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An introduction to topological degree in Euclidean spaces

Pierluigi Benevieri, Massimo Furi,
Maria Patrizia Pera and Marco Spadini

1 Introduction

This paper aims to provide a careful and self-contained introduction to the theory of topological degree in Euclidean spaces. It is intended for people mostly interested in analysis and, in general, a heavy background in algebraic or differential topology is not required.

Roughly speaking, our construction of the topological degree can be summarized in a few steps. We first define a notion of degree for the special case of *regular triples* that is for triples (f, U, y) where f is an $\mathbb{R}^k$-valued smooth function defined (at least) on the closure $\overline{U}$ of the open set $U \subseteq \mathbb{R}^k$ and proper on $\overline{U}$, and $y \in \mathbb{R}^k$ is a regular value for f in U. We then proceed to the definition of degree in the general case of *admissible triples* when f is assumed only continuous and proper on $\overline{U}$, and y is any point in $\mathbb{R}^k \setminus f(\partial U)$. Lastly, we consider the so-called extended case of the *weakly admissible triples*, that is when f is defined (and continuous) at least on U and $y \in \mathbb{R}^k$ is such that $f^{-1}(y) \cap U$ is compact.

Our approach emphasizes the importance of three fundamental properties of topological degree: Normalization, Additivity, and Homotopy Invariance (see below). Actually, these properties determine the notion of degree in a unique way yielding a computation formula for the degree valid for admissible triples (f, U, y) such that f is Fréchet differentiable in any $x \in f^{-1}(y)$. This allows an alternative approach.

This paper is organized as follows: Section 2 gathers some results and notions needed for the following sections. In Section 3 the notion of degree both for regular triples and for admissible triples is defined, and the main consequences of the three above mentioned fundamental properties are explored. Section 4 is devoted to the notion of degree for weakly admissible triples, while Section 5 contains the (lengthy) proof of the Homotopy Invariance Property for regular triples.

A word of caution: unless differently stated, *all the maps considered in this paper are continuous.* We also recall that we say that a function defined on an arbitrary subset X of $\mathbb{R}^k$ is C^∞ (resp. C^r with $1 \leq r < \infty$), if it admits a C^∞ (resp. C^r) extension to an open neighborhood of X.

2 Preliminaries

Let $f : U \to \mathbb{R}^s$ be a C^1 map defined on an open subset of $\mathbb{R}^k$. An element $x \in U$ is called a *critical point* (*of* f) if the Fréchet derivative $f'(x) \in \mathrm{L}(\mathbb{R}^k, \mathbb{R}^s)$ is not onto; otherwise x is a *regular point.* An element $y \in \mathbb{R}^s$ is a *critical value* if $f^{-1}(y)$ contains critical points; otherwise y is a *regular value.*

To avoid confusion, *points* are in the source space and *values* in the target space.

Observe that if $k < s$, then any $x \in U$ is a critical point. Consequently, $f(U)$ coincides with the set of critical values of f.

A very important special case is when $k = s$. In this context, $x \in U$ is a regular point (of f) if and only if the Jacobian of f at x, $\det(f'(x))$, is nonzero. When this holds, the sign of $\det(f'(x))$ is called the *index of* f *at* x and denoted $\mathrm{i}(f, x)$. Actually, the index $\mathrm{i}(f, x)$ is defined as $\mathrm{sign}(\det(f'(x)))$ even if f is simply continuous, provided it is Fréchet differentiable at x with invertible derivative.

Exercise 2.1. Let p be a complex polynomial and regard p as a map from $\mathbb{R}^2$ into itself. Show that $z \in \mathbb{C} \cong \mathbb{R}^2$ is a critical point of p if and only if it is a root of the polynomial p'. Prove that $\mathrm{i}(p, z) = 1$ for any regular point $z \in \mathbb{C}$.

The following result is of crucial importance in degree theory. (See e.g., [11] or [8].)

Sard's Lemma. *Let $f : U \to \mathbb{R}^s$ be a C^n map defined on an open subset of $\mathbb{R}^k$. If $n > \max\{0, k-s\}$, then the set of critical values of f has (s-dimensional) Lebesgue measure zero. In particular, the set of regular values of f is dense in $\mathbb{R}^s$.*

Observe that, in view of Sard's Lemma, a C^1 curve $\alpha : [a, b] \to \mathbb{R}^s$, $s > 1$, cannot be a Peano curve (i.e. a curve whose image contains interior points).

