# Uniqueness of Local Control Sets

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**Abstract**

The local controllability behavior near an equilibrium is discussed. If the Jacobian of the linearized system is hyperbolic, uniqueness of local control sets is established.

## 1 Introduction

Local controllability properties have been studied for a long time in control theory. In this paper we concentrate on controllability properties near an equilibrium point $x_0$ corresponding to a constant control value $u_0$. We assume that the linearized control system given by the Jacobians at $(x_0, u_0)$ is controllable. Thus the nonlinear system is locally controllable near the equilibrium. This also holds if control constraints $u(t) \in U$ are present and $u_0 \in \mathrm{int} U$. Thus the equilibrium is in the interior of a maximal subset of complete controllability, i.e., a control set, which, naturally, depends on the control range. However, it turns out that already in this apparently simple situation the controllability behavior can be very complicated: In Example 2.2, below, the number of control sets near the equilibrium point tends to infinity as the control ranges decrease. The underlying philosophy of the approach taken here is that hyperbolicity assumptions should exclude this and instead yield "simple" behavior, just as in dynamical systems theory hyperbolicity implies structural stability. Here we say that a "simple" behavior

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occurs if there exists a neighborhood $V$ of the equilibrium such that for all control ranges small enough the control set around the equilibrium is the unique one in $V$. The main result of this paper shows that hyperbolicity of the Jacobian with respect to $x$ does, in fact, guarantee locally uniqueness of the subset of complete controllability; due to the local nature of the problem, we have to consider local control sets which are defined as locally maximal subsets of complete controllability.

The analogy to the role of hyperbolicity in dynamical systems can be made more precise, if one considers control systems $\dot{x} = f(x, u)$ as dynamical systems or control flows where the set $\mathcal{U}$ of admissible control functions $u$ is considered as a part of the state and the dynamics on $\mathcal{U}$ are given by the time-shift; compare Colonius/Kliemann [4] for a systematic exposition. Then the control sets are characterized via the maximal limit sets as time tends to infinity (i.e., the topologically transitive subsets of the control flow). Thus our result shows that locally around a hyperbolic equilibrium $x_0$ of the nominal system (i.e., $\dot{x} = f(x, u_0)$) all small control ranges yield a unique maximal limit set. Naturally, our controllability assumption for the linearized system implies that the eigenvalues can be shifted by feedback. In particular, hyperbolicity can be achieved; see Remark 2.1 for a discussion in our context.

A similar relation of controllability to hyperbolicity was observed by Grünvogel [7] in an opposite case: For singular points, i.e., equilibria which remain fixed for all controls, the existence of control sets is connected with the Lyapunov spectrum of the linearized system (which, in this case, is a bilinear control system). Here hyperbolicity excludes the existence of control sets near the equilibrium, which is a one-point control set. The present paper is an analogue of his results in the regular situation. The importance of hyperbolicity assumptions in this context is emphasized by the results in Colonius/Du [1] showing that hyperbolic control sets depend continuously in the Hausdorff metric on parameters. Related work on controllability behavior near equilibria is given for one-dimensional systems in Colonius/Kliemann [3].

Section 2 recalls some basic facts on control sets. For perturbed linear systems, Sections 3 and 4 give conditions which guarantee global uniqueness of control sets. Section 5 uses these results to show uniqueness of local control sets for nonlinear systems near an equilibrium.

**Notation 1.1.** *Besides the function space $L^\infty(\mathbb{R}, \mathbb{R}^d)$ with norm $\|\cdot\|_\infty$, we shall consider the Sobolev space $W^{1,\infty}(\mathbb{R}, \mathbb{R}^d)$ endowed with the norm*

*$\|x\|_{W^{1,\infty}} = \|x\|_\infty + \|\dot{x}\|_\infty$. Moreover, given $T > 0$, we will consider the corresponding (Banach) subspaces of $T$-periodic functions $L^\infty_T(\mathbb{R}^d)$ and $W^{1,\infty}_T(\mathbb{R}^d)$ respectively.*

## 2 Preliminaries

In this section, we introduce some notions and prove preliminary results on control sets.

Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \qquad u \in \mathcal{U}, \tag{1}$$

where $\mathcal{U}$ denotes the set of all piecewise continuous functions taking values in the compact subset $U$ of $\mathbb{R}^m$, and $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is $C^1$. We will endow $\mathcal{U}$ with the topology inherited by the inclusion $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$. By $\mathcal{U}_T$ we will denote the subset of $\mathcal{U}$ consisting of all its $T$-periodic elements. We assume that unique solutions $\varphi(t, x_0, u)$, $t \in \mathbb{R}$, exist for all $x_0 \in \mathbb{R}^d$ and all piecewise continuous controls $u$.

System (1) is locally accessible in $x \in \mathbb{R}^d$ if for all $T > 0$ the positive orbit up to time $T$

$$\mathcal{O}^+_{\leq T}(x) := \{\varphi(t, x, u),\ 0 < t \leq T \text{ and } u \in \mathcal{U}\}$$

and the negative orbit up to time $T$

$$\mathcal{O}^-_{\leq T}(x) := \{\varphi(t, x, u),\ -T \leq t < 0 \text{ and } u \in \mathcal{U}\},$$

have nonvoid interior. It is called locally accessible in a subset $A \subset \mathbb{R}^d$ if it is locally accessible in every $x \in A$.

Local accessibility holds if a rank condition for the Lie algebra generated by the vector fields $f(\cdot, u)$, $u \in U$, holds. In the sequel, we will consider small perturbations of linear controllable systems. Then local accessibility always holds (see Remark 3.1).

We now turn to the main notions discussed in this paper.

**Definition 2.1.** *A subset $D$ of $\mathbb{R}^d$ with nonvoid interior is a control set of (1) if for all $x \in D$ one has*

$$D \subset \operatorname{cl}\left\{\varphi(t, x, u),\ t > 0 \text{ and } u \in \mathcal{U}\right\},$$

*and $D$ is a maximal subset of $\mathbb{R}^d$ with this property.*

Note that this definition does not change if piecewise continuous controls are replaced by locally integrable ones (cp. [4, Section 3.2]). For a point $x \in D$, large excursions may be necessary in order to return to $x$. Hence we refer to control sets also as to *global* control sets. A local version is introduced next.

