BRANCHES OF FORCED OSCILLATIONS FOR A CLASS OF CONSTRAINED ODES: A TOPOLOGICAL APPROACH

ALESSANDRO CALAMAI AND MARCO SPADINI

Abstract. We apply topological methods to obtain global continuation results for harmonic solutions of some periodically perturbed ordinary differential equations on a m-dimensional differentiable manifold $M \subseteq \mathbb{R}^k$. We assume that M is globally defined as the zero set of a smooth map and, as a first step, we determine a formula which reduces the computation of the degree of a tangent vector field on M to the Brouwer degree of a suitable map in $\mathbb{R}^m$. As further applications, we study the set of harmonic solutions to periodic semi-esplicit differential-algebraic equations.

1. Introduction and preliminaries

In this paper we study T-periodic solutions of some parametrized families of T-periodic constrained ordinary differential equations (ODEs). More precisely, we study periodically perturbed autonomous ODEs on a differentiable submanifold of some Euclidean space, under the assumption that such a manifold is globally defined as the zero set of a smooth map. We consider the two different cases of nontrivial unperturbed equation and of perturbation of the zero vector field. We adopt a topological approach and we make use of results which are based on the fixed point index. However, our techniques require just the notion of the degree (often called characteristic or rotation) of tangent vector fields to differentiable manifolds, which in the 'flat' case, that is when the manifold is an open subset of an Euclidean space, is essentially the well known Brouwer degree. As an application of our results, we study T-periodic solutions of particular parametrized differential-algebraic equations (DAEs), for which we will prove global continuation results.

Recently, differential-algebraic equations have received an increasing interest due, in particular, to applications in engineering and have been the subject of extensive study (see e.g. [10] for a comprehensive treatment) aimed mostly (but not only) to numerical methods. Our approach here, inspired by [1] and [13], is directed towards qualitative theory of some particular DAEs which are studied by means of topological methods, making use of the equivalence of the given equations and suitable ODEs on manifolds. Relatively to [1, 13], here we operate a change of perspective: assuming the viewpoint of ODEs on manifold allows us to present the matter in a general and extensively studied framework (see e.g. [4]).

Our first goal is to obtain a formula for the computation of the degree of tangent fields to a k-dimensional differentiable submanifold M of $\mathbb{R}^m$, in the particular case when the manifold is defined implicitly as the zero set of a smooth function $g : U \to \mathbb{R}^{m-k}$, $U \subseteq \mathbb{R}^m$ open and connected, and assuming that with an appropriate choice of (orthonormal) coordinates one can decompose $\mathbb{R}^m$ as $\mathbb{R}^k \times \mathbb{R}^{m-k}$ in such a way that the Jacobian matrix of g with respect to the last k variables, $\partial_2 g(x, y)$,

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is nonsingular for all $(x,y) \in U$. Notice that, then, M has codimension $s := m-k$ in $\mathbb{R}^m$.

The formula we find (see Theorem 4.1 below) reduces the computation of the degree of a tangent vector field on M to that of an appropriate map in $\mathbb{R}^m$. More precisely, let $\varphi : M \to \mathbb{R}^m$ be tangent to M, in the sense that $\varphi(\xi)$ belongs to the tangent space $T_\xi M$ of M at ξ for any $\xi \in M$. Let also $\widetilde{\varphi}$ be *any* extension of φ to U. With a small abuse of notation, we will write, according to the above decomposition,

$$\widetilde{\varphi}(\xi) = \widetilde{\varphi}(x,y) = \big(\widetilde{\varphi}_1(x,y), \widetilde{\varphi}_2(x,y)\big),$$

and define $\mathcal{F} : U \to \mathbb{R}^m$ as $\mathcal{F}(x,y) = \big(\widetilde{\varphi}_1(x,y), g(x,y)\big)$, for any $(x,y) \in U$. We will prove that

$$\deg(\varphi, M) = \mathfrak{s} \deg(\mathcal{F}, U) \tag{1.1}$$

where $\mathfrak{s}$ is the (constant) sign of $\det(\partial_2 g)$ on the connected set U. Observe that the just defined vector field $\mathcal{F}$ on U may well not be tangent to M. In fact, on M, the second component of $\mathcal{F}$ is forced to be zero regardless of the shape of M.

The above formula (1.1) is equivalent to a result proved in [13] but we provide here a simplified proof. Notice that (1.1) does not depend on the chosen extension of φ. Notice also that, since in Euclidean spaces vector fields can be regarded as maps and vice versa, the degree of the vector field $\mathcal{F}$ that appears in the second member of (1.1) is essentially the well known Brouwer degree, with respect to 0, of $\mathcal{F}$ seen as a map. Hence the degree of $\mathcal{F}$, having a simpler nature than that of φ, is 'morally' easier to compute.

As an application we study the set of harmonic solutions of the following differential equations on a manifold $M \subseteq \mathbb{R}^m$, with $M = g^{-1}(0)$ and g as above:

$$\dot{\xi} = f(\xi) + \lambda h(t,\xi) \tag{1.2a}$$

and

$$\dot{\xi} = \lambda h(t,\xi), \tag{1.2b}$$

where $h : \mathbb{R} \times M \to \mathbb{R}^m$ and $f : M \to \mathbb{R}^m$ are continuous maps with the property that $f(\xi)$ and $h(t,\xi)$ belong to $T_\xi M$ for any $(t,\xi) \in \mathbb{R} \times M$, and h is T-periodic in the first variable.

Notice that locally M can be represented as graph of some map from an open subset of $\mathbb{R}^{m-s}$ to $\mathbb{R}^s$. Thus equations (1.2) can be locally simplified. In view of this fact one might think that it is possible to reduce equations (1.2) to ordinary differential equations in $\mathbb{R}^{m-s}$. It is not so. In fact, globally, M may not be the graph of a map from an open subset of $\mathbb{R}^{m-s}$ to $\mathbb{R}^s$ as, for instance, when $U = \mathbb{R}^3$ and $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$g(x,y) = g(x; y_1, y_2) = \big(e^{y_1}\cos(y_2) - x, e^{y_1}\sin(y_2) - x\big).$$

In this case, although $\det \partial_2 g(x,y) \neq 0$, one clearly has that the 1-dimensional manifold $M = g^{-1}(0)$ is not the graph of a function $x \mapsto \big(y_1(x), y_2(x)\big)$. In fact, M consists of infinitely many connected components each lying in a plane $y_2 = \frac{\pi}{4} + k\pi$ for $k \in \mathbb{Z}$. (See also Examples 5.2 and 5.8 below.)

Observe also that even when M is a (global) graph of some map Γ, the expression of Γ might be too complicated or impossible to determine analytically, so that the decoupled versions of equations (1.2) may be too difficult to use. A simple example of this fact is obtained by taking $k = s = 1$, $U = \mathbb{R}\times\mathbb{R}$ and $g(x,y) = y^7 + y - x^2 + x^5$.