Definition 2.2. A map $f : X \to Y$ between two metric spaces is *proper* if $f^{-1}(K)$ is compact whenever $K \subseteq Y$ is compact.

Clearly, if X is compact, then f is proper (any map is assumed to be continuous).

Exercise 2.3. Show that if $f : X \to Y$ is proper, then it is a closed map (that is, $f(A)$ is closed whenever $A \subseteq X$ is closed).

Exercise 2.4. Let $X \subseteq \mathbb{R}^k$ be closed and unbounded. Prove that a map $f : X \to \mathbb{R}^s$ is proper if and only if

$$\lim_{x \in X,\ |x| \to +\infty} |f(x)| = +\infty.$$

Example 2.5. Let $p : \mathbb{C} \to \mathbb{C}$ be a non-constant complex polynomial. Then $\lim_{|z| \to +\infty} |p(z)| = +\infty$. Thus, p is a proper map.

In the following we will need to approximate continuous functions with more regular ones. To do that, we shall make use of the following approximation theorem

Smooth Approximation Theorem. *Let $U \subseteq \mathbb{R}^k$ be open, and let f be an $\mathbb{R}^s$-valued (continuous) function defined on the closure $\overline{U}$ of U in $\mathbb{R}^k$. Then, given a continuous function $\varepsilon : \overline{U} \to (0, \infty)$, there exists a C^∞ function $g : \overline{U} \to \mathbb{R}^s$ such that $|f(x) - g(x)| < \varepsilon(x)$ for any $x \in \overline{U}$.*

This fact could be proved directly. However, since any continuous function defined on a closed subset of $\mathbb{R}^k$ with values in $\mathbb{R}^s$ can be extended to a continuous function on $\mathbb{R}^k$ (this is a consequence of the well-known Tietze extension Theorem, see e.g., [4]), the approximation result just stated can be deduced from more known theorems valid for maps defined on open sets, see e.g., [6].

3 Brouwer degree in Euclidean spaces

3.1 The special case

Let U be an open subset of $\mathbb{R}^k$, f an $\mathbb{R}^k$-valued map defined (at least) on the closure $\overline{U}$ of U, and $y \in \mathbb{R}^k$.

Definition 3.1. The triple (f, U, y) is said to be *admissible* (for the Brouwer degree in $\mathbb{R}^k$) provided that f is proper on $\overline{U}$ and $f(x) \neq y$, $\forall x \in \partial U$.

Notice that, according to Exercise 2.3, $f(\partial U)$ is a closed subset of $\mathbb{R}^k$.

Definition 3.2. An admissible triple (f, U, y) is said to be *regular* if f is C^∞, and y is a regular value for f in U.

We point out that if (f, U, y) is a regular triple, then the set $f^{-1}(y) \cap U$ is finite. In fact, $f^{-1}(y) \cap \overline{U}$ is compact (f being proper on $\overline{U}$), it is contained in U (since $y \notin f(\partial U)$) and it is discrete (because of the Inverse Function Theorem). This justifies the following definition of degree for the special case of a regular triple.

Definition 3.3. The *Brouwer degree* of a regular triple (f, U, y) is the integer

$$\deg(f, U, y) := \sum_{x \in f^{-1}(y) \cap U} \mathrm{i}(f, x) \tag{3.1}$$

In some sense the Brouwer degree of a regular triple (f, U, y) is an algebraic count of the number of solutions in U of the equation $f(x) = y$. This integer, as we shall see, turns out to depend only on the connected component of $\mathbb{R}^k \setminus f(\partial U)$ containing the regular value y. This is not so for the absolute count of the solutions (i.e. the cardinality $\# f^{-1}(y)$ of the set $f^{-1}(y)$), as it happens, for example, to the proper map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Incidentally, observe that in this case we have $\deg(f, \mathbb{R}, y) = 0$ for any regular value $y \in \mathbb{R}$ (i.e. for any $y \neq 0$).