**Definition 2.2.** *A subset $D$ of $\mathbb{R}^d$ with nonempty interior is a local control set if there exists a neighborhood $V$ of* $\mathrm{cl}\, D$ *such that for each $x, y \in D$ and every $\varepsilon > 0$ there exist $T > 0$ and $u \in \mathcal{U}$ such that*

$$\varphi(t,x,u) \in V \text{ for all } t \in [0,T] \text{ and } \quad d\big(\varphi(T,x,u), y\big) < \varepsilon$$

*and for every $D'$ with $D \subset D' \subset V$ which satisfies this property, one has $D' = D$.*

Thus for local control sets the maximality property of control sets is replaced by a local maximality property. The neighborhood $V$ in the definition above will also be called an *isolating neighborhood* of $D$.

**Lemma 2.1.** *Let $D$ be a local control set of (1), and assume that local accessibility holds in* $\mathrm{cl}\, D$. *Then for every $x_0 \in \mathrm{int} D$ there are $T_0 > 0$ and a $T_0$-periodic control function $u_0 \in \mathcal{U}$ such that $\varphi(\cdot, x_0, u_0)$ is $T_0$-periodic and contained in $D$.*

*Proof.* Let $x_0 \in \mathrm{int} D$. By local accessibility and by boundedness of the control range $U$, there exists $T_- > 0$ such that $\emptyset \neq \mathrm{int}\mathcal{O}^-_{\leq T_-}(x_0) \subset \mathrm{int} D$. Choose $\delta > 0$ and a point $x_1$ such that $B(x_1, \delta) \subset \mathrm{int}\mathcal{O}^-_{\leq T_-}(x_0)$. By approximate controllability in $D$ there exist $u_1 \in \mathcal{U}$ and $T_1 > 0$ such that $\varphi(t, x_0, u_1) \in V$, $0 \leq t \leq T_1$, and $x_2 := \varphi(T_1, x_0, u_1) \in B(x_1, \delta)$. Hence we also find $u_2 \in \mathcal{U}$ and $0 < T_2 < T_-$ such that $\varphi(t, x_2, u_2) \in D$, $0 \leq t \leq T_2$, and $\varphi(T_2, x_2, u_2) = x_0$. Concatenation of $u_1$ and $u_2$ and periodic continuation yields the desired piecewise continuous control $u_0$ with $\varphi(T_1 + T_2, x_0, u_0) = x_0$. By maximality in $V$, this trajectory is contained in $D$. $\square$

**Lemma 2.2.** *Let $D$ be a local control set. Then*
*(i) $D$ is connected;*
*(ii) if local accessibility holds in a neighborhood of* $\mathrm{cl}\,\mathrm{int} D$ *then* $\mathrm{cl} D = \mathrm{cl}\,\mathrm{int} D$.

*Proof.* (i) Assume by contradiction that there are two open subsets $A$, $B \subset \mathbb{R}^d$ such that $A \cap D$ and $B \cap D$ are nonvoid and disjoint and their union is $D$. Since $D$ has nonvoid interior we may assume that there is a point $x \in \operatorname{int}(A \cap D)$. Pick $y \in B \cap D$. Then, within an isolating neighborhood $V$ for $D$, the point $y$ can be steered into every neighborhood of $x$. Hence there are $T > 0$ and $u \in \mathcal{U}$ with $\varphi(T, x, u) \in \operatorname{int}(A \cap D)$. It follows that every point $z = \varphi(t, y, u),\ t \in [0, T]$, is in $D$ contradicting the assumption: In fact, the point $z$ can be steered arbitrarily close to any point in $D$, without leaving $V$. On the other hand let $N$ be a neighborhood of $z$. By continuous dependence on initial values, there is a neighborhood $W$ of $y$ such that $\varphi(t, W, u) \subset N$. By approximate controllability in $D$ every point in $D$ can be steered into $W$ and hence into $N$.

(ii) Assume by contradiction that there exists $x_0 \in D \setminus \operatorname{cl}\operatorname{int} D$. Let $W$ be a neighborhood of $\operatorname{cl}\operatorname{int} D$ where local accessibility holds. Then, by connectedness, there exists $x \in W \cap D \setminus \operatorname{cl}\operatorname{int} D$. Then $x$ can be steered within an isolating neighborhood $V$ into int $D$. Thus there are $T > 0,\ u \in \mathcal{U}$ and an open neighborhood $N \subset V \cap W$ of $x$ with $\varphi(T, N, u) \subset \operatorname{int} D$. For $y = \varphi(T, x, u)$ there are $S > 0$ and $v \in \mathcal{U}$ with $\varphi(S, y, v) \in N$. Because local accessibility holds at $\varphi(S, y, v)$, the sets $\{\varphi(t, x, u),\ 0 < t \leq \tau\},\ \tau > 0$, have nonvoid interiors and, for $\tau$ small enough, they are contained in $N$. Clearly, these sets are contained in int $D$. Since $N$ is an arbitrary neighborhood of $x$, it follows that $x \in$ cl int $D$. This is a contradiction. □

**Proposition 2.1.** *Let $D$ be a local control set, and assume that local accessibility holds in a neighborhood of* $\operatorname{cl}\operatorname{int} D$. *Then for every $x, y \in D$ there exist a control $u \in \mathcal{U}$ and a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to +\infty$ such that*

$$\varphi\big([0, \infty), x, u\big) \subset D \text{ and } \varphi(t_n, x, u) \to y.$$

*Proof.* Suppose first that $y \in \operatorname{int} D$. By boundedness of $U$ there is $T > 0$ with $\operatorname{int} \mathcal{O}^-_{\leq T}(y) \subset D$. Hence one can first steer, within $V$, the system from $x$ into $\operatorname{int} \mathcal{O}^-_{\leq T}(y)$ and then to $y$. This trajectory is contained in $D$ by maximality. For a point $y \in \partial D$ one can find $y_n \in \operatorname{int} D$ with $y_n \to y$, since, by Lemma 2.1 (ii), cl int $D = \operatorname{cl} D$. Then one argues as before. □

An example of a local control set which is not global can be obtained in the following situation: Suppose that $x_0$ is a hyperbolic equilibrium of the uncontrolled system with a homoclinic orbit. Then for small control range one will expect a local control set around $x_0$ which is a proper subset of a (global) control set containing also the homoclinic orbit. An explicit example is the Takens-Bogdanov oscillator discussed below. We also note that in the interior of local control sets exact controllability holds, if the system is locally accessible.

The following example –the Takens-Bogdanov oscillator– illustrates the differences between local and global control sets. Its properties have been discussed by Häckl and Schneider in [8]; see also [4, Section 9.4].