As further applications, we will deduce the results of [1, 13] about harmonic solutions of periodic semi-explicit differential-algebraic equations that have either the form

$$\begin{cases} \dot{x} = \gamma(x,y) + \lambda\sigma(t,x,y), & \lambda \geq 0, \\ g(x,y) = 0, \end{cases} \tag{1.3a}$$

or

$$\begin{cases} \dot{x} = \lambda\sigma(t,x,y), & \lambda > 0, \\ g(x,y) = 0. \end{cases} \tag{1.3b}$$

where $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is a connected open set, $g : U \to \mathbb{R}^s$ is as above and $\gamma : U \to \mathbb{R}^k$ and $\sigma : U \to \mathbb{R}^k$ are continuous maps with σ T-periodic in t for a given $T > 0$. In fact, as we shall see, equations (1.3a) and (1.3b) are equivalent to (1.2a) and (1.2b), respectively, for appropriate vector fields f and h on the manifold $g^{-1}(0)$. Notice that, as remarked above, although the set $g^{-1}(0)$ is locally the graph of some map of an open set of $\mathbb{R}^{m-s}$ to $\mathbb{R}^s$ so that equations (1.3a) and (1.3b) can be locally decoupled, it is not always possible or convenient to do globally so.

2. Tangent vector fields and the notion of degree

We now recall some basic notions about tangent vector fields on manifolds, and introduce the notion of degree of an admissible tangent vector field.

Let $M \subseteq \mathbb{R}^m$ be a manifold. Let w be a tangent vector field on M, that is, a continuous map $w : M \to \mathbb{R}^m$ with the property that $w(\zeta) \in T_\zeta M$ for any $\zeta \in M$. If w is (Fréchet) differentiable at $\zeta \in M$ and $w(\zeta) = 0$, then the differential $dw_\zeta : T_\zeta M \to \mathbb{R}^m$ maps $T_\zeta M$ into itself (see e.g. [11]), so that the determinant $\det dw_\zeta$ of dw_ζ is defined. If, in addition, ζ is a nondegenerate zero (i.e. $dw_\zeta : T_\zeta M \to \mathbb{R}^m$ is injective) then ζ is an isolated zero and $\det dw_\zeta \neq 0$.

Let W be an open subset of M in which we assume w *admissible* (for the degree); that is, the set $w^{-1}(0) \cap W$ is compact. Then, one can associate to the pair (w, W) an integer, $\deg(w, W)$, called the *degree (or characteristic) of the vector field w in W*, which, in a sense, counts (algebraically) the zeros of w in W (see e.g. [5, 9, 11] and references therein). In fact, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap W$ is finite and

$$\deg(w, W) = \sum_{\zeta \in w^{-1}(0) \cap W} \operatorname{sign} \det dw_\zeta. \tag{2.1}$$

Observe that in the flat case, i.e. when $M = \mathbb{R}^m$, $\deg(w, W)$ is just the classical Brouwer degree with respect to zero, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap W$ whose closure is contained in W.

The notion of degree of an admissible tangent vector field plays a crucial role throughout this paper. It enjoys a number of properties some of which we report here for the sake of future reference.

Additivity: *Let w be admissible in W. If W_1 and W_2 are two disjoint open subsets of W whose union contains $w^{-1}(0) \cap W$, then*

$$\deg(w, W) = \deg(w, W_1) + \deg(w, W_2).$$

Homotopy Invariance: *Let $h : M \times [0,1] \to \mathbb{R}^m$ be an admissible homotopy (of tangent vector fields) in W; that is, $h(\zeta, \lambda) \in T_\zeta M$ for all $(\zeta, \lambda) \in M \times [0,1]$ and $h^{-1}(0) \cap W \times [0,1]$ is compact. Then $\deg\big(h(\cdot, \lambda), W\big)$ is independent of λ.*

Solution: *If w is admissible in W and $\deg(w, W) \neq 0$, then w has a zero in W.*

The Additivity Property implies the following important property:

Excision: *Let (w, W) be admissible. If $V \subseteq W$ is open and contains $w^{-1}(0) \cap W$, then* $\deg(w, W) = \deg(w, V)$.

Another property that plays an important role in this paper is the following one which allows the comparison between the degrees of vector fields that correspond under diffeomorphisms. Recall that if $v : N \to \mathbb{R}^n$ and $w : M \to \mathbb{R}^m$ are tangent vector fields on the differentiable manifolds $N \subseteq \mathbb{R}^n$ and $M \subseteq \mathbb{R}^m$, and if $\rho : W \to V$ is a diffeomorphism from an open subset W of M onto an open subset V of N, we say that $v|_V$ and $w|_W$ *correspond under* ρ when $v(\rho(\zeta)) = d\rho_\zeta\big(w(\zeta)\big)$ for all $\zeta \in W$.

Invariance under diffeomorphisms: *Let $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ be differentiable manifolds and let $v : N \to \mathbb{R}^n$ and $w : M \to \mathbb{R}^m$ be tangent vector fields. Assume that $v|_V$ and $w|_W$ correspond under some diffeomorphism. Then, if either v is admissible in V or or w is admissible in W, so is the other and*

$$\deg(v, V) = \deg(w, W).$$

Remark 2.1. *Let $W \subseteq M$ be open and relatively compact. If $w : M \to \mathbb{R}^m$ is such that $w(\zeta) \neq 0$ on the boundary $\mathrm{Fr}\,(W)$ of W, then (w, W) is admissible. Let $\varepsilon = \min_{\zeta \in \mathrm{Fr}\,(W)} |w(\zeta)|$. Then, for any $v : M \to \mathbb{R}^m$ such that $\max_{\zeta \in \mathrm{Fr}\,(W)} |w(\zeta) - v(\zeta)| < \varepsilon$, we have that (v, W) is admissible and that the homotopy $h : M \times [0, 1] \to \mathbb{R}^m$ given by*

$$h(\zeta, \lambda) = \lambda w(\zeta) + (1 - \lambda)v(\zeta)$$

is admissible in W. Hence, by the Homotopy Invariance Property,

$$\deg(w, W) = \deg(v, W).$$

The Excision Property allows the introduction of the notion of index of an isolated zero of a tangent vector field. Let $\zeta \in M$ be an isolated zero of w. Clearly, $\deg(w, V)$ is well defined for each open $V \subseteq M$ such that $V \cap w^{-1}(0) = \{\zeta\}$. By the Excision Property $\deg(w, V)$ is constant with respect to such V's. This common value of $\deg(w, V)$ is, by definition, the *index of w at ζ*, and is denoted by $\mathrm{i}\,(w, \zeta)$. Using this notation, if (w, W) is admissible, by the Additivity Property we get that if all the zeros in W of w are isolated, then

$$\deg(w, W) = \sum_{\zeta \in w^{-1}(0) \cap W} \mathrm{i}\,(w, \zeta). \tag{2.2}$$

By formula (2.1) we have that if ζ is a nondegenerate zero of w, then

$$\mathrm{i}\,(w, \zeta) = \operatorname{sign}\det dw_\zeta. \tag{2.3}$$

Notice that (2.1) and (2.2) differ in the fact that, in the latter, the zeros of w are not necessarily nondegenerate as they have to be in the former. In fact, in (2.2), w need not be differentiable at its zeros.