Notice that the notation $\deg(f, U, y)$ is not redundant, since $\overline{U}$ can be strictly contained in the domain of f (which is uniquely associated with f). For example, if $\deg(f, U, y)$ is defined and V is an open subset of U such that $f^{-1}(y) \cap \partial V = \emptyset$, then also $\deg(f, V, y)$ is defined (and depends only on the restriction of f to V).

Observe also that (f, U, y) is a regular triple if and only if so is $(f - y, U, 0)$, where $f - y$ stands for the map $x \mapsto f(x) - y$. Obviously, when this holds, one has

$$\deg(f, U, y) = \deg(f - y, U, 0).$$

Example 3.4. Given a positive integer n, let $p_n : \mathbb{C} \to \mathbb{C}$ be the map defined by $p_n(z) = z^n$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, $0 \in \mathbb{C}$ is the only critical point of p_n (see Exercise 2.1). Therefore, $0 = p_n(0)$ is the unique critical value and, consequently, recalling that p_n is a proper map (see Exercise 2.4), $\deg(p_n, \mathbb{C}, w)$ is defined for any $w \in \mathbb{C} \setminus \{0\}$. Since any $w \neq 0$ admits exactly n different n-roots, Exercise 2.1 shows that $\deg(p_n, \mathbb{C}, w) = n$ for all $w \neq 0$. It is therefore natural to extend the function $w \mapsto \deg(p_n, \mathbb{C}, w)$ by putting $\deg(p_n, \mathbb{C}, w) = n$ even for $w = 0$ (this will be a consequence of the general definition of degree).

Theorem 3.6 below collects *the three fundamental properties of the degree for regular triples*. The first two, the *Normalization* and the *Additivity*, are a straightforward consequence of the definition; the third one, the *Homotopy Invariance*, is crucial for the construction of the degree in the general case, is nontrivial and (to please the impatient reader) will be proved in the appendix to this chapter. As we shall see later, there exists at most one integer-valued function (defined on the set of all regular triples) satisfying the three fundamental properties.

In order to simplify the statement of Theorem 3.6, it is convenient to introduce the following notion.

Definition 3.5. Let U be an open subset of $\mathbb{R}^k$, H an $\mathbb{R}^k$-valued map defined (at least) on $\overline{U} \times [0,1]$, and $\alpha : [0,1] \to \mathbb{R}^k$ a path. The triple (H, U, α) is said to be a *homotopy of triples* (on U, joining $(H(\cdot,0), U, \alpha(0))$ with $(H(\cdot,1), U, \alpha(1))$). If, in addition, H is proper on $\overline{U} \times [0,1]$ and $H(x,\lambda) \neq \alpha(\lambda)$ for all $(x,\lambda) \in \partial U \times [0,1]$, then (H, U, α) is called an *admissible homotopy* (of triples). If both H and α are smooth maps, then (H, U, α) is said to be *smooth*.

Theorem 3.6. *The degree for regular triples satisfies the following three fundamental properties:*

(Normalization) $\deg(I, \mathbb{R}^k, 0) = 1$, *where I denotes the identity on $\mathbb{R}^k$;*

(Additivity) *if (f, U, y) is regular, and U_1 and U_2 are two disjoint open subsets of U such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then*

$$\deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y);$$

(Homotopy Invariance) *if (H, U, α) is a smooth admissible homotopy joining two regular triples, then*

$$\deg(H(\cdot,0), U, \alpha(0)) = \deg(H(\cdot,1), U, \alpha(1)).$$

3.2 The general case

The Brouwer degree, preliminarily defined for regular triples, can be extended to the larger class of admissible triples; where, we recall, a triple (f, U, y) is admissible (for the degree in Euclidean spaces) provided that U is an open subset of $\mathbb{R}^k$, f is an $\mathbb{R}^k$-valued map which is proper on $\overline{U}$, and $y \in \mathbb{R}^k$ does not belong to the (possibly empty) set $f(\partial U)$.

The passage from the regular to the admissible case can be made in one big step or, as usual, in two small steps (the intermediate stage regarding admissible triples (f, U, y) with f smooth). We will reach the goal in just one step, but in a way that the reader interested only in the smooth case can easily imagine how to perform the first small step, which consists in removing the assumption that the value y in the triple (f, U, y) is regular.

Before giving the definition of degree in the general case, we need some preliminaries.

Let f and g be two $\mathbb{R}^s$-valued maps defined (at least) on a subset X of $\mathbb{R}^k$. Given $\epsilon > 0$, we say that *f is ϵ-close to g in X* if $|f(x) - g(x)| \leq \epsilon$, $\forall x \in X$. Moreover, given $y, z \in \mathbb{R}^s$, *y is ϵ-close to z*, provided that $|y - z| \leq \epsilon$.

Observe that if (f, U, y) is an admissible triple and g is ϵ-close to f in $\overline{U}$ for some $\epsilon > 0$, then g is proper on $\overline{U}$ (see Exercise 2.4). If, in addition, $\epsilon < \operatorname{dist}(y, f(\partial U))$,[1] then $y \notin g(\partial U)$, and in this case also the triple (g, U, y) is admissible. More generally, if $z \in \mathbb{R}^k$ is σ-close to y and $\epsilon + \sigma < \operatorname{dist}(y, f(\partial U))$, then (g, U, z) is admissible as well.

Definition 3.7. The degree of an admissible triple (f, U, y), also called *degree of f in U at y*, is the integer

$$\deg(f, U, y) := \deg(g, U, z),$$

where (g, U, z) is any regular triple with the following properties:

1. g is ϵ-close to f;
2. z is σ-close to y;
3. $\epsilon + \sigma < \operatorname{dist}(y, f(\partial U))$.

Clearly, given (f, U, y) admissible, the existence of a regular triple (g, U, z) as in Definition 3.7 is ensured by the Smooth Approximation Theorem (which shows the existence of g) and Sard's Lemma (which shows the existence of z).

The following consequence of Theorem 3.6 guarantees that this Definition is actually well posed.

Corollary 3.8. *Let (f, U, y) be an admissible triple. Then,*

$$\deg(g_0, U, z_0) = \deg(g_1, U, z_1)$$

for any pair of regular triples (g_0, U, z_0) and (g_1, U, z_1) satisfying the following conditions:

1. *g_0 and g_1 are ϵ-close to f;*
2. *z_0 and z_1 are σ-close to y;*
3. *$\epsilon + \sigma < \operatorname{dist}(y, f(\partial U))$.*

Proof. Let (g_0, U, z_0) and (g_1, U, z_1) be as in the statement, and define the smooth homotopy of triples (H, U, α) joining (g_0, U, z_0) and (g_1, U, z_1) by

$$H(x, \lambda) = (1 - \lambda) g_0(x) + \lambda g_1(x), \quad \alpha(\lambda) = (1 - \lambda) z_0 + \lambda z_1.$$

We have

$$H(x, \lambda) - f(x) = (1 - \lambda)(g_0(x) - f(x)) + \lambda (g_1(x) - f(x)).$$

[1] Recall the convention $\inf \emptyset = +\infty$, which implies $\operatorname{dist}(y, \emptyset) = +\infty$

Thus,

$$|H(x,\lambda) - f(x)| \le \epsilon, \quad \forall (x,\lambda) \in \overline{U} \times [0,1],$$

which implies that H is proper on $\overline{U}$, on the basis of Exercise 2.4. Analogously,

$$|\alpha(\lambda) - y| \le \sigma, \quad \forall \lambda \in [0,1].$$

Let us show that

$$H(x,\lambda) \ne \alpha(\lambda), \quad \forall (x,\lambda) \in \partial U \times [0,1].$$

In fact, given $(x,\lambda) \in \partial U \times [0,1]$, we have

$$H(x,\lambda) - \alpha(\lambda) = H(x,\lambda) - f(x) + f(x) - y + y - \alpha(\lambda)$$

and, consequently,

$$|H(x,\lambda) - \alpha(\lambda)| \ge |f(x) - y| - \epsilon - \sigma > 0.$$

The assertion now follows from the Homotopy Invariance Property for regular triples (see Theorem 3.6). □

The following important result is an extension, a consequence, and the analogue of Theorem 3.6 for the general case.

Theorem 3.9. *The Brouwer degree in $\mathbb{R}^k$ satisfies the following three Fundamental Properties:*

(Normalization) $\deg(I, \mathbb{R}^k, 0) = 1$, *where I denotes the identity on $\mathbb{R}^k$;*

(Additivity) *if (f, U, y) is admissible, and U_1 and U_2 are two disjoint open subsets of U such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then*

$$\deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y);$$

(Homotopy Invariance) *if (H, U, α) is an admissible homotopy, then*

$$\deg(H(\cdot, 0), U, \alpha(0)) = \deg(H(\cdot, 1), U, \alpha(1)).$$

Proof. Only the last two properties need to be proved.

(Additivity) Since f if proper on $\overline{U}$, the subset $C = f(\overline{U} \setminus (U_1 \cup U_2))$ of $\mathbb{R}^k$ is closed. Moreover, the assumption $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$ implies $\operatorname{dist}(y, C) > 0$. Let g be any smooth map which is ϵ-close to f, with $\epsilon < \operatorname{dist}(y, C)$. It is easy to check that $g^{-1}(y) \cap U \subseteq U_1 \cup U_2$. The assertion now follows from Definition 3.7 and the Additivity Property of the degree for regular triples (stated in Theorem 3.6).

(Homotopy Invariance) Observe that, on the basis of Exercise 2.4, the map

$$(x, \lambda) \mapsto H(x, \lambda) - \alpha(\lambda)$$

is proper on $\overline{U} \times [0,1]$. Thus the image, under this map, of the set $\partial U \times [0,1]$ is closed in $\mathbb{R}^k$. Consequently, since this set does not contain the origin of $\mathbb{R}^k$, the extended real number

$$\delta := \inf \Big\{ |H(x,\lambda) - \alpha(\lambda)| : (x,\lambda) \in \partial U \times (0,1) \Big\}$$

is nonzero. Let (G, U, β) be any smooth homotopy of triples, joining two regular triples, and satisfying the following properties:

1. G is ϵ-close to H (on $\overline{U} \times [0,1]$);
2. β is σ-close to α (on $[0,1]$);
3. $\epsilon + \sigma < \delta$.

The existence of such a triple is ensured by the Smooth Approximation Theorem and Sard's Lemma. As in the proof of Corollary 3.8 one can show that (G, U, β) is an admissible homotopy. Therefore, because of the Homotopy Invariance Property for regular triples, we get

$$\deg(G(\cdot, 0), U, \beta(0)) = \deg(G(\cdot, 1), U, \beta(1)).$$

The assertion now follows from Definition 3.7. □

3.3 Direct consequences of the Fundamental Properties

We will prove now some important additional properties of the Brouwer degree. Even if they could be easily deduced from the definition of degree, we prefer to prove them starting from the three *Fundamental Properties* stated in Theorem 3.9: *Normalization*, *Additivity* and *Homotopy Invariance*. The advantage of this method will be evident in the next subsection, which is devoted to the axiomatic approach.

First of all we observe that, given a map $f : X \to \mathbb{R}^k$ defined on a subset X of $\mathbb{R}^k$ and given $y \in \mathbb{R}^k$, the triple $(f, \emptyset, y)$ is admissible. Therefore, $\deg(f, \emptyset, y)$ is defined. We claim that this degree is zero.

Indeed, from the Additivity Property, putting $U = \emptyset$, $U_1 = \emptyset$ and $U_2 = \emptyset$, we get

$$\deg(f, \emptyset, y) = \deg(f, \emptyset, y) + \deg(f, \emptyset, y),$$

which implies our assertion.

The following property, which is evident in the regular case, shows that the degree of an admissible triple (f, U, y) depends only on the behavior of f in any neighborhood of the set of solutions of the equation $f(x) = y$, $x \in U$.

Theorem 3.10 (Excision Property). *If (f, U, y) is admissible and V is an open subset of U such that $f^{-1}(y) \cap U \subseteq V$, then (f, V, y) is admissible and*

$$\deg(f, U, y) = \deg(f, V, y).$$

Proof. The admissibility of (f, V, y) is clear. To show the equality apply the Additivity Property with $U_1 = V$ and $U_2 = \emptyset$. □

As the above property, also the following one is evident in the regular case.

Theorem 3.11 (Existence Property). *If $\deg(f, U, y) \neq 0$, then the equation $f(x) = y$ admits at least one solution in U.*

Proof. Assume that $f^{-1}(y) \cap U$ is empty. By the Excision Property, taking $V = \emptyset$, we get

$$\deg(f, U, y) = \deg(f, \emptyset, y) = 0,$$

which is a contradiction. □

Given an admissible triple (f, U, y), since the target space of f is $\mathbb{R}^k$, the equation $f(x) = y$ is equivalent to $f(x) - y = 0$. In terms of degree this fact is expressed by the following property, which is evident in the regular case.

Theorem 3.12 (Translation Invariance Property). *If (f, U, y) is admissible, then so is $(f - y, U, 0)$, and*

$$\deg(f, U, y) = \deg(f - y, U, 0).$$

Proof. Consider the family of equations

$$f(x) - \lambda y = (1 - \lambda)y, \quad \lambda \in [0, 1],$$

and apply the Homotopy Invariance Property with $H(x, \lambda) = f(x) - \lambda y$ and $\alpha(\lambda) = (1 - \lambda)y$. $\square$

The next result is a straightforward consequence of the Homotopy Invariance Property, and the proof is left to the reader.

Theorem 3.13 (Continuous Dependence Property). *Let f be proper on the closure of an open subset U of $\mathbb{R}^k$. Then the map $y \mapsto \deg(f, U, y)$, which is defined on the open set $\mathbb{R}^k \setminus f(\partial U)$, is locally constant. Thus, $\deg(f, U, y)$ depends only on the connected component of $\mathbb{R}^k \setminus f(\partial U)$ containing y.*

Because of the above property, given an open $U \subseteq \mathbb{R}^k$, f proper on $\overline{U}$ and a connected subset V of $\mathbb{R}^k \setminus f(\partial U)$, we will use the notation $\deg(f, U, V)$ to indicate the degree of f in U at any $y \in V$. In particular, if $U = \mathbb{R}^k$, the integer $\deg(f, \mathbb{R}^k, \mathbb{R}^k)$ will be denoted by $\deg(f)$.

The following property means that, given $y \in \mathbb{R}^k$ and $U \subseteq \mathbb{R}^k$ open and *bounded*, the degree of a map $f : \overline{U} \to \mathbb{R}^k$ (in U at y) depends only on the restriction of f to the boundary of U (assuming the condition $y \notin f(\partial U)$, which, U being bounded, is sufficient for the degree to be defined). This is important since, in many cases, it allows us to deduce the existence of solutions in U of the equation $f(x) = y$ only from the inspection of the behavior of f along the boundary of U; as in the case of $U = (a, b) \subseteq \mathbb{R}$, where the condition $f(a)f(b) < 0$ implies $f(x) = 0$ has a solution in (a, b).

Theorem 3.14 (Boundary Dependence Property). *Let $U \subseteq \mathbb{R}^k$ be open and bounded, and let $f, g : \overline{U} \to \mathbb{R}^k$ be such that $f(x) = g(x)$ for all $x \in \partial U$. Then, given $y \in \mathbb{R}^k \setminus f(\partial U)$, one has*

$$\deg(f, U, y) = \deg(g, U, y).$$

Proof. The assertion follows from the fact that the homotopy of triples (H, U, α) defined by

$$H(x, \lambda) = \lambda f(x) + (1 - \lambda)g(x), \quad \alpha(\lambda) = y$$

is admissible. $\square$

We point out that, in the above result, the assumption that U is bounded cannot be dropped. To see this, take $U = (0, +\infty)$, $f(x) = x$, $g(x) = -x$, $y = 1$.

3.4 The axiomatic approach

From an axiomatic point of view, the topological degree (in Euclidean spaces) is a map which to any admissible triple (f, U, y) assigns an integer, $\deg(f, U, y)$, satisfying the three *Fundamental Properties* (stated in Theorem 3.9): *Normalization*, *Additivity* and *Homotopy Invariance*.

A famous result by Amann-Weiss [1] (1973) asserts *the uniqueness of the topological degree*. That is, there exists at most one integer-valued map (defined on the class of the admissible triples) which verifies the three Fundamental Properties.

There are several methods for the construction of degree (see, for example, [2, 3, 5, 7, 8, 9, 10, 12, