also, given $W \subseteq \widetilde{M}$, we put $\mathring{W} = \big\{p \in M : \widehat{p} \in W\big\}$.

Observe that, for any given $U \subseteq M$, one has $\widehat{U} \subseteq \widetilde{U}$ and $\overset{\smile}{\widehat{U}} = U$. It is known (see e.g. [Ee66]) that $\widetilde{M}$ is a smooth infinite dimensional manifold. Actually, it turns out that it is a C^1–ANR (see e.g. [EF76]), as a C^1 retract of the open subset $\widetilde{U}$ of $\widetilde{\mathbb{R}}^k$, U being a tubular neighborhood of M in $\mathbb{R}^k$.

Assume now, till further notice, that g is C^1. Consider the map Q in $\widetilde{M}$ defined by $Q(\varphi)(\theta) = x\big(\varphi(0), T + \theta\big)$, $\theta \in [-r, 0]$, where $x(p, \cdot)$ denotes the unique solution

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of the Cauchy problem

$$\dot{x}(t) = g\big(x(t)\big), \tag{3.5a}$$

$$x(0) = p. \tag{3.5b}$$

Well known properties of differential equations imply that the domain $\mathcal{D}(Q)$ of Q is an open subset of $\widetilde{M}$. Also, since $T \geq r$, the Ascoli-Arzelà Theorem implies that Q is a locally compact map (see, e.g. [Ol69]).

Observe that the T-periodic solutions of (3.5a) are in a one-to-one correspondence with the fixed points of Q. We will prove a formula (Theorem 3.2 below) for the computation of the fixed point index of the admissible pairs (Q, W), where W is an open subset of $\mathcal{D}(Q)$. Clearly, Q is strictly related to the M-valued Poincaré map P, given by $P(p) = x(p, T)$, whose domain is the open subset $\mathcal{D}(P)$ of M consisting of those points p such that the solution $x(p, \cdot)$ of the above Cauchy problem is defined up to T.

We shall need the following result of [FS96] about the fixed point index of P.

Theorem 3.1. *Let g be as above and let $U \subseteq M$ be open and such that $\mathrm{ind}(P, U)$ is defined. Then, $\deg(-g, U)$ is defined as well and*

$$\mathrm{ind}(P, U) = \deg(-g, U).$$

There is a simple relation between the domain $\mathcal{D}(Q)$ of Q and the domain $\mathcal{D}(P)$ of P. In fact $\mathcal{D}(Q) = \{\varphi \in \widetilde{M} : \varphi(0) \in \mathcal{D}(P)\}$. In particular, $\widetilde{\mathcal{D}(P)} \subseteq \mathcal{D}(Q)$. Observe also that $P(p) = Q(\widehat{p})(0)$ for all $p \in \mathcal{D}(P)$.

Theorem 3.2. *Let g, T and Q be as above, and let $W \subseteq \widetilde{M}$ be open. If the fixed point index $\mathrm{ind}(Q, W)$ is defined, then so is $\deg(-g, \check{W})$ and*

$$\mathrm{ind}(Q, W) = \deg(-g, \check{W}).$$

Proof. The assumption that $\mathrm{ind}(Q, W)$ is defined means that $W \subseteq \mathcal{D}(Q)$ and that $\mathrm{Fix}(Q, W)$ is compact. Let us show that $\deg(-g, \check{W})$ is defined too. We need to prove that $g^{-1}(0) \cap \check{W}$ is compact. If $p \in g^{-1}(0) \cap \check{W}$, then the constant function $\widehat{p}$ is a fixed point of Q. Thus $g^{-1}(0) \cap \check{W}$ is compact since it can be regarded as a closed subset of the compact set $\mathrm{Fix}(Q, W)$.

We now use the Commutativity Property of the fixed point index in order to deduce the desired formula for the fixed point index of Q from the analogous one for P, expressed in Theorem 3.1. In order to do so, we define the maps $h : \mathcal{D}(P) \to \widetilde{M}$ and $k : \widetilde{M} \to M$ by $h(p)(\theta) = x(p, \theta + T)$ and $k(\varphi) = \varphi(0)$, respectively. Clearly, we have

$$(h \circ k)(\varphi)(\theta) = x\big(\varphi(0), \theta + T\big) = Q(\varphi)(\theta), \qquad \varphi \in \mathcal{D}(Q), \quad \theta \in [-r, 0], \tag{3.6}$$

and

$$(k \circ h)(p) = x(p, T) = P(p), \qquad p \in \mathcal{D}(P). \tag{3.7}$$

Define $\gamma = k|_W$. By the Commutativity Property of the fixed point index, $\text{ind}\left(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P))\right)$ is defined if and only if so is $\text{ind}\left(\gamma \circ h, h^{-1}(W)\right)$, and

$$\text{ind}\left(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P))\right) = \text{ind}\left(\gamma \circ h, h^{-1}(W)\right). \tag{3.8}$$

Since $W \subseteq \mathcal{D}(Q)$, then $\gamma^{-1}(\mathcal{D}(P)) = W$. Hence, from formulas (3.6)–(3.7), it follows that

$$\text{ind}(Q, W) = \text{ind}\left(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P))\right), \tag{3.9}$$

$$\text{ind}\left(P, h^{-1}(W)\right) = \text{ind}\left(\gamma \circ h, h^{-1}(W)\right). \tag{3.10}$$

Thus, by (3.8), we get

$$\text{ind}(Q, W) = \text{ind}\left(P, h^{-1}(W)\right). \tag{3.11}$$

From Theorem 3.1, we obtain

$$\text{ind}\left(P, h^{-1}(W)\right) = \deg\left(-g, h^{-1}(W)\right). \tag{3.12}$$

From the definition of h it follows immediately that

$$g^{-1}(0) \cap \check{W} = g^{-1}(0) \cap h^{-1}(W).$$

Therefore, from the Excision Property of the degree of a vector field, one has

$$\deg\left(-g, h^{-1}(W)\right) = \deg(-g, \check{W}) \tag{3.13}$$

and the assertion follows from equations (3.11), (3.12) and (3.13). $\qquad\square$

Let $W \subseteq \mathcal{D}(Q)$ be open in $\widetilde{M}$. We point out that Theorems 3.1 and 3.2 imply that the fixed point index of Q in W actually reduces to the fixed point index of the finite dimensional operator P in $\check{W}$. Namely,

$$\text{ind}(Q, W) = \text{ind}(P, \check{W}). \tag{3.14}$$

In fact, P is defined on $\check{W}$ and $\text{Fix}(P, \check{W})$ can be regarded as a closed subset of $\text{Fix}(Q, W)$. Therefore, if $\text{ind}(Q, W)$ is defined, then so is $\text{ind}(P, \check{W})$ and, applying Theorems 3.2 and 3.1, we get $\text{ind}(Q, W) = \deg(-g, \check{W}) = \text{ind}(P, \check{W})$.

Let us remark that the mappings h and k, defined in the proof of Theorem 3.2, establish a bijection between the fixed point sets of Q and P. However, we should not think of formula (3.14) as a trivial consequence of this correspondence. In fact, given W as in Theorem 3.2, we see that h and k induce a one-to-one correspondence between the fixed points of Q in W and those of P in $h^{-1}(W)$ but, in general, $\text{Fix}\left(P, h^{-1}(W)\right) \neq \text{Fix}\left(P, \check{W}\right)$. Observe also that the "finite dimensional reduction formula" (3.14) has a clear advantage over the more crude reduction formula (3.11) obtained in the proof of Theorem 3.2 by means of the Commutativity Property of the fixed point index (and that derives from the correspondence we just mentioned). In fact, differently from the set $h^{-1}(W)$ that appears in (3.11), the open set $\check{W}$ does not depend on the equation (3.5a).

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4 Branches of starting pairs

Any pair $(\lambda, \varphi) \in [0, \infty) \times \widetilde{M}$ is said to be a *starting pair* (for (3.4)) if the following initial value problem has a T-periodic solution:

$$\begin{cases} \dot{x}(t) = g\big(x(t)\big) + \lambda f\big(t, x(t), x(t-r)\big) & t > 0, \\ x(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \tag{4.15}$$

A pair of the type $(0, \widehat{p})$ with $g(p) = 0$ is clearly a starting pair and will be called a *trivial starting pair*. The set of all starting pairs for (3.4) will be denoted by S.