3. Tangent vector fields on implicitly defined manifolds

Let $\Psi : \mathbb{R} \times M \to \mathbb{R}^m$ be a (time-dependent) tangent vector field on $M \subseteq \mathbb{R}^m$, that is a continuous map with the property that $\Psi(t, \zeta) \in T_\zeta M$ for each $(t, \zeta) \in \mathbb{R} \times M$. Assume that there is a connected open subset U of $\mathbb{R}^m$ and a C^2 map $g : U \to \mathbb{R}^s$ with the property that $M = g^{-1}(0)$. Suppose that up to change of coordinates, writing $\mathbb{R}^m = \mathbb{R}^{m-s} \times \mathbb{R}^s$, one has that the partial derivative of g with respect to the second variable, $\partial_2 g(x, y)$, is invertible for each $(x, y) \in U$.

According to the above decomposition of $\mathbb{R}^m$ we can write, for any $\zeta \in \mathbb{R}^m$, $\zeta = (x, y)$ and, for any $t \in \mathbb{R}$

$$\Psi(t, \zeta) = \Psi(t, x, y) = \big(\Psi_1(t, x, y), \Psi_2(t, x, y)\big).$$

Notice that one must have

$$\Psi_2(t,x,y) = -(\partial_2 g(x,y))^{-1}\partial_1 g(x,y)\Psi_1(t,x,y). \tag{3.1}$$

In fact, $\Psi(t,\zeta) \in T_\zeta M$ being equivalent to $\Psi(t,\zeta) \in \ker g'(x,y)$, one has for each $(t,x,y) \in \mathbb{R}\times M$ that

$$0 = g'(x,y)\Psi(t,x,y) = \partial_1 g(x,y)\Psi_1(t,x,y) + \partial_2 g(x,y)\Psi_2(t,x,y),$$

which implies (3.1); here $g'(x,y)$ denotes the Fréchet derivative of g at (x,y).

Let us now consider the following differential equation on M:

$$\dot{\xi} = \Psi(t,\xi). \tag{3.2}$$

This, by setting $\xi = (x,y)$, can conveniently be written as the following system:

$$\begin{cases} \dot{x} = \Psi_1(t,x,y), \\ \dot{y} = \Psi_2(t,x,y). \end{cases} \tag{3.3}$$

We claim that (3.3) is equivalent to the following Differential-Algebraic equation:

$$\begin{cases} \dot{x} = \Psi_1(t,x,y), \\ g(x,y) = 0. \end{cases} \tag{3.4}$$

In fact, if $x : J \to \mathbb{R}^{m-s}$ and $y : J \to \mathbb{R}^s$ are C^1 maps defined on an interval J with the property that $t \mapsto \big(x(t),y(t)\big)$ is a solution of (3.3), then one has $\big(x(t),y(t)\big) \in M$ for all $t \in J$. Thus $g\big(x(t),y(t)\big) = 0$ for all $t \in J$. Conversely, if $t \mapsto \big(x(t),y(t)\big)$ is a solution of (3.4) then, differentiating $g\big(x(t),y(t)\big) = 0$ at any $t \in J$, one gets

$$\partial_1 g\big(x(t),y(t)\big)\dot{x}(t) + \partial_2 g\big(x(t),y(t)\big)\dot{y}(t) = 0.$$

So that

$$\begin{aligned} \dot{y}(t) = -(\partial_2 g\big(x(t),y(t)\big))^{-1}&\partial_1 g\big(x(t),y(t)\big)\dot{x}(t) \\ &= -(\partial_2 g\big(x(t),y(t)\big))^{-1}\partial_1 g\big(x(t),y(t)\big)\Psi_1\big(t,x(t),y(t)\big), \end{aligned}$$

which, by (3.1), implies the claim.

4. Computation of the degree

As in the previous section, let $M \subseteq \mathbb{R}^m$ be a differentiable manifold that is globally defined as a zero set of a suitable map $g : U \to \mathbb{R}^s$, $U \subseteq \mathbb{R}^m$. Here we give a formula for the degree of tangents vector fields on M in terms of (potentially easier to compute) degree of appropriate vector fields on U. The main result of this section is Theorem 4.2 below, which is equivalent to a result of [13]. Here we provide a simplified proof.

Throughout this section $\varphi : M \to \mathbb{R}^m$ will be a tangent vector field on M.

Remark 4.1. *Since $M = g^{-1}(0)$ is a closed subset of the metric space U, the well known Tietze's Theorem (see e.g.* [2]*) implies that there exists an extension $\widetilde{\varphi} : U \to \mathbb{R}^m$ of φ.*

Remark 4.1 shows that it is not restrictive to assume, as we sometimes do, that the given tangent vector fields are actually defined on a convenient neighborhood of the manifold M. In fact, although an arbitrary extension of φ may have many zeros outside M, we are interested in the degree of φ on M which only takes into account those zeros of φ that lie on M.

Theorem 4.2. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, let $g : U \to \mathbb{R}^s$ be a smooth function such that $\partial_2 g(x,y)$ is nonsingular for any $(x,y) \in U$ and let $M = g^{-1}(0)$. Assume that $\varphi : M \to \mathbb{R}^k \times \mathbb{R}^s$ is tangent to M and let $\widetilde{\varphi}_1$ be the projection on $\mathbb{R}^k$ of an arbitrary continuous extension $\widetilde{\varphi}$ of φ to U. Define $\mathcal{F} : U \to \mathbb{R}^k \times \mathbb{R}^s$ by*

$\mathcal{F}(x,y) = \big(\widetilde\varphi_1(x,y), g(x,y)\big)$. Then, $\mathcal{F}$ is admissible in U if and only if so is φ in M, and

$$\deg(\varphi, M) = \mathfrak{s}\deg(\mathcal{F}, U), \tag{4.1}$$

where $\mathfrak{s}$ is the constant sign of $\det\partial_2 g(x,y)$ for all $(x,y)\in U$.

Before we provide the proof of Theorem 4.2, we consider a special case. Observe that a point $(p,q)\in M$ is a zero of φ if and only if it is a zero of $\mathcal{F}$.