**Example 2.1.** *Consider the second order system*

$$\ddot{x} = \lambda_1 + \lambda_2 x + x^2 + x\dot{x} + u(t),\ u(t) \in \mathcal{U}^\rho := [-\rho, \rho].$$

*The equivalent first order system is*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \lambda_1 + \lambda_2 x + x^2 + xy \end{pmatrix} + u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

*For the parameter values $\lambda_1 = -0.2$ , $\lambda_2 = -1$ the uncontrolled system has a hyperbolic equilibrium $q_0 = (x_0, y_0)$ with a homoclinic orbit. For small $\rho > 0$, one finds around the hyperbolic equilibrium a local control set $D^{loc,\rho}$, which is a proper subset of a global control set $D^\rho$, which also contains the homoclinic orbit $\varphi(\cdot, q, 0)$. Furthermore*

$$\bigcap_{\rho>0} D^{loc,\rho} = \{(x_0, y_0)\} \text{ and } \bigcap_{\rho>0} D^\rho = \{(x_0, y_0)\} \cup \{\varphi(t, q, 0),\ t \in \mathbb{R}\}.$$

The next example shows, as announced in the introduction, the complicated controllability behavior which may occur in the absence of hyperbolicity. It is taken from Colonius/Kliemann [2].

**Example 2.2.** *Consider a system in $\mathbb{R}$ of the form*

$$\dot{x} = f_0(x) - 3u_1 + 6u_2 =: f(x, u),\ x \in \mathbb{R},$$

*Then, as in [2, Example 5.5], a $C^\infty$ vector field $f_0$ can be constructed such that the following holds: For the control range $U^{\frac{1}{N}} = [-\frac{1}{N}, \frac{1}{N}] \times [-\frac{1}{N}, \frac{1}{N}]$ there are at least $\frac{N}{2} + 2$ control sets. For $N \to \infty$ the number of control sets tends to infinity, and they cluster at $x = \pi$. Thus one obtains an ever more complex controllability behavior near the equilibrium as the control range decreases. The system linearized at $(x_0 = \pi,\ u_1^0 = u_2^0 = 0)$ is obviously controllable. However, the Jacobian $A = \frac{\partial f}{\partial x}(x, u)_{|x=\pi, u=0}$ with respect to $x$ vanishes and hence $A$ is not hyperbolic (i.e., no eigenvalue is on the imaginary axis).*

We will show that the kind of degenerate behavior near an equilibrium as discussed above cannot occur if $A$ is hyperbolic. For controllable linearization $(A, B)$ with hyperbolic $A$ we show that there exists a neighborhood of the equilibrium containing a unique local control set provided that the control range is small enough.

**Remark 2.1.** *Controllability of the linearized system implies that the eigenvalues can be arbitrarily shifted by a feedback $F$. In particular, one can obtain hyperbolicity by applying the preliminary feedback $F$ resulting in the system*

$$\dot{x} = f(x, F(x - x_0) + v(t)).$$

*If one keeps track of the original control constraint $u(t) \in U$, one has to require that the new control $v$ is restricted by a state dependent set,*

$$v(t) \in U - F(x - x_0).$$

*Thus the results presented below, in particular, Theorem 5.1, do not apply to this system (observe that also $F(x - x_0) \in U$ must hold).*

# 3 Perturbed Linear Systems

In this section, we analyze the reachability behavior of systems which are nonlinear perturbations of linear control systems. In particular, we provide sufficient conditions for the trajectories to end in the interior of the reachable set.

We consider control processes of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + F(x(t), u(t)),\ u(t) \in U, \tag{3}$$

where $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$, and $F : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is a $C^1$-function with

$$\|\partial_1 F(x, u)\| \le M \qquad \text{and} \qquad \|\partial_2 F(x, u)\| \le M,$$

uniformly for some $M > 0$.

We denote by $\varphi(\cdot, x_0, u)$, $x_0 \in \mathbb{R}^d$, $u \in \mathcal{U}$, the solution of the Cauchy problem

$$\dot{x}(t) = Ax(t) + Bu(t) + F\left(x(t), u(t)\right),\ x(0) = x_0. \tag{4}$$

For $T > 0$ consider the Banach space

$$C([0,T],\mathbb{R}^m) := \{v : [0,T] \to \mathbb{R}^m,\ v \text{ is continuous}\}$$

endowed with the supremum-norm. Let a piecewise continuous control function $u_0$ with $u_0(t) \in \operatorname{int} U$ for all $t \in \mathbb{R}$ be given and define a nonempty open subset of $C([0,T],\mathbb{R}^m)$ by

$$\mathcal{V}(u_0) := \{v \in C([0,T],\mathbb{R}^m),\ u_0(t) + v(t) \in \operatorname{int} U \text{ for all } t \in [0,T]\}.$$

Given $x_0 \in \mathbb{R}^d$ define a $C^1$ map $\Theta : \mathcal{V}(u_0) \to \mathbb{R}^d$ by

$$\Theta(v) = \varphi(T, x_0, u_0 + v).$$

We want to show that, under suitable assumptions on $A$ and $B$ and for $M$ small enough, there exists a neighborhood of $x_1 := \Theta(0)$ which consists of images of $\Theta$. This follows from the rank theorem (see e.g. Sontag [10, Theorem 52]), if $\Theta'(0)$ is surjective.

Define a bounded linear map $\Gamma : C([0,T],\mathbb{R}^m) \to \mathbb{R}^d$ by

$$\Gamma v = \int_0^T e^{(T-s)A} B v(s)\, ds.$$

For a controllable pair $(A, B)$ the map $\Gamma$ is surjective. Since the surjective linear maps form an open subset of the space of continuous linear maps $\mathcal{L}(C([0,T],\mathbb{R}^m),\mathbb{R}^d)$, there exists $r = r(A,B,T) > 0$, depending on $A$, $B$, and $T$, such that every $H \in \mathcal{L}(C([0,T],\mathbb{R}^m),\mathbb{R}^d)$ with $\|H - \Gamma\| \le r$ is surjective.