Throughout this section we shall assume that f and g are C^1, so that (4.15) admits a unique solution that we shall denote by $\xi^\lambda(\varphi, \cdot)$. Observe that $\xi^0\big(\varphi(0), \cdot\big) = x\big(\varphi(0), \cdot\big)$, where, we recall, $x(p, \cdot)$ is the unique solutions of the Cauchy problem (3.5). By known continuous dependence properties of delay differential equations the set $\mathcal{V} \subseteq [0, \infty) \times \widetilde{M}$ given by

$$\mathcal{V} := \big\{ (\lambda, \varphi) : \xi^\lambda(\varphi, \cdot) \text{ is defined on } [0, T] \big\}$$

is open. Clearly $\mathcal{V}$ contains the set S of all starting pairs for (3.4). Observe that S is closed in $\mathcal{V}$, even if it could be not so in $[0, +\infty) \times \widetilde{M}$. Moreover, by the Ascoli-Arzelà Theorem it follows that S is locally compact.

In the sequel, given $A \subseteq \mathbb{R} \times \widetilde{M}$ and $\lambda \in \mathbb{R}$, we will denote the slice $\{x \in \widetilde{M} : (\lambda, x) \in A\}$ by the symbol A_λ. Observe that $\widecheck{\mathcal{V}}_0 = \mathcal{D}(P)$ where P is the Poincaré operator defined in the previous section.

In order to study the T-periodic solutions of (1.1), it will be convenient to introduce, for each $\lambda \geq 0$, the map $Q_\lambda : \mathcal{V}_\lambda \to \widetilde{M}$ given by

$$Q_\lambda(\varphi)(\theta) = \xi^\lambda(\varphi, \theta + T), \quad \theta \in [-r, 0].$$

Notice that Q_0 coincides with the map Q defined in the previous section.

We will need the following global connectivity result of [FP93].

Lemma 4.1. *Let Y be a locally compact metric space and let Z be a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then $Y \setminus Z$ contains a connected set whose closure (in Y) intersects Z and is not compact.*

Proposition 4.1. *Assume that f, g, S are as above. Given $W \subseteq [0, \infty] \times \widetilde{M}$ open, if $\deg(g, \widecheck{W}_0)$ is (defined and) nonzero, then the set*

$$(S \cap W) \setminus \big\{ (0, \widehat{p}) \in W : g(p) = 0 \big\}$$

of nontrivial starting pairs in W, admits a connected subset whose closure in $S \cap W$ meets $\big\{ (0, \widehat{p}) \in W : g(p) = 0 \big\}$ and is not compact.

Proof. Let us define the open set $U = W \cap \mathcal{V}$. Since $g^{-1}(0) \cap \widecheck{U}_0 = g^{-1}(0) \cap \widecheck{W}_0$, and $S \cap U = S \cap W$, we need to prove that the set of nontrivial starting pairs in

U admits a connected subset whose closure in $S \cap U$ meets $\left\{(0, \widehat{p}) \in U : g(p) = 0\right\}$ and is not compact.

As pointed out before, S is locally compact, thus, U being open, $S \cap U$ is locally compact. Moreover the assumption that $\deg(g, \widecheck{W}_0)$ is defined means that the set

$$\left\{p \in \widecheck{W}_0 : g(p) = 0\right\} = \left\{p \in \widecheck{U}_0 : g(p) = 0\right\}$$

is compact. Thus the homeomorphic set $\left\{(0, \widehat{p}) \in U : g(p) = 0\right\}$ is compact as well.

The assertion will follow applying Lemma 4.1 to the pair

$$(Y, Z) = \left(S \cap U, \left\{(0, \widehat{p}) \in U : p \in g^{-1}(0)\right\}\right).$$

In fact, if Σ is a connected set as in the assertion of Lemma 4.1, its closure satisfies the requirement.

Assume, by contradiction, that there exists a compact subset C of the set $S \cap U$ of starting pairs of (4.15) in U containing Z and with empty boundary in $S \cap U$. Thus C is a relatively open subset of $S \cap U$. As a consequence, $(S \cap U) \setminus C$ is closed in $S \cap U$, so the distance, $\delta = \operatorname{dist}\left(C, (S \cap U) \setminus C\right)$, between C and $(S \cap U) \setminus C$ is nonzero (recall that C is compact). Consider the set

$$A = \left\{(\lambda, \varphi) \in U : \operatorname{dist}\left((\lambda, \varphi), C\right) < \delta/2\right\},$$

which, clearly, does not meet $(S \cap U) \setminus C$.