Lemma 4.3. *Let U, $\mathfrak{s}$, φ be as in Theorem 4.2. Assume that φ is C^1 and let $\widetilde\varphi = \big(\widetilde\varphi_1, \widetilde\varphi_2\big)$ be a C^1 extension of φ to U. Let $\mathcal{F}: U\to\mathbb{R}^k\times\mathbb{R}^s$ be given by $\mathcal{F}(x,y) = \big(\widetilde\varphi_1(x,y), g(x,y)\big)$, as in Theorem 4.2, and suppose that all the zeros of $\mathcal{F}$ are nondegenerate. Then,*

$$\deg(\varphi, M) = \mathfrak{s}\deg(\mathcal{F}, U).$$

Proof. Observe that since the zeros of $\mathcal{F}$ are nondegenerate, they are also isolated. This implies that the zeros of φ are isolated as well. Let (p,q) be a zero of $\mathcal{F}$. As a first step, we will show that

$$\mathrm{i}\,\big(\varphi,(p,q)\big) = \mathfrak{s}\,\mathrm{sign}\det d\mathcal{F}_{(p,q)}. \tag{4.2}$$

Since $\det\partial_2 g(p,q)\neq 0$, the so-called generalized Gauss algorithm (see e.g. [7]) yields

$$\begin{aligned}\det d\mathcal{F}_{(p,q)} &= \det\begin{pmatrix}\partial_1\widetilde\varphi_1(p,q) & \partial_2\widetilde\varphi_1(p,q)\\ \partial_1 g(p,q) & \partial_2 g(p,q)\end{pmatrix} =\\ &= \det\partial_2 g(p,q)\cdot\det\Big(\partial_1\widetilde\varphi_1(p,q) - \partial_2\widetilde\varphi_1(p,q)\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)\Big).\end{aligned} \tag{4.3}$$

As remarked above, since (p,q) is a nondegenerate zero of $\mathcal{F}$ then it is also isolated, as a zero, of both $\mathcal{F}$ and of φ on M. Let $B = W\times V$, with $W\subseteq\mathbb{R}^k$ and $V\subseteq\mathbb{R}^s$ open, be an isolating neighborhood of (p,q) in $\mathbb{R}^k\times\mathbb{R}^s$ i.e. B is such that $\mathcal{F}(p,q)\neq(0,0)$ for any $(x,y)\in B\setminus\{(p,q)\}$, and $\varphi(x,y)\neq(0,0)$ for any $(x,y)\in B\cap M\setminus\{(p,q)\}$.

Since $\partial_2 g(p,q)$ is invertible, the implicit function theorem implies that, taking a smaller W if necessary, we can assume that there exists a C^1 function $\gamma: W\to\mathbb{R}^s$ such that $g\big(p,\gamma(p)\big) = 0$ for any $p\in W$ and $\gamma(W)\subseteq V$. Thus the map $G: x\mapsto \big(x,\gamma(x)\big)$ is a diffeomorphism of W onto $B\cap M$ whose inverse is the projection $\pi: B\cap M\to W$ given by $\pi(p,q) = p$.

The property of invariance under diffeomorphisms of the degree of tangent vector fields implies that

$$\deg(\varphi, B\cap M) = \deg\big(\pi\circ\varphi\circ G, B\big).$$

Notice that p is an isolated zero of $\pi\circ\varphi\circ G$. Thus, the above relation becomes

$$\mathrm{i}\,\big(\varphi,(p,q)\big) = \mathrm{i}\,\big(\pi\circ\varphi\circ G, p\big) \tag{4.4}$$

The differential of $\pi\circ\varphi\circ G$ at p is given by

$$\partial_1\varphi_1(p,q) - \partial_2\varphi_1(p,q)\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)$$

(recall that $q=\gamma(p)$), which is equal to

$$\partial_1\widetilde\varphi_1(p,q) - \partial_2\widetilde\varphi_1(p,q)\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q) \tag{4.5}$$

because the differential of φ at (p,q) coincides with the restriction to $T_{(p,q)}M$ of the differential of $\widetilde\varphi$ at the same point. By (4.3) and the fact that (p,q) is a nondegenerate zero of $\mathcal{F}$, it follows that the map in (4.5) is invertible. Therefore, by (4.4) and (2.3), we have

$$\begin{aligned}&\mathrm{i}\,\big(\varphi,(p,q)\big) =\\ &\quad= \mathrm{sign}\det\Big(\partial_1\widetilde\varphi_1(p,q) - \partial_2\widetilde\varphi_1(p,q)\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)\Big).\end{aligned} \tag{4.6}$$

Formula (4.2) follows from (4.3) and (4.6).

To complete the proof, let $(p_1,q_1),\dots,(p_n,q_n)$ be the zeros of $\mathcal{F}$. Since $\mathfrak{s}$ is constant on the connected set U, from (2.2), Lemma 4.3 and (2.1) we have

$$\deg(\varphi,M)=\sum_{i=1}^{n}\mathrm{i}\left(\varphi,(p_i,q_i)\right)=$$
$$=\sum_{i=1}^{n}\mathfrak{s}\,\mathrm{sign}\det d\mathcal{F}_{(p_i,q_i)}=\mathfrak{s}\deg(\mathcal{F},U),$$

that proves the assertion. □

Proof of Theorem 4.2. The assertion that $\mathcal{F}$ is admissible in U if and only if so is φ in M follows from the identity

$$\big\{(p,q)\in M:\varphi(p,q)=0\big\}=\big\{(p,q)\in U:\mathcal{F}(p,q)=0\big\},$$

which can be deduced from the definition of $\mathcal{F}$ and the fact that, according to (3.1), the projection $\varphi_2(p,q)$ of $\varphi(p,q)=\big(\varphi_1(p,q),\varphi(p,q)_2\big)$, at $(p,q)\in M$ onto $\mathbb{R}^s$, is given by

$$-\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)\varphi_1(p,q).$$

Assume now that $\mathcal{F}$ is admissible in U. Let V be an open and bounded subset of U with the property that the closure $\overline{V}$ of V is contained in U and that $\mathcal{F}^{-1}(0,0)\subseteq V$. Clearly, $\varphi^{-1}(0,0)\cap M$ is contained in V as well and, by the excision property of the degree of a vector field, we get

$$\deg(\mathcal{F},U)=\deg(\mathcal{F},V),\qquad \deg(\varphi,M)=\deg(\varphi,V\cap M).$$

Therefore, it is sufficient to prove that

$$\deg(\varphi,V\cap M)=\mathfrak{s}\deg(\mathcal{F},V). \tag{4.7}$$

We shall deduce equation (4.7) from Lemma 4.3 via an approximation procedure. Given $\varepsilon>0$, Sard's Lemma implies that one can find a C^1 map $\mathcal{F}^\varepsilon:U\to\mathbb{R}^k\times\mathbb{R}^s$, $\mathcal{F}^\varepsilon=(\mathcal{F}_1^\varepsilon,\mathcal{F}_2^\varepsilon)$, that has $(0,0)$ as a regular value and such that

$$\max_{(p,q)\in\mathrm{Fr}\,(V)}\big|\mathcal{F}^\varepsilon(p,q)-\mathcal{F}(p,q)\big|<\varepsilon.$$

Define $\widetilde{\psi}^\varepsilon:U\to\mathbb{R}^k\times\mathbb{R}^s$ by

$$\widetilde{\psi}^\varepsilon(p,q)=\left(\mathcal{F}_1^\varepsilon(p,q),-\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)\mathcal{F}_1^\varepsilon(p,q)\right),$$

and denote by ψ^ε the restriction of $\widetilde{\psi}^\varepsilon$ to M. As in Section 3, we see immediately that ψ^ε is a tangent vector field on M. Recalling formula (3.1), one has

$$\sup_{(p,q)\in\mathrm{Fr}\,(V\cap M)}\big|\psi^\varepsilon(p,q)-\varphi(p,q)\big|\le\sup_{(p,q)\in\mathrm{Fr}\,(V\cap M)}\Big|\mathcal{F}_1^\varepsilon(p,q)-\mathcal{F}_1(p,q)\Big|+$$
$$+\sup_{(p,q)\in\mathrm{Fr}\,(V\cap M)}\Big|\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)\big(\mathcal{F}_1^\varepsilon(p,q)-\mathcal{F}_1(p,q)\big)\Big|$$
$$<\varepsilon\left(1+\sup_{(p,q)\in\mathrm{Fr}\,(V\cap M)}\Big\|\big(\partial_2 g(p,q)\big)^{-1}\partial_1 g(p,q)\Big\|\right)$$

where $|\cdot|$ denotes, according to the space where applied, the Euclidean norm in $\mathbb{R}^k$, $\mathbb{R}^s$ or $\mathbb{R}^{k+s}$, and $\|\cdot\|$ denotes the norm of linear operators from $\mathbb{R}^k$ to $\mathbb{R}^s$. Thus, by the continuity of the partial derivatives of g and the compactness of $\overline{V}\cap M$, it follows that one can choose ε so small that

$$\max_{(p,q)\in\mathrm{Fr}\,(V)}\big|\mathcal{F}^\varepsilon(p,q)-\mathcal{F}(p,q)\big|<\min\{|\mathcal{F}(p,q)|:(p,q)\in\mathrm{Fr}\,(V)\},$$

and

$$\max_{(p,q)\in \mathrm{Fr}\,(V\cap M)} \left|\psi^\varepsilon(p,q)-\varphi(p,q)\right| < \min\{|\varphi(p,q)| : (p,q)\in \mathrm{Fr}\,(V\cap M)\}.$$

For such a choice of ε it is easily checked that $\mathcal{F}^\varepsilon$ and ψ^ε are admissibly homotopic to $\mathcal{F}$ on V and to φ on $V\cap M$, respectively (compare Remark 2.1). Thus,

$$\deg(\mathcal{F}^\varepsilon, V) = \deg(\mathcal{F}, V). \tag{4.8}$$

and

$$\deg(\psi^\varepsilon, V\cap M) = \deg(\varphi, V\cap M) \tag{4.9}$$

Observe also that because of the assumptions on g, any zero of $\mathcal{F}^\varepsilon$ is nondegenerate. By Lemma 4.3 it follows that

$$\deg(\psi^\varepsilon, V\cap M) = \mathfrak{s}\deg(\mathcal{F}^\varepsilon, V). \tag{4.10}$$

Now, Equations (4.9), (4.10) and (4.8) imply (4.7). This completes the proof. □

Example 4.4. *Let $k=s=1$, $U=\mathbb{R}^2$ and $g(x,y)=x^3-y^3-3y$. Consider the tangent vector field on $M=g^{-1}(0)$ given by $\varphi(x,y)=\big(x(y^2+1),x^3\big)$. Define $\mathcal{F}:U\to\mathbb{R}^2$ by $\mathcal{F}(x,y)=\big(x(y^2+1),x^3-y^3-3y\big)$. From the above theorem one gets immediately that* $\deg(\varphi,M)=-1\cdot\deg(\mathcal{F},U)=+1$.

Example 4.5. *Let $s=1$, $k=2$, $U=\mathbb{R}^3$ and $g(x_1,x_2,y)=x_1^2-y$, $\varphi(x_1,x_2,y)=(x_1,1+x_2^3,2x_1^2)$. Put $\varphi_1(x_1,x_2,y)=(x_1,1+x_2^3)$ and $\varphi_2(x_1,x_2,y)=2x_1^2$. Define $\mathcal{F}(x_1,x_2,y)=\big(\varphi_1(x_1,x_2,y),g(x_1,x_2,y)\big)=(x_1,1+x_2^3,x_1^2-y)$. The unique zero of $\mathcal{F}$ is $(0,-1,0)$. From the above theorem one gets that* $\deg(f,M)=\frac{\partial g}{\partial y}(x_1,x_2,y)\cdot\deg(\mathcal{F},U)=+1$.

Theorem 4.2 and the Additivity Property can be combined to get a formula for the degree of a tangent vector field tangent valid in a slightly more general situation.

Corollary 4.6. *Let $U\subseteq\mathbb{R}^k\times\mathbb{R}^s$ be open, $g:U\to\mathbb{R}^s$ a smooth function having $0\in\mathbb{R}^s$ as a regular value and let $M=g^{-1}(0)$. Assume $\varphi:M\to\mathbb{R}^k\times\mathbb{R}^s$ is tangent to M and suppose that there are pairwise disjoint open and connected subsets $U_1,\ldots,U_N$ of U such that*

(1) *$\varphi^{-1}(0)$ is compact and contained in $\bigcup_{i=1}^N U_i$;*
(2) *$\partial_2 g(x,y)$ is nonsingular for all $(x,y)\in U_i$, $i=1,\ldots,N$.*

$$\deg(\varphi,M)=\sum_{i=1}^N \mathfrak{s}_i\deg(\mathcal{F},U_i) \tag{4.11}$$

where $\mathcal{F}:U\to\mathbb{R}^k\times\mathbb{R}^s$ is defined as in Theorem 4.2 and $\mathfrak{s}_i$ denotes the constant sign of $\det\partial_2 g(x,y)$ *in U_i, for $i=1,\ldots,N$.*

This corollary can be sometimes used to rule out the existence of tangent vector fields of particular forms. The following example is a simple illustration of this idea:

Example 4.7. *Let ψ, σ, and δ be continuous $\mathbb{R}$-valued functions defined on the unit sphere $S^2\subset\mathbb{R}^3$ centered at the origin. Assume that ψ is nonzero on S^2 and that σ and δ are zero at $(0,0,\pm1)$. Corollary 4.6 implies that the map $\varphi\colon S^2\to\mathbb{R}^3$ given by $\varphi(\xi)=\big(\xi_1+(\xi_1^2+\xi_2^2)\sigma(\xi),-\xi_2+(\xi_1^2+\xi_2^2)\delta(\xi),(\xi_1^2+\xi_2^2)\psi(\xi)\big)$ cannot be a tangent vector field. To see this, assume by contradiction that φ is a tangent vector field on S^2. Let U_1 and U_2 be disjoint neighborhood of $(0,0,1)$ and of $(0,0,-1)$, respectively. Take $k=2$ and $s=1$ in Corollary 4.6 and let $\mathcal{F}$ be as in Theorem 4.2. Using the properties of the degree and taking smaller U_i's if necessary, it is not difficult to show that* $\deg(\mathcal{F},U_1)=-1$ *and* $\deg(\mathcal{F},U_2)=1$. *Formula* (4.11) *yields,*

$$\deg(\varphi,S^2)=\deg(\mathcal{F},U_1)-\deg(\mathcal{F},U_2)=-2,$$

but the Euler-Poincaré Theorem yields $\deg(\varphi, S^2) = \chi(S^2) = 2$. *This contradiction shows that* φ *is not a tangent vector field.*

5. Applications and examples

This sections is devoted to the study of the set of T-periodic solutions of equations (1.2), for $\lambda \geq 0$.

For the sake of simplicity we make some conventions. We will regard every space as its image in the following diagram of natural inclusions

$$\begin{array}{ccc} [0,\infty)\times M & \longrightarrow & [0,\infty)\times C_T(M) \\ \uparrow & & \uparrow \\ M & \longrightarrow & C_T(M) \end{array}$$

In particular, we will identify M with its image in $C_T(M)$ under the embedding which associates to any $\zeta \in M$ the map $\hat{\zeta} \in C_T(M)$ constantly equal to ζ. Moreover we will regard M as the slice $\{0\} \times M \subset [0,\infty) \times M$ and, analogously, $C_T(M)$ as $\{0\} \times C_T(M)$. We point out that the images of the above inclusions are closed.

According to these identifications, if Ω is an open subset of $[0,\infty) \times C_T(M)$, by $\Omega \cap M$ we mean the open subset of M given by all $\zeta \in M$ such that the pair $(0, \hat{\zeta})$ belongs to Ω. If $\mathcal{O}$ is an open subset of $[0,\infty) \times M$, then $\mathcal{O} \cap M$ represents the open set $\big\{\zeta \in M : (0,\zeta) \in \mathcal{O}\big\}$.

Throughout this section U will be an open and connected subset of $\mathbb{R}^k \times \mathbb{R}^s$. We will always assume that $g : U \to \mathbb{R}^s$ is a smooth function such that $\partial_2 g(x,y)$ is nonsingular for any $(x,y) \in U$, and $M = g^{-1}(0)$. It will also be convenient, given a continuous tangent vector field, $f\colon M \to \mathbb{R}^k \times \mathbb{R}^s$, to denote by $\widetilde{f}$ an arbitrary extension of f to U (see Remark 4.1) and to let $\widetilde{f}_1(x,y)$ be the projection of $\widetilde{f}(x,y)$ on $\mathbb{R}^k$ for any $(x,y) \in U$.

Theorem 5.1. *Let* $f : M \to \mathbb{R}^k \times \mathbb{R}^s$, *and* $h : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ *be continuous tangent vector fields, with* h *of a given period* $T > 0$ *in the first variable. Define* $\mathcal{F} : U \to \mathbb{R}^k \times \mathbb{R}^s$ *by* $\mathcal{F}(x,y) = \big(\widetilde{f}_1(x,y), g(x,y)\big)$ *for any* $\xi = (x,y) \in U$. *Given an open set* $\Omega \subseteq [0,\infty) \times C_T(M)$, *let* $\mathcal{O} \subseteq \mathbb{R}^m$ *be open with the property that* $\mathcal{O} \cap M = \Omega \cap M$. *Assume that* $\deg(\mathcal{F}, \mathcal{O})$ *is well defined and nonzero. Then there exists a connected set* Γ *of nontrivial solution pairs for* (1.2) *in* Ω *whose closure in* Ω *meets* $f^{-1}(0) \cap \Omega$ *and is not compact. In particular, if* $\Omega = [0,\infty) \times C_T(M)$, *then* Γ *is unbounded.*

Proof. By Theorem 4.2 we have

$$|\deg(f, \Omega \cap M)| = |\deg(f, \mathcal{O} \cap M)| = |\deg(\mathcal{F}, \mathcal{O})|.$$

Thus, $\deg(f, \Omega \cap M) \neq 0$ and the assertion follows from Theorem 3.3 of [6]. $\square$

Example 5.2. *Let* $s = 2$, $k = 1$, $U = \mathbb{R}^3$ *and consider* $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$g(x,y) = g(x; y_1, y_2) = \big(e^{y_1}\cos(y_2) - x, e^{y_1}\sin(y_2) + x - 1\big).$$

where $y = (y_1, y_2)$. *Clearly, although for each* $(x,y) \in \mathbb{R} \times \mathbb{R}^2$

$$\det \partial_2 g(x,y) = \det \begin{pmatrix} e^{y_1}\cos(y_2) & e^{y_1}\sin(y_2) \\ -e^{y_1}\sin(y_2) & e^{y_1}\cos(y_2) \end{pmatrix} = e^{2y_1} > 0,$$

$M = g^{-1}(0)$ *is not the graph of a map* $x \mapsto y(x)$.

Consider the following ODE on M:

$$\dot{\xi} = f(\xi),$$

where $\xi = (x, y_1, y_2)$ *and* f *is the tangent vector field given by*

$$f(x, y_1, y_2) = \big(y_2, y_2(\cos y_2 + \sin y_2)e^{-y_1}, -y_2(\cos y_2 - \sin y_2)e^{-y_1}\big).$$

Define $\mathcal{F}(x,y_1,y_2) = \big(y_2, e^{y_1}\cos(y_2) - x, e^{y_1}\sin(y_2) + x - 1\big)$*, for* $(x,y_1,y_2) \in \mathbb{R}^3$*. From Theorem 4.2 we get* $\deg(f,M) = \deg(\mathcal{F},\mathbb{R}^3) = 1$*.*

Clearly $f^{-1}(0) = \{(1,0,0)\}$*. Thus, letting* $\Omega = [0,\infty) \times C_T(M)$ *in Theorem 5.1, one has that given any* T*-periodic vector field* $h : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ *tangent to* M *there exists an unbounded connected set* Γ *of nontrivial solution pairs of equation*

$$\dot{\xi} = f(\xi) + \lambda h(t,\xi), \qquad \lambda \geq 0,$$

whose closure in $[0,\infty) \times C_T(M)$ *meets* $\{(0,\hat{\zeta})\}$ *where* $\hat{\zeta} \in C_T(M)$ *is the function constantly equal to* $(1,0,0)$*.*

Let us now consider Equation (1.2b). Let g and h be as above, and suppose that h is T-periodic in the first variable for a given $T > 0$. We want to derive a continuation result for (1.2b), analogous to Theorem 5.1 above, following [1].

For this purpose, define first the vector field w^h on M by

$$w^h(\xi) = \frac{1}{T}\int_0^T h(t,\xi)dt.$$

The following result concerns Equation (1.2b).

Theorem 5.3. *Let* $h : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ *be a continuous tangent vector field, of a given period* $T > 0$ *in the first variable. Define* $\Phi : U \to \mathbb{R}^m$ *by* $\Phi(x,y) = \big(\widetilde{w}_1^h(x,y), g(x,y)\big)$ *for any* $\xi = (x,y) \in U$*. Given an open set* $\Omega \subseteq [0,\infty) \times C_T(U)$*, let* $\mathcal{O} \subset \mathbb{R}^m$ *be an open subset with the property that* $\Omega \cap M = \mathcal{O} \cap M$*. Assume that* $\deg(\Phi,\mathcal{O})$ *is well defined and nonzero. Then there exists a connected set* Γ *of nontrivial solution pairs in* Ω *whose closure in* Ω *is not compact and meets the set* $(w^h)^{-1}(0) \cap \Omega$*. In particular, if* $\Omega = [0,\infty) \times C_T(M)$*, then* Γ *is unbounded.*

Proof. By Theorem 4.2 we have

$$|\deg(w^h,\Omega \cap M)| = |\deg(w^h,\mathcal{O} \cap M)| = |\deg(\Phi,\mathcal{O})|.$$

Thus, $\deg(w^h,\Omega \cap M) \neq 0$ and the assertion follows from Theorem 2.2 of [3]. □

Example 5.4. *Let* $k = s = 1$ *and let* $U = \mathbb{R}^2$*. Consider the map*

$$g(x,y) = y^3 + y - x^2.$$

Clearly, $\partial_2 g(x,y) = 3y^2 + 1 > 0$ *for all* $(x,y) \in \mathbb{R}^2$*. Consider the following ODE on* $M = g^{-1}(0)$*:*

$$\dot{x} = \lambda h(t,x,y) \tag{5.1}$$

where the tangent vector field h *is given by* $h(t,x,y) = \left(x + y + \sin t, \frac{2x(x+y+\sin t)}{3y^2+1}\right)$*.*

Define

$$\Phi(x,y) = (x + y, y^3 + y - x^2).$$

Observe that $\Phi^{-1}(0,0) = \{(0,0)\}$ *and* $\deg(\Phi,\mathbb{R}^2) = 1$*, so that Theorem 5.3 applies yielding the existence of an unbounded branch of solution pairs of* (5.1)*.*

5.1. **Applications to a class of Differential-Algebraic Equations (DAEs).** Let us now consider applications to semi-explicit Differential-Algebraic equations of the form (1.3).

It is convenient to recall a few basic facts about these equations. In particular, as in [1, 13], compare also [10], we will consider the case when $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is open and connected and $g : U \to \mathbb{R}^s$ is C^∞ and such that $\partial_2 g(p,q)$ is invertible for all $(p,q) \in U$. Notice that, in this case, $0 \in \mathbb{R}^s$ is a regular value of g so that $M := g^{-1}(0)$ is a C^∞ submanifold of $\mathbb{R}^m := \mathbb{R}^k \times \mathbb{R}^s$. We will show that there is an equivalence of (1.3a) with (1.2a) and of (1.3b) with (1.2b), respectively, for appropriate tangent vector fields f and h.

Let us consider equations on an open connected set $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ of the following form:

$$\begin{cases} \dot{x} = F(t,x,y), \\ g(x,y) = 0. \end{cases} \tag{5.2}$$

where $F : \mathbb{R} \times U \to \mathbb{R}^k$ is continuous and $g : U \to \mathbb{R}^s$ is C^∞ and such that $\partial_2 g(p,q)$ is invertible for all $(p,q) \in U$.

A solution of (5.2) consists of a pair of functions $x \in C^1(I, \mathbb{R}^k)$ and $y \in C(I, \mathbb{R}^s)$, I an interval, with the property that

$$\begin{cases} \dot{x}(t) = F\big(t, x(t), y(t)\big), \\ g\big(x(t), y(t)\big) = 0, \end{cases}$$

for each $t \in I$. Notice that the assumptions on g and the Implicit Function Theorem imply that y is actually a C^1 function.

It is well known (compare [10, §4.5]) and easy to see that in this situation, equation (5.2) induces a tangent vector field Ψ on M, that is, it gives rise to an ordinary differential equation on $M = g^{-1}(0) \subseteq \mathbb{R}^k \times \mathbb{R}^s$. In fact, one can see that setting

$$\Psi(t;x,y) = \big(F(t,x,y)\,,\, -(\partial_2 g(x,y))^{-1}\partial_1 g(x,y)F(t,x,y)\big)\,,$$

equation (5.2) is equivalent to

$$\dot{\zeta} = \Psi(t,\zeta)$$

on M. Conversely, given a tangent vector field $\Psi : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$, with $\Psi(t,x,y) = \big(\Psi_1(t;x,y), \Psi_2(t;x,y)\big) \in T_{(x,y)}M$ for any $(x,y) \in M$ one has

$$\Psi_2(t;x,y) = -(\partial_2 g(x,y))^{-1}\partial_1 g(x,y)\Psi_1(t;x,y).$$

Consequently (compare [1, 13]), given $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ open and connected, and maps $g : U \to \mathbb{R}^s$, $\gamma : U \to \mathbb{R}^k$ and $\sigma : \mathbb{R} \times U \to \mathbb{R}^k$ such that γ and σ are continuous, and g is C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$, one has that (1.3a) and (1.3b) are equivalent to (1.2a) and (1.2b), respectively, where $f\colon M \to \mathbb{R}^k \times \mathbb{R}^s$ and $h\colon M \to \mathbb{R}^k \times \mathbb{R}^s$ are the tangent vector fields on M given by

$$f(x,y) = \big(\gamma(x,y), -(\partial_2 g(x,y))^{-1}\partial_1 g(x,y)\gamma(x,y)\big), \tag{5.3}$$

and

$$h(t,x,y) = \big(\sigma(t,x,y), -(\partial_2 g(x,y))^{-1}\partial_1 g(x,y)\sigma(t,x,y)\big). \tag{5.4}$$

We are going to use this equivalence to deduce some of the results of [13] and of [1] for equations of the form (1.3a) and (1.3b), respectively. Let us begin with equations of the form (1.3a).

We need to introduce some notation: given $T > 0$ denote by $C_T(U)$ the metric subspace of the Banach space $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ of all the continuous T-periodic functions taking values in U. We say that $(\mu; x, y) \in [0,\infty) \times C_T(U)$ is a *solution pair of* (1.3a) if (x,y) satisfies (1.3a) for $\lambda = \mu$; here the pair (x,y) is thought of as a single element of $C_T(U)$. It is convenient, given any $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s$, to denote by $(\hat{p},\hat{q})$ the map in $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ that is constantly equal to (p,q). A solution pair of the form $(0;\hat{p},\hat{q})$ is called *trivial*.

Let $\mathcal{F} : U \to \mathbb{R}^k \times \mathbb{R}^s$ be given by $\mathcal{F}(p,q) = \big(\gamma(p,q), g(p,q)\big)$. As one immediately checks, $(\hat{p},\hat{q})$ is a constant solution of (1.3a) corresponding to $\lambda = 0$ if and only if $\mathcal{F}(p,q) = (0,0)$. Thus, with this notation, the set of trivial solution pairs can be written as

$$\{(0;\hat{p},\hat{q}) \in [0,\infty) \times C_T(U) : \mathcal{F}(p,q) = (0,0)\}.$$

Given $\Omega \subseteq [0,\infty) \times C_T(U)$, with $U \cap \Omega$ we denote the set of points of U that, regarded as constant functions, lie in Ω. Namely,

$$U \cap \Omega = \{(p,q) \in U : (0; \hat{p}, \hat{q}) \in \Omega\}.$$

We are now ready to state and prove a result concerning the T-periodic solutions of (1.3a).