**Proposition 3.1.** *Assume that the pair $(A, B)$ is controllable and let $T > 0$. Then, there exists a constant $M := M(A,B,T) > 0$, such that for every $C^1$ function $F$ with $\|\partial_1 F(x,u)\| \le M$ and $\|\partial_2 F(x,u)\| \le M$ uniformly, the following holds: For all $x_0 \in \mathbb{R}^d$ and $u_0 \in \operatorname{int}\mathcal{U}$ there exists a neighborhood $V$ of $0$ in $C([0,T],\mathbb{R}^m)$ with $\Theta(0) = \varphi(T,x_0,u_0) \in \operatorname{int}\Theta(V)$.*

*Proof.* By the preceding remarks, we have to prove that

$$\Theta'(0) = \partial_3\varphi(T, x_0, u_0)$$

is surjective. For $v \in C([0,T],\mathbb{R}^m)$ we put

$$\alpha(t) = \partial_3\varphi(t,x_0,u_0)v,\ \beta(t) = \partial_3\psi(t,x_0,u_0)v,$$

where $\psi(t, x_0, u_0)$ is the solution of the unperturbed equation

$$\dot{x} = Ax + Bu_0(t),\ x(0) = x_0.$$

The chain rule implies

$$\alpha(t) = \int_0^t [A\alpha(s) + Bv(s) + \partial_1 F(\varphi(s, x_0, u_0), u_0(s))\alpha(s) \\ + \partial_2 F(\varphi(s, x_0, u_0), u_0(s))v(s)]\, ds \tag{5}$$

and, analogously,

$$\beta(t) = \int_0^t [A\beta(s) + Bv(s)]\ ds. \tag{6}$$

Hence

$$|\alpha(t)| \leq \|v\|_\infty (\|B\| + M) + \int_0^t (M + \|A\|)\ |\alpha(s)|\ ds.$$

Gronwall's inequality yields

$$|\alpha(t)| \leq \|v\|_\infty (\|B\| + M)\, e^{t(\|A\|+M)}. \tag{7}$$

Moreover, (5) and (6) imply for all $t \in [0, T]$

$$|\alpha(t) - \beta(t)| \leq M \left( \|v\|_\infty + \int_0^T |\alpha(s)|\ \ ds \right) + \int_0^t \|A\| \, |\alpha(s) - \beta(s)|\ \ ds. \tag{8}$$

Plugging (7) into (8) we get

$$\begin{aligned} &|\alpha(t) - \beta(t)| \leq M \|v\|_\infty (1 + T\|B\| + TMe^{T(\|A\|+M)}) \\ &+ \int_0^t \|A\| \, |\alpha(s) - \beta(s)|\ \ ds \\ &\leq c(M) \|v\|_\infty + \int_0^t \|A\| \, |\alpha(s) - \beta(s)|\ \ ds, \end{aligned} \tag{9}$$

where

$$c(M) := M(1 + T\|B\| + TMe^{T(\|A\|+M)})$$

Note that the estimate (9) is independent of $u_0$ and $x_0$. Applying Gronwall's inequality we get

$$\sup_{t\in[0,T]} |\alpha(t) - \beta(t)| \leq c(M) \|v\|_\infty \ e^{T\|A\|}.$$

Since $c(M) \to 0$ as $M \to 0^+$, there exists $M = M(A, B, T) > 0$ such that

$$|\alpha(T) - \beta(T)| \le r \, \|v\|_\infty .$$

Recalling $\alpha(T) = \Theta'(0)v$ and $\beta(T) = \Gamma v$, we find

$$\|\Theta'(0) - \Gamma\| \le r$$

independently of $u_0$ and $x_0$. This yields the surjectivity of $\Theta'(0)$ for all $u_0$ and $x_0$. □

Next we show that the constant $M = M(A, B, T)$ of Proposition 3.1 can be chosen independently of $T$ for $T > 1$. We shall abbreviate

$$M_{A,B} = M(A, B, 1). \tag{10}$$

For $u \in C(\mathbb{R}, \mathbb{R}^m)$ and $\tau \in \mathbb{R}$, we put $(\vartheta_\tau u)(t) = u(t + \tau)$, $t \in [0, 1]$. For $x_0 \in \mathbb{R}^d$, $u_0 \in \mathcal{U}$ and $T > 1$, define $\Psi_T : C([0, 1], \mathbb{R}^m) \to \mathbb{R}^d$ by

$$\Psi_T(v) = \varphi(T - 1, \varphi(1, x_0, u_0 + v), \vartheta_1 u_0).$$

We obtain the following corollary.

**Corollary 3.1.** *Assume that the pair $(A, B)$ is controllable and that $F$ is a $C^1$ function with $\|\partial_1 F(x, u)\| \le M_{A,B}$ and $\|\partial_2 F(x, u)\| \le M_{A,B}$ uniformly. Then for all $x_0 \in \mathbb{R}^d$ and $u_0 \in \mathcal{U}$ there exists a neighborhood $V$ of $0 \in C([0, 1], \mathbb{R}^m)$, such that for every $T > 1$ one has $\varphi(T, x_0, u_0) \in \operatorname{int} \Psi_T(V)$.*

*Proof.* Proposition 3.1 implies $\varphi(1, x_0, u_0) \in \operatorname{int}\{\varphi(1, x_0, u_0 + v), v \in V\}$. Then the assertion follows since the solution of a differential equation defines a homeomorphism. □

Observe that under the assumptions of Corollary 3.1 one has

$$\varphi(T, x_0, u_0) \in \operatorname{int}\{\varphi(T, x_0, w),\ w \in \operatorname{int} \mathcal{U}\}.$$

Here the controls $w$ are not necessarily continuous.

**Remark 3.1.** *Consider a nonlinear system given by*

$$\dot{x}(t) = f\big(x(t), u(t)\big),\ u(t) \in \rho U,$$

*where $\rho > 0$ is given and $U \subset \mathbb{R}^m$. Let $x_0 \in \mathbb{R}^d$ be an equilibrium corresponding to $u_0 \in \operatorname{int} U$ such that $f(x_0, u_0) = 0$. Assume that the linearized system*

$$\dot{x}(t) = \partial_1 f(x_0, u_0)x(t) + \partial_2 f(x_0, u_0)u(t)$$

*is controllable. Then there exists a neighborhood $N$ of the equilibrium $x_0$, such that the nonlinear system is locally accessible in $N$. This follows, since the accessibility rank condition which holds by assumption for the linearized system, remains true under small variations of the involved vector fields. Also the local controllability problem around trajectories studied in this section can be analyzed using similar arguments (based on a Lie algebraic criterion). We prefer the functional analytic arguments above, because they fit with the analysis of periodic solutions given in the next section.*

# 4 Global Uniqueness for Perturbations of Linear Systems

In this section we prove a 'global' uniqueness result for control sets under the assumptions that $A$ is hyperbolic, $(A, B)$ is controllable and $F$ has bounded partial derivatives. See [5] for examples where the number of control sets varies dramatically when a 'small' nonzero term is added to a linear control process; cp. also Paice/Wirth [9].

We start with the following result about periodic solutions of linear differential equations.

**Lemma 4.1.** *Let $A$ be hyperbolic. Then there exists a constant $K_A > 0$, depending only on $A$, such that for all $T > 0$ and $y \in L_T^\infty(\mathbb{R}^d)$, the (unique) $T$-periodic solution $\xi$ of*

$$\dot{x} = Ax + y, \tag{11}$$

*satisfies $\|\xi\|_{W^{1,\infty}} < K_A \|y\|_\infty$.*

*Proof.* Since $A$ is hyperbolic, for any $T$-periodic $y$ one finds the unique $T$-periodic solution of (11)

$$\xi(t) = e^{tA}(1 - e^{TA})^{-1} \int_0^T e^{(T-s)A} y(s)\, ds + \int_0^t e^{(t-s)A} y(s)\, ds.$$

It remains to show the boundedness assertion. First we claim that for every $y \in L^\infty(\mathbb{R}^d)$ there exists an essentially bounded solution of (11). In fact, it is readily proven that the following inequalities hold:

$$\left\| e^{tA}(I-P)e^{-sA} \right\| \le K e^{-a(t-s)} \quad \text{for } t \ge s,$$
$$\left\| e^{tA} P e^{-sA} \right\| \le L e^{-b(s-t)} \quad \text{for } s \ge t,$$

where $K, L, a, b$ are positive constants and $P$ is the projection onto the direct sum of all the generalized eigenspaces corresponding to the eigenvalues of $A$ having negative real part. Thus

$$\xi(t) = \int_{-\infty}^{t} e^{tA}(I-P)e^{-sA} y(s)\, ds - \int_{t}^{\infty} e^{tA} P e^{-sA} y(s)\, ds.$$

is an essentially bounded solution. Since 0 is the only essentially bounded solution of $\dot{x} - Ax = 0$, the linear mapping $\Gamma : W^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \to L^\infty(\mathbb{R}, \mathbb{R}^d)$ which takes $x$ into $\dot{x} - Ax$ is injective. It is obviously continuous and, by the claim, also surjective. Hence, by the open mapping theorem, $K_A := \|\Gamma^{-1}\| < +\infty$; i.e., for every essentially bounded $y$, the solution $\xi$ of $\dot{x} = Ax + y$ satisfies $\|\xi\|_{W^{1,\infty}} \le K_A \|y\|_\infty$. □

**Remark 4.1.** *One could prove Lemma 4.1 following with only minor changes the proof of Theorem 3.1 in [5]. However, due to is greater simplicity and better insight into the problem, we prefer, as suggested by Prof. M. Furi (Florence), the arguments presented above which were inspired by Coppel [6].*

**Corollary 4.1.** *Let $A$ be hyperbolic and $c$ be a positive given number. For any given $T$-periodic function $y \in L^\infty_T(\mathbb{R}^d)$, let $\xi$ denote the unique $T$-periodic solution of*

$$\dot{\xi} = cA\xi + y. \tag{12}$$

*Then, it holds $\|\xi\|_\infty \le \frac{K_A}{c}\|y\|_\infty$, where $K_A$ is the constant –depending only on $A$– given in Lemma 4.1.*

*Proof.* Denote by $x$ the unique $cT$-periodic solution of the following equation

$$\dot{x}(t) = Ax(t) + \frac{1}{c} y\left(\frac{t}{c}\right).$$

Then, $\xi(t) := x(ct)$ is the (obviously unique) $T$-periodic solution to (12). Since by Lemma 4.1, $\|x\|_{W^{1,\infty}} \le \frac{K_A}{c}\|y\|_\infty$, one has

$$\|\xi\|_\infty = \|x\|_\infty \le \|x\|_{W^{1,\infty}} \le \frac{K_A}{c}\|y\|_\infty$$

as claimed. ∎

A crucial step towards uniqueness of control sets is the following result:

**Lemma 4.2.** *For system (3) there exists a constant $K_A > 0$, depending only on $A$, such that the following holds. Assume*

$$\|\partial_1 F(x,u)\| \leq \min\left\{1, \frac{1}{2K_A}\right\}, \text{ uniformly.} \tag{13}$$

*Then for every $T > 0$ equation (3) has a unique $T$-periodic solution $x(\cdot, u)$ for $u \in \mathcal{U}_T$, and the map $\mathcal{U}_T \to W_T^{1,\infty}(\mathbb{R}^d)$ given by $u \mapsto x(\cdot, u)$ is continuous. If, additionally, $U$ contains the origin of $\mathbb{R}^m$ in its interior, one has for every $u \in \mathcal{U}_T$*

$$\sup_{t\in[0,T]} |x(t,u)| \leq$$
$$2K_A \left[ c_U \left( \|B\| + \sup_{(v,p)\in U\times\mathbb{R}^d} \|\partial_2 F(p,v)\| \right) + |F(0,0)| \right], \tag{14}$$

*where $c_U := \max\{|v| : v \in U\}$.*

*Proof.* We write the $T$-periodic problem for (3) in the form:

$$Lx - \bar{A}x - \bar{B}u - \bar{F}(x,u) = 0 \tag{15}$$

where we put

$$\begin{array}{llll}
L : W_T^{1,\infty}(\mathbb{R}^d) \to L_T^\infty(\mathbb{R}^d) & \text{with} & (Lx)(t) = \dot{x}(t), \\
\bar{A} : W_T^{1,\infty}(\mathbb{R}^d) \to L_T^\infty(\mathbb{R}^d) & \text{with} & (\bar{A}x)(t) = Ax(t), \\
\bar{B} : L_T^\infty(\mathbb{R}^m) \to L_T^\infty(\mathbb{R}^d) & \text{with} & (\bar{B}u)(t) = Bu(t), \\
\bar{F} : W_T^{1,\infty}(\mathbb{R}^d) \times \mathcal{U}_T \to L_T^\infty(\mathbb{R}^d) & \text{with} & \bar{F}\big(x,u\big)(t) = F\big(x(t),u(t)\big).
\end{array}$$

By Lemma 4.1, $\left(L - \bar{A}\right) x = y$ implies $\|x\|_{W^{1,\infty}} \leq K_A \|y\|_\infty$ and hence

$$\left\| \left(L - \bar{A}\right)^{-1} \right\| \leq K_A.$$

Let $\Phi : \mathcal{U}_T \times W_T^{1,\infty}(\mathbb{R}^d) \to W_T^{1,\infty}(\mathbb{R}^d)$, be given by

$$\Phi(u,x) = \left(L - \bar{A}\right)^{-1} \left(\bar{B}u + \bar{F}(x,u)\right).$$