Because of the compactness of $S \cap U \cap A = C$, there exists $\overline{\lambda} > 0$ such that $\left(\{\overline{\lambda}\} \times A_{\overline{\lambda}}\right) \cap S \cap U = \emptyset$. Moreover, the set $S \cap U \cap A$ coincides with $\{(\lambda, \varphi) \in A : Q_\lambda(\varphi) = \varphi\}$. Then, from the Generalized Homotopy Invariance Property of the fixed point index,

$$0 = \operatorname{ind}\left(Q_{\overline{\lambda}}, A_{\overline{\lambda}}\right) = \operatorname{ind}\left(Q_\lambda, A_\lambda\right),$$

for all $\lambda \in [0, \overline{\lambda}]$. But, by Theorem 3.2 and by the Excision Property of the degree, we get

$$\operatorname{ind}(Q, A_0) = \deg(-g, \widecheck{A}_0) = \deg(-g, \widecheck{W}_0) \neq 0.$$

That contradicts the previous formula, since $Q = Q_0$. $\qquad \square$

5 Branches of T-periodic pairs

Let us introduce the function space where most of the work of this section is done. We will denote by $C_T(M)$ the metric subspace of the Banach space $\left(C_T(\mathbb{R}^k), \|\cdot\|\right)$ of all the T-periodic continuous maps $x : \mathbb{R} \to M$ with the usual C^0 norm. Observe that $C_T(M)$ is not complete unless M is complete (i.e. closed in $\mathbb{R}^k$). Nevertheless, since M is locally compact, $C_T(M)$ is always locally complete.

For the sake of simplicity, we will identify M with its image in $[0, \infty) \times C_T(M)$ under the embedding which associates to any $p \in M$ the pair $(0, \bar{p})$, $\bar{p} \in C_T(M)$ being the map constantly equal to p. According to these identifications, if E is a subset of $[0, \infty) \times C_T(M)$, by $E \cap M$ we mean the subset of M given by all $p \in M$

such that the pair $(0, \bar{p})$ belongs to E. Observe that if $\Omega \subseteq [0, \infty) \times C_T(M)$ is open, then so is $\Omega \cap M$.

A pair $(\lambda, x) \in [0, \infty) \times C_T(M)$, where x a solution of (3.4), is called a *T-periodic pair* (for (3.4)). Those T-periodic pairs that are of the particular form $(0, \bar{p})$ are said to be *trivial*. Observe that $(0, \bar{p}) \in [0, \infty) \times C_T(M)$ is a trivial T-periodic pair if and only if $g(p) = 0$. We point out that if x is a nonconstant T-periodic solution of the unperturbed equation $\dot{x}(t) = g(x(t))$, then $(0, x)$ is a nontrivial T-periodic pair.

We are now in a position to state our main result. The proof is inspired by [FS96, FP93].

Theorem 5.1. *Let $g : M \to \mathbb{R}^k$ be a tangent vector field on M and, given $T > 0$, let $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be T-periodic in the first variable and tangent to M in the second one. Let Ω be an open subset of $[0, \infty) \times C_T(M)$, and assume that $\deg(g, \Omega \cap M)$ is defined and nonzero. Then Ω contains a connected set of nontrivial T-periodic pairs whose closure in Ω meets the set $\{(0, \bar{p}) \in \Omega : g(p) = 0\}$ and is not compact.*

In particular, the set of T-periodic pairs for (3.4) contains a connected component that meets $\{(0, \bar{p}) \in \Omega : g(p) = 0\}$ and whose intersection with Ω is not compact.

Proof. Denote by X the set of T-periodic pairs of (3.4) and by S the set of starting pairs of the same equation; that is, of all pairs $(\lambda, x|_{[-r,0]})$ with $(\lambda, x) \in X$, $x|_{[-r,0]}$ being the restriction to $[-r, 0]$ of x.

Assume first that f and g are smooth. Define the map $h : X \to S$ by $h(\lambda, x) = (\lambda, x|_{[-r,0]})$ and observe that h is continuous, onto and, since f and g are smooth, it is also one to one. Furthermore, by the continuous dependence on data, $h^{-1} : S \to X$ is continuous as well.

Take

$$S_\Omega = \{(\lambda, \varphi) \in S : \text{the solution of (3.4) is contained in } \Omega\}.$$

So that $X \cap \Omega$ and S_Ω correspond under the homeomorphism $h : X \to S$. Thus, S_Ω is an open subset of S and, consequently, we can find an open subset W of $[0, \infty) \times \widetilde{M}$ such that $S \cap W = S_\Omega$. This implies

$$\{p \in \check{W}_0 : g(p) = 0\} = \{p \in M : (0, \hat{p}) \in W, \ g(p) = 0\} =$$
$$= \{p \in M : (0, \bar{p}) \in \Omega, \ g(p) = 0\} = \{p \in \Omega \cap M : g(p) = 0\}.$$

Thus, by excision, $\deg(g, \check{W}_0) = \deg(g, \Omega \cap M) \neq 0$. Applying Proposition 4.1, we get the existence of a connected set

$$\Sigma \subseteq (S \cap W) \setminus \{(0, \hat{p}) \in W : g(p) = 0\}$$

whose closure in $S \cap W$ meets $\{(0, \hat{p}) \in W : g(p) = 0\}$ and is not compact.

Observe that the trivial T-periodic pairs correspond to the trivial starting pairs under the homeomorphism h. Thus, $\Gamma = h^{-1}(\Sigma) \subseteq X \cap \Omega$ is a connected set of

nontrivial T-periodic pairs whose closure in $X \cap \Omega$ meets $\{(0, \overline{p}) \in \Omega : g(p) = 0\}$ and is not compact. Since X is closed in $[0, \infty) \times C_T(M)$, the closures of Γ in $X \cap \Omega$ and in Ω coincide. This proves that Γ satisfies the requirements of the first part of the assertion.

Let us remove the smoothness assumption on g and f. As above, it is enough to show the existence of a connected set Γ of nontrivial T-periodic pairs whose closure in $X \cap \Omega$ meets $\{(0, \overline{p}) \in \Omega : g(p) = 0\}$ and is not compact.

Observe that the closed subset X of $[0, \infty) \times C_T(M)$ is locally compact because of Ascoli-Arzelà Theorem. It is convenient to introduce the following subset of X:

$$\Upsilon = \big\{(0, \overline{p}) \in [0, \infty) \times C_T(M) : g(p) = 0\big\}.$$

Take

$$Y = X \cap \Omega \quad \text{and} \quad Z = \Upsilon \cap \Omega$$

and notice that Y is locally compact as an open subset of X. Moreover, Z is a compact subset of Y (recall that, by assumption, $\deg(g, M \cap \Omega)$ is defined). Since Y is closed in Ω, we only have to prove that the pair (Y, Z) satisfies the hypothesis of Lemma 4.1. Assume the contrary. Thus, we can find a relatively open compact subset C of Y containing Z. Similarly to the proof of Proposition 4.1, given $0 < \rho < \mathrm{dist}(C, Y \setminus C)$, we consider the set A^ρ of all pairs $(\lambda, \varphi) \in \Omega$ whose distance from C is smaller than ρ. Thus, $A^\rho \cap Y = C$ and $\partial A^\rho \cap Y = \emptyset$. We can also assume that the closure $\overline{A^\rho}$ of A^ρ in $[0, \infty) \times C_T(M)$ is contained in Ω. Since C is compact and $[0, \infty) \times M$ is locally compact, we can take A^ρ in such a way that the set

$$\big\{(\lambda, x(t), x(t - r)) \in [0, \infty) \times M \times M : (\lambda, x) \in A^\rho, \ t \in [0, T]\big\}$$

is contained in a compact subset of $[0, \infty) \times M \times M$. This implies that A^ρ is bounded with complete closure and $A^\rho \cap M$ is a relatively compact subset of $\Omega \cap M$. In particular g is nonzero on the boundary of $A^\rho \cap M$ (relative to M). By well known approximation results on manifolds, we can find sequences $\{g_i\}$ and $\{f_i\}$ of smooth maps uniformly approximating g and f, and such that the following properties hold for all $i \in \mathbb{N}$:

- $g_i(p) \in T_p M$ for all $p \in M$;
- $f_i(t, p, q) \in T_p M$ for all $(t, p, q) \in \mathbb{R} \times M \times M$;
- f_i is T-periodic in the first variable.

For $i \in \mathbb{N}$ large enough, we get

$$\deg(g_i, A^\rho \cap M) = \deg(g, A^\rho \cap M).$$

Furthermore, by excision,

$$\deg(g, A^\rho \cap M) = \deg(g, \Omega \cap M) \neq 0.$$

Therefore, given i large enough, the first part of the proof can be applied to the equation

$$\dot{x}(t) = g_i\big(x(t)\big) + \lambda f_i\big(t, x(t), x(t - r)\big). \tag{5.16}$$

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Let X_i denote the set of T-periodic pairs of (5.16) and put

$$\Upsilon_i = \big\{(0, \overline{p}) \in [0, \infty) \times C_T(M) : g_i(p) = 0\big\}.$$

Because of the first part of the proof, there exists a connected subset Γ_i of A^ρ whose closure in A^ρ meets $\Upsilon_i \cap A^\rho$ and is not compact. Let us denote by $\overline{\Gamma}_i$ and $\overline{A^\rho}$ the closures in $[0, \infty) \times C_T(M)$ of Γ_i and A^ρ, respectively.

Let us show that, for i large enough, $\overline{\Gamma}_i \cap \partial A^\rho \neq \emptyset$. Thus, X_i being closed, we get $\overline{\Gamma}_i \subseteq X_i$. This will imply the existence of a T-periodic pair $(\lambda_i, x_i) \in \partial A^\rho$ of (5.16). It is enough to prove that $\overline{\Gamma}_i$ is compact. In fact, if this is true and if we assume $\overline{\Gamma}_i \cap \partial A^\rho = \emptyset$, then $\overline{\Gamma}_i \subseteq A^\rho$ which implies that the closure of Γ_i in A^ρ coincides with the compact set $\overline{\Gamma}_i$, and this is a contradiction. The compactness of $\overline{\Gamma}_i$, for i large enough, follows from the completeness of $\overline{A^\rho}$ and the fact that, by the Ascoli-Arzelà Theorem, $\overline{\Gamma}_i$ is totally bounded, when i is sufficiently large. Thus, for i large enough, there exists a T-periodic pair $(\lambda_i, x_i) \in \partial A^\rho$ of (5.16).

Again by Ascoli-Arzelà Theorem, we may assume that $x_i \to x_0$ in $C_T(M)$ and $\lambda_i \to \lambda_0$ with $(\lambda_0, x_0) \in \partial A^\rho$. Passing to the limit in equation (5.16), it is not difficult to show that (λ_0, x_0) is a T-periodic pair of (3.4) in ∂A^ρ. This contradicts the assumption $\partial A^\rho \cap Y = \emptyset$ and proves the first part of the assertion.

Let us prove the last part of the assertion. Consider the connected component Ξ of X that contains the connected set Γ of the first part of the assertion. We shall now show that Ξ has the required properties. Clearly, Ξ meets the set $\big\{(0, \overline{p}) \in \Omega : g(p) = 0\big\}$ because the closure $\overline{\Gamma}^\Omega$ of Γ in Ω does. Moreover, $\Xi \cap \Omega$ cannot be compact, since $\Xi \cap \Omega$, as a closed subset of Ω, contains $\overline{\Gamma}^\Omega$, and $\overline{\Gamma}^\Omega$ is not compact. $\qquad\square$

The following corollary, in the case of a compact boundaryless manifolds, extends a result of [BCFP07] in which g is identically zero.

Corollary 5.1. *Let f and g be as in Theorem 5.1 and let $M \subseteq \mathbb{R}^k$ be compact with nonzero Euler-Poincaré characteristic $\chi(M)$. Then, there exists an unbounded connected set of nontrivial T-periodic pairs whose closure meets $\{(0, \overline{p}) \in [0, \infty) \times C_T(M) : g(p) = 0\}$. In particular, equation (4.15) has a solution for any $\lambda \geq 0$.*

Proof. Since M is compact, $[0, \infty) \times C_T(M)$ is a complete metric space. Moreover, the Ascoli-Arzelà Theorem implies that any bounded set of T-periodic pairs is totally bounded. The Poincaré-Hopf Theorem yields $\deg(g, M) = \chi(M) \neq 0$. Thus, Theorem 5.1 implies the existence of an unbounded connected set Γ of nontrivial T-periodic pairs whose closure in $[0, \infty) \times C_T(M)$ meets $\{(0, \overline{p}) \in [0, \infty) \times C_T(M) : g(p) = 0\}$. The last assertion follows from the fact that $C_T(M)$ is bounded while Γ is unbounded. $\qquad\square$

Corollary 5.2. *Let f and g be as in Theorem 5.1. Assume that M is closed as a subset of $\mathbb{R}^k$. Let $\Omega \subseteq [0, \infty) \times C_T(M)$ be open and such that $\deg(g, \Omega \cap M)$ is defined and nonzero. Then there exists a connected component Γ of T-periodic pairs that meets $\{(0, \overline{p}) \in \Omega : g(p) = 0\}$ and cannot be both bounded and contained in Ω. In particular, if Ω is bounded, then $\Gamma \cap \partial\Omega \neq \emptyset$.*

Proof. Since M is a closed subset of $\mathbb{R}^k$, $[0, \infty) \times C_T(M)$ is complete. Moreover, the Ascoli-Arzelà Theorem implies that any bounded set of T-periodic pairs is totally bounded. Thus, the first part of the assertion follows from Theorem 5.1. The last part of the assertion follows from the fact that Γ is connected and that $\emptyset \neq \Gamma \cap \Omega \neq \Gamma$. $\qquad\square$

To better understand the meaning of Corollary 5.2, consider for example the case when $M = \mathbb{R}^m$. If $g^{-1}(0)$ is compact and $\deg(g, \mathbb{R}^m) \neq 0$, then there exists an unbounded connected set of T-periodic pairs in $[0, \infty) \times C_T(\mathbb{R}^m)$ which meets the set $\{(0, \overline{p}) \in [0, \infty) \times C_T(M) : g(p) = 0\}$, that can be identified with $g^{-1}(0)$. The existence of this unbounded connected set cannot be destroyed by a particular choice of f. However it is possibly "completely vertical", i.e. contained in the slice $\{0\} \times C_T(M)$. This peculiarity is exhibited, for instance, by the set of T-periodic pairs of the equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \lambda \sin t, \end{cases}$$

where $M = \mathbb{R}^2$ and $T = 2\pi$.

A somewhat opposite behavior is shown by the set X of T-periodic pairs for (3.4) in the "degenerate" situation when $f(t, p, q) \equiv 0$. In this case, X consists of the pairs (λ, x), where $\lambda \geq 0$ and x is a T-periodic solution to $\dot{x} = g(x)$. In particular, given any $p \in M$ such that $g(p) = 0$, the connected component Γ of X containing $\{0\} \times \overline{p}$ contains the "horizontal" set $[0, +\infty) \times \{\overline{p}\}$ and, clearly, satisfies the requirement of Corollary 5.2.

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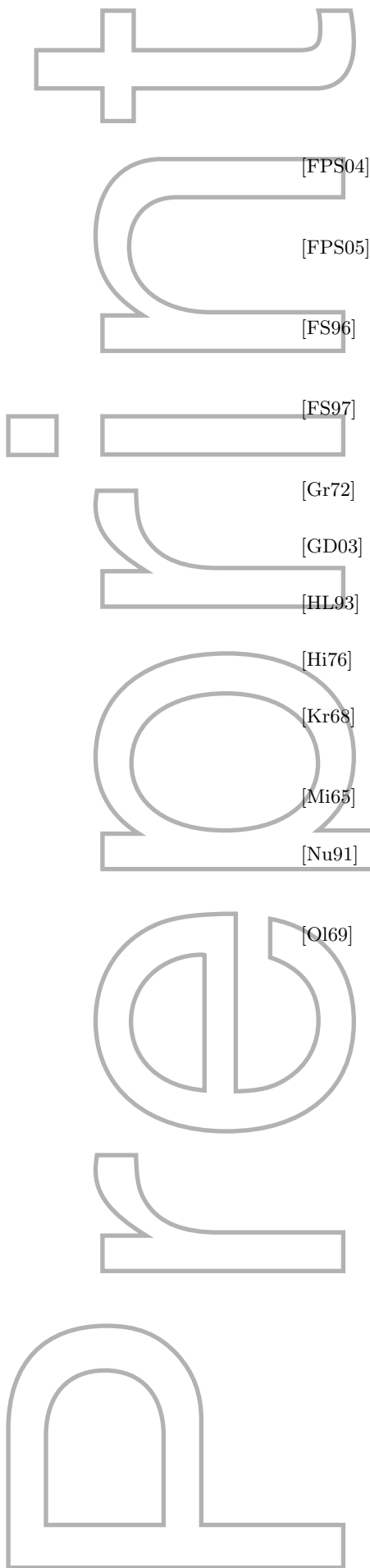
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