Theorem 5.5 ([13]). *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected. Let $g : U \to \mathbb{R}^s$, $\gamma : U \to \mathbb{R}^k$, $\sigma : \mathbb{R} \times U \to \mathbb{R}^k$ and $T > 0$ be such that γ and σ are continuous, σ being T-periodic in the first variable, and g is C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$. Let also $\mathcal{F}(p,q) = \big(\gamma(p,q), g(p,q)\big)$. Given $\Omega \subseteq [0,\infty) \times C_T(U)$ open, assume $\deg(\mathcal{F}, U \cap \Omega)$ is well-defined and nonzero. Then, there exists a connected set Γ of nontrivial solution pairs of (1.3a) whose closure in Ω is not compact and meets the set $\{(0,\hat{p},\hat{q}) \in \Omega : \mathcal{F}(p,q) = (0,0)\}$.*

Proof. Let $f\colon M \to \mathbb{R}^k \times \mathbb{R}^s$ and $h\colon M \to \mathbb{R}^k \times \mathbb{R}^s$ be given by (5.3) and (5.4), respectively. Then, as remarked above (1.3a) is equivalent to (1.2a) on $M = g^{-1}(0)$. This equivalence implies that each pair $(\lambda; x, y)$ can be thought as a solution pair of (1.2a) and vice versa. The assertion follows from Theorem 5.1. □

The following example illustrates this result.

Example 5.6. *Consider the second order DAE*

$$\begin{cases} \ddot{x} = -y - \alpha\dot{x} + \lambda\sigma(t,x,\dot{x}), & \lambda \geq 0 \\ y^3 + y - x^5 - x = 0 \end{cases} \tag{5.5}$$

that represents the motion with friction $-\alpha\dot{x}$, $\alpha > 0$, of a unit mass particle constrained to the real axis and attached to the origin with an initially 'stiff' nonlinear spring (such that the displacement x and the reaction force $-y$ are related implicitly by $y^3 + y = x^5 + x$), and acted on by a T-periodic force σ depending on position and velocity. Let us rewrite equivalently (5.5) as a first order DAE of the form (1.3a).

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -y - \alpha x_2 + \lambda\sigma(t; x_1, x_2), & \lambda \geq 0 \\ y^3 + y - x_1^5 - x_1 = 0. \end{cases} \tag{5.6}$$

Take $U = \mathbb{R}^2 \times \mathbb{R}$ and define $\mathcal{F}(x_1, x_2, y) = (x_2, -\alpha x_2 - y, y^3 + y - x_1^5 - x_1)$. Since $\deg(\mathcal{F}, U) = 1$, Theorem 5.5 yields an unbounded connected set Γ of nontrivial solution pairs of (5.6) emanating from the solution constantly equal to $(0,0,0)$. Clearly, each element of Γ corresponds to a nonconstant T-periodic solution of (5.5). In fact, an energy argument shows that (5.5) has only constant periodic solutions for $\lambda = 0$. Thus, Γ has no intersection with the slice $\{0\} \times C_T(U)$.

In a similar way we deduce a continuation result for equation (1.3b) from Theorem 5.3 above. In the following we will say that $(\mu; x, y) \in [0,\infty) \times C_T(U)$ is a *solution pair of* (1.3b) if (x,y) satisfies (1.3b) for $\lambda = \mu$. A solution pair of the form $(0; \hat{p}, \hat{q})$ will be called *trivial*.

Theorem 5.7 ([1]). *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected. Let $g : U \to \mathbb{R}^s$ be C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in U$, and $\sigma : \mathbb{R} \times U \to \mathbb{R}^k$ continuous and T-periodic in the first variable. Let also $\Phi : U \to \mathbb{R}^k \times \mathbb{R}^s$ be given by $\Phi(p,q) = \big(\Sigma(p,q), g(p,q)\big)$, where*

$$\Sigma(p,q) = \frac{1}{T}\int_0^T \sigma(t,p,q)dt.$$

Given $\Omega \subseteq [0,\infty) \times C_T(U)$ open, assume $\deg(\Phi, U \cap \Omega)$ is well-defined and nonzero. Then, there exists a connected set Γ of nontrivial solution pairs of (1.3b) whose closure in Ω is not compact and meets the set $\{(0,\hat{p},\hat{q}) \in \Omega : \Phi(p,q) = (0,0)\}$.

Proof. Let $h\colon M \to \mathbb{R}^k \times \mathbb{R}^s$ be the tangent vector field on M given by (5.4). Then, equation (1.3b) is equivalent to (1.2b) on $M = g^{-1}(0)$, and the assertion follows from Theorem 5.3. □

Example 5.8. *Consider the following DAE in the form* (1.3b) *with* $T = 2\pi$*:*

$$\left\{\begin{array}{ll} \dot x_1 = \lambda(y_2 + \cos t) & \lambda > 0 \\ \dot x_2 = \lambda\big(y_1 - 2\cos^2 t\big) & \\ x_1 - y_1 \cos y_2 = 0 & \\ x_2 - y_1 \sin y_2 = 0 & y_1 > 0 \end{array}\right. \tag{5.7}$$

Here, $s = 2$*,* $k = 2$*,* $U = \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2, y_1 > 0\}$*. Let* $g : U \to \mathbb{R}^2$ *be given by*

$$g(x, y) = g(x_1, x_2; y_1, y_2) = \big(x_1 - y_1 \cos(y_2), x_2 - y_1 \sin(y_2)\big).$$

where $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$*. One has,*

$$\det \partial_2 g(x, y) = \det \begin{pmatrix} \cos(y_2) & y_1 \sin(y_2) \\ \sin(y_2) & -y_1 \cos(y_2) \end{pmatrix} = y_1 > 0.$$

Clearly, the 2*-dimensional manifold* M *cannot be written as the graph of a function* $(x_1, x_2) \mapsto \big(y_1(x_1, x_2), y_2(x_1, x_2)\big)$*. Let* $\Phi : U \to \mathbb{R}^4$ *be given by*

$$\Phi(x, y) = \Phi(x_1, x_2; y_1, y_2) = \big(y_2, y_1 - 1, x_1 - y_1 \cos y_2, x_2 - y_1 \sin y_2\big).$$

A straightforward computation shows that $\Phi^{-1}(0) = (1, 0, 1, 0)$ *and that* $\deg(\Phi, U) = -1$*. Then, Theorem 5.7 applies with* $\Omega = [0, \infty) \times C_T(U)$*, yielding the existence of an unbounded branch of nontrivial solution pairs of* (5.7)*.*

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(Alessandro Calamai) Dipartimento di Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

(Marco Spadini) Dipartimento di Matematica Applicata, Università di Firenze, Via Santa Marta 3, 50139 Firenze, Italy