Then equation (15) is equivalent to

$$\Phi(u,x) = x. \tag{16}$$

Let us show that equation (16) admits exactly one solution for every $u \in \mathcal{U}_T$. Since

$$\begin{aligned}
\|\Phi(u,x_1) - \Phi(u,x_2)\|_{W^{1,\infty}} &\leq \left\|\left(L-\bar{A}\right)^{-1}\right\| \left\|\bar{F}(x_1,u) - \bar{F}(x_2,u)\right\|_\infty \\
&\leq K_A \sup_{(p,v)\in\mathbb{R}^d\times U} \|\partial_1 F(p,v)\| \, \|x_1 - x_2\|_{W^{1,\infty}} \\
&\leq \frac{1}{2}\|x_1 - x_2\|_{W^{1,\infty}}
\end{aligned}$$

for every $u \in \mathcal{U}_T$, the map $\Phi(u,\cdot)$ is a contraction. Then the Banach contraction theorem yields the existence of a unique fixed point which we denote by $x(\cdot,u)$. Furthermore, for fixed $T > 0$, the solution $x(\cdot,u)$ depends continuously on $u \in \mathcal{U}_T$ (see e.g. [11, Proposition 1.2]). To prove the last assertion, notice that for a fixed point $x$ of $\Phi(u,\cdot)$ one has

$$\begin{aligned}
\|x\|_{W^{1,\infty}} &\leq \|\Phi(u,x) - \Phi(u,0)\|_{W^{1,\infty}} \\
&\quad + \|\Phi(u,0) - \Phi(0,0)\|_{W^{1,\infty}} + \|\Phi(0,0)\|_{W^{1,\infty}} \leq \frac{1}{2}\|x\|_{W^{1,\infty}} \\
&\quad + \left\|(L-\bar{A})^{-1}\right\| \left(c_U \|B\| + \left\|\bar{F}(0,u) - \bar{F}(0,0)\right\|_\infty + \|\bar{F}(0,0)\|_{W^{1,\infty}}\right) \\
&\leq \frac{1}{2}\|x\|_{W^{1,\infty}} + K_A\left[c_U\left(\|B\| + \sup_{(p,v)\in\mathbb{R}^d\times U}\|\partial_2 F(p,v)\|\right) + |F(0,0)|\right]
\end{aligned}$$

This implies inequality (14). ☐

Define (recall (10))

$$M^{\#}_{A,B} = \min\left\{1,\ \frac{1}{2K_A},\ M_{A,B}\right\}. \tag{17}$$

The lemma above yields a bound on the control sets.

**Corollary 4.2.** *Let $A$, $B$ and $F$ be as in Lemma 4.2, assume that $U$ contains the origin of $\mathbb{R}^m$ in its interior, and that*

$$\|\partial_1 F(x,u)\| \leq M^{\#}_{A,B}, \quad \text{and} \quad \|\partial_2 F(x,u)\| \leq M^{\#}_{A,B},$$

*for all $(x,u) \in \mathbb{R}^d \times \mathcal{U}$. Then for every control set of (3) its interior is contained in the ball of $\mathbb{R}^d$ centered at the origin and having radius*

$$2K_A \left[ c_U \big( \|B\| + M^{\#}_{A,B} \big) + |F(0,0)| \right].$$

*This ball contains all control sets if local accessibility holds in a neighborhood of its closure.*

*Proof.* Assume that there exist a point $p$ outside the $2K_A\big[c_U\big(\|B\|+M^{\#}_{A,B}\big)+|F(0,0)|\big]$-ball centered at the origin, but belonging to the interior of a control set. Then, by Lemma 2.1 there exists a periodic solution of (3) whose image contains $p$. This contradicts inequality (14). This shows that the interior of the control sets is contained in the ball. Local accessibility implies by Lemma 2.2 that $\mathrm{cl}\ D = \mathrm{cl}\,\mathrm{int} D$, hence also the last assertion follows. $\square$

**Lemma 4.3.** *Let $U$ have nonempty interior. Assume that $A$ is hyperbolic, that the pair $(A,B)$ is controllable and that $F$ is a $C^1$ map with*

$$\|\partial_1 F(x,u)\| \leq M^{\#}_{A,B}, \quad \text{and} \quad \|\partial_2 F(x,u)\| \leq M^{\#}_{A,B},$$

*for all $(x,u) \in \mathbb{R}^d \times \mathcal{U}$. Then, given $T > 0$ and $u_0 \in \mathrm{int}\,\mathcal{U}_T$, equation (3) has a unique $T$-periodic solution. Furthermore this solution is contained in the interior of a control set of (3).*

*Proof.* Observe that a $T$-periodic function is also $nT$-periodic, $n \in \mathbb{N}$. Hence, without loss of generality, we can assume $T > 1$. Lemma 4.2 yields the existence of a unique $T$-periodic solution of (3) for $u_0 \in \mathrm{int}\,\mathcal{U}_T$. Fix $u_0 \in \mathrm{int}\,\mathcal{U}_T$ and let $x_0$ be the starting point of the unique periodic $T$-periodic solution of (3). From Corollary 3.1 it follows that there exists a neighborhood $V$ of $x_0$ in $\mathbb{R}^d$ such that for any $q \in V$ there exists $w \in \mathrm{int}\,\mathcal{U}_T$ such that $q = \varphi(T, x_0, w)$. Considering the time reversed system and reducing $V$, if necessary, we can assume that every point in $V$ can be steered to every other point of $V$. Hence $V$ is contained in the interior of a control set. Take now any point $q \in \varphi([0,T], x_0, u_0)$ and let $t_0 \in [0,T]$ be such that $q = \varphi(t_0, x_0, u_0)$. By the continuity of $\varphi(t_0, \cdot, u_0)$ there exists a neighborhood $W$ of $q$ such that

$$\varphi(t_0, \cdot, u_0)^{-1}(W) \subset V.$$

Analogously, by the continuity of the time reversed system, shrinking $W$ if necessary, we can assume that

$$\varphi(t_0, W, u_0) \subset V.$$

Hence, every point of $W$ can be driven to every other point of $W$ and hence $W$ is contained in a control set. The assertion now follows from the compactness of $\varphi([0,T],x_0,u_0)$. $\square$

**Remark 4.2.** *Assume, in addition to the hypotheses of Lemma 4.3, that $U$ contains $0$ in its interior and that $F(0,0)=0$. Then the origin of $\mathbb{R}^d$ is contained in the interior of a control set. In fact, the origin can be regarded as a 1-periodic solution of (3).*

We are now in a position to state and prove the main result of this section.

**Theorem 4.1.** *Let $U$ be compact and convex with nonempty interior. Assume that the pair $(A,B)$ in (3) is controllable and $A$ is hyperbolic. Let $F$ be a $C^1$ function with*

$$\|\partial_1 F(x,u)\| \leq M^{\#}_{A,B}, \quad \textit{and} \quad \|\partial_2 F(x,u)\| \leq M^{\#}_{A,B},$$

*for all $(x,u)\in\mathbb{R}^d\times\mathcal{U}$, and assume that the system (3) is locally accessible. Then it admits exactly one control set $D$. Its interior is contained in the $2K_A\big[c_U\big(\|B\|+M^{\#}_{A,B}\big)+|F(0,0)|\big]$-ball of $\mathbb{R}^d$ centered at the origin. If $F(0,0)=0$, then the origin is an element of the interior of $D$.*

*Proof.* Let $T>1$ and $u_0\in\operatorname{int}\mathcal{U}_T$. Lemma 4.2 guarantees the existence of a $T$-periodic solution of (3), whose image is, by Lemma 4.3, contained in the interior of a control set. This proves the existence of at least one control set. In order to prove the uniqueness assertion consider control sets $D_0$ and $D_1$. Then, by Lemma 2.1, there exists $u_i\in\operatorname{int}\mathcal{U}_{T_i}$, $i\in\{0,1\}$, such that the corresponding $T_i$-periodic trajectory of (3) is contained in the interior of $D_i$. Naturally, we can assume that $T_0$, $T_1>1$. Put $T_\lambda=\lambda T_1+(1-\lambda)T_0$ and define

$$v_\lambda(t)=\lambda u_1(tT_1)+(1-\lambda)u_0(tT_0).$$

These functions are 1-periodic and, since $U$ is assumed convex, $v_\lambda\in\operatorname{int}\mathcal{U}_1$. Consider the differential equation

$$\dot y(\tau)=T_\lambda\left[Ay(\tau)+Bv_\lambda(\tau)+F\big(y(\tau),v_\lambda(\tau)\big)\right]. \tag{18}$$

We claim that, for any $\lambda\in[0,1]$, this equation has a unique 1-periodic solution $y_\lambda$ and that the map $[0,1]\to L_1^\infty(\mathbb{R}^d)$ given by $\lambda\mapsto y_\lambda$ is continuous. To prove the claim we proceed similarly to the first part of the proof to

Lemma 4.2: The existence of 1-periodic solutions to (18) is equivalent to the existence of solutions to the equation

$$Lx - T_\lambda \bar{A}x - T_\lambda \bar{B}u - T_\lambda \bar{F}(x,u) = 0 \tag{19}$$

where $L$, $\bar{A}$, $\bar{B}$ and $\bar{F}$ are as in Lemma 4.2, with $T = 1$. By Lemma 4.1, $L - T_\lambda \bar{A}$ is invertible for any $\lambda \in [0,1]$. Considering $(L - T_\lambda \bar{A})^{-1}$ as a map $L_1^\infty(\mathbb{R}^d) \to L_1^\infty(\mathbb{R}^d)$, Corollary 4.1 yields

$$\left\| \left(L - T_\lambda \bar{A}\right)^{-1} \right\| \le \frac{K_A}{T_\lambda}.$$

Let $\Psi : [0,1] \times L_1^\infty(\mathbb{R}^d) \to L_1^\infty(\mathbb{R}^d)$, be given by

$$\Psi(\lambda, x) = \left(L - T_\lambda \bar{A}\right)^{-1} \left(T_\lambda \bar{B} v_\lambda + T_\lambda \bar{F}(x, v_\lambda)\right).$$

Then equation (19) for $u = v_\lambda$ is equivalent to

$$\Psi(\lambda, x) = x. \tag{20}$$

Notice that any fixed point of $\Psi(\lambda, \cdot)$ actually belongs to $W_1^{1,\infty}(\mathbb{R}^d)$. Let us show that equation (20) admits exactly one solution for every $\lambda \in [0,1]$. Since for every $\lambda \in [0,1]$

$$\begin{aligned} \|\Psi(\lambda, x_1) - \Psi(\lambda, x_2)\|_\infty &\le \left\| \left(L - T_\lambda \bar{A}\right)^{-1} \right\| \; T_\lambda \; \left\| \bar{F}(x_1, v_\lambda) - \bar{F}(x_2, v_\lambda) \right\|_\infty \\ &\le \frac{K_A}{T_\lambda} T_\lambda \sup_{(p,v) \in \mathbb{R}^d \times U} \|\partial_1 F(p,v)\| \, \|x_1 - x_2\|_\infty \\ &\le \frac{1}{2} \|x_1 - x_2\|_\infty , \end{aligned}$$

the map $\Psi(\lambda, \cdot)$ is a contraction uniformly in $\lambda$. Then the claim follows as in Lemma 4.2.

Set $u_\lambda(t) = v_\lambda(t/T_\lambda)$. By Lemma 4.3, the equation

$$\dot{x}(t) = Ax(t) + Bu_\lambda(t) + F(x(t), u_\lambda(t)),$$

admits a unique $T_\lambda$-periodic solution $x_\lambda$, and the image of $x_\lambda$ and hence every $x_\lambda(0)$ is contained in the interior of a control set. By a time transformation, one has $x_\lambda(t) = y_\lambda(t/T_\lambda)$ for all $t$.

By the claim, the map $\lambda \mapsto y_\lambda$ is continuous, so the map $[0,1] \to \mathbb{R}^d$, given by $\lambda \mapsto y_\lambda(0) = x_\lambda(0)$ is continuous as well. Thus $\{x_\lambda(0),\ \lambda \in [0,1]\}$ is connected and therefore contained in the interior of a single control set. It also meets $D_0$ and $D_1$; consequently, $D_0 = D_1$. □

# 5 Uniqueness of Local Control Sets

Consider the following control process:

$$\dot{x}(t) = f(x(t), u(t)),\ u(t) \in \rho U, \tag{21}$$

where $\rho > 0$ is given and $U \subset \mathbb{R}^m$ is compact, convex and contains the origin in its interior. We consider the behavior near an isolated equilibrium $x_0$ of the nonlinear system; more precisely we assume that there exists $x_0 \in \mathbb{R}^d$ such that $f(x_0, 0) = 0$ and that $\partial_1 f(x_0, 0)$ is hyperbolic.

In the next theorem we will find conditions ensuring that there exists $\delta_0 > 0$ such that, for every small control range, the ball $B(p_0, \delta_0)$ contains a unique local control set of (21).

**Theorem 5.1.** *Let $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be $C^1$. Consider an equilibrium $x_0 \in \mathbb{R}^d$ such that $f(x_0, 0) = 0$ and assume that the pair $\big(\partial_1 f(x_0, 0), \partial_2 f(x_0, 0)\big)$ is controllable and the operator $\partial_1 f(x_0, 0)$ is hyperbolic.*

*Then there exist $\rho_0 > 0$ and $\delta_0 > 0$ such that, for all $0 < \rho < \rho_0$, the ball $B(x_0, \delta_0)$ contains exactly one local control set $D^\rho$ for (21).*

*Proof.* Without loss of generality we can assume $x_0 = 0$. By Remark 3.1 we can choose $\delta_0$ small enough such that local accessibility holds in the $2\delta_0$-ball around the origin. The proof proceeds via a cutting-off technique. For any $\delta > 0$, let $\sigma_\delta : [0, \infty) \to \mathbb{R}$ be a $C^1$ function such that for $r \in [0, \infty)$

$$\sigma_\delta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \delta, \\ 0 & \text{if } r \ge 2\delta, \end{cases}$$

and $0 \le \sigma_\delta(r) \le 1,\ |\sigma'_\delta(r)| \le \frac{1}{\delta}$. For instance, we can take

$$\sigma_\delta(r) = \begin{cases} 0 & \text{if } r \ge 2\delta, \\ \dfrac{1 + \cos\left(\frac{\pi}{\delta}(r - \delta)\right)}{2} & \text{if } r \in [\delta, 2\delta], \\ 1 & \text{if } r \in [0, \delta]. \end{cases}$$

Then, putting

$$A = \partial_1 f(0,0),\ B = \partial_2 f(0,0),\ F_\delta(p, v) = \sigma_\delta\left(|p|^2 + |v|^2\right)(f(p, v) - Ap - Bv),$$

one has that

$$
\begin{aligned}
&[\partial_1 F_\delta(p,v)]\,\eta \\
&= \begin{cases}
0 & \text{if } |p|^2+|v|^2 \geq 2\delta \\
\begin{aligned} &2\langle p,\eta\rangle \sigma_\delta'\left(|p|^2+|v|^2\right)\Big(f(p,v)-Ap-Bv\Big) \\ &\qquad +\sigma_\delta\left(|p|^2+|v|^2\right)\Big(\partial_1 f(p,v)\eta - A\eta\Big)\end{aligned} & \text{if } \delta \leq |p|^2+|v|^2 \leq 2\delta \\
\partial_1 f(p,v)\eta - A\eta & \text{if } 0 \leq |p|^2+|v|^2 \leq \delta.
\end{cases}
\end{aligned}
$$

An analogous formula holds for $\partial_2 F_\delta(p,v)$. Hence, using the fact that $f$ is $C^1$ and the definitions of $A$ and $B$, one has $\|\partial_1 F_\delta(p,v)\| \to 0$ and $\|\partial_2 F_\delta(p,v)\| \to 0$ as $\delta \to 0$. Therefore, taking $\delta_1$ small enough, we can assume

$$\|\partial_1 F_{\delta_1}(p,v)\| \leq M^{\#}_{A,B} \text{ and } \|\partial_2 F_{\delta_1}(p,v)\| \leq M^{\#}_{A,B},$$

where $M^{\#}_{A,B}$ is as in (17). Consider now the control process

$$\dot{x}(t) = Ax(t) + Bu(t) + F_{\delta_1}\big(x(t),u(t)\big),\ u(t) \in \rho U. \tag{22}$$

Put $\delta_0 = \delta_1/\sqrt{2}$ and notice that (22) coincides with (21) when $(x,u) \in B(0,\delta_0)\times B(0,\delta_0)$. By assumption, $(A,B)$ is controllable and $A$ is hyperbolic. From Theorem 4.1 it follows that (22) has a unique control set $D^\rho$ and that it is contained in the ball of radius $2\rho c_U K_A\left(\|B\| + M^{\#}_{A,B}\right)$ centered at the origin of $\mathbb{R}^d$; here $c_U = \sup\{|v| : v \in U\}$. Hence, taking

$$\rho_0 \leq \min\left\{\frac{\delta_0}{c_U},\ \frac{\delta_0}{2c_U K_A\Big(\|B\| + M^{\#}_{A,B}\Big)}\right\},$$

one has that $D^\rho$ is contained in $B(0,\delta_0)$ when $\rho \leq \rho_0$. Since (21) and (22) coincide in $B(0,\delta_0)$, $D^\rho$ is a local control set for (21). In fact, from the uniqueness of the control set of (22) it follows that only one local control set of (21) can be contained in $B(0,\delta_0)$. □

Finally, we discuss the implications of this result for bifurcation questions. Consider a parameter-dependent family of control systems

$$\dot{x}(t) = f(x(t),u(t),\alpha),\ u(t) \in \rho U, \tag{23}$$

where $\alpha \in \mathbb{R}$, $\rho > 0$ and $U \subset \mathbb{R}^m$ is bounded, convex and contains the origin in its interior. We consider the behavior near an equilibrium of the uncontrolled system with $\alpha = \alpha_0$ and show that under the assumptions of Theorem 5.1 no "bifurcation" of local control sets can occur.

**Theorem 5.2.** *Let $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^d$ be a continuous map which is $C^1$ with respect to the first two variables. Consider a continuous family of equilibria $x_\alpha \in \mathbb{R}^d$ such that $f(x_\alpha, 0, \alpha) = 0$ and assume that for $\alpha = \alpha_0$ the pair $\big(\partial_1 f(x_{\alpha_0}, 0, \alpha_0), \partial_2 f(x_{\alpha_0}, 0, \alpha_0)\big)$ is controllable and the operator $\partial_1 f(x_{\alpha_0}, 0, \alpha_0)$ is hyperbolic.*

*Then there exist $\varepsilon_0 > 0$, $\rho_0 > 0$ and $\delta_0 > 0$ such that, for all $|\alpha - \alpha_0| < \varepsilon_0$ and all $0 < \rho < \rho_0$, the ball $B(x_{\alpha_0}, \delta_0)$ contains exactly one local control set for (23) with parameter value $\alpha$.*

*Proof.* The assumptions on $f$ in Theorem 5.1 are satisfied for all $\alpha$ near $\alpha_0$. Hence the assertion of Theorem 5.1 holds for all $|\alpha - \alpha_0| < \varepsilon_0$ and all $0 < \rho < \rho_0$ in a ball $B(x_{\alpha_0}, \delta_0)$. $\square$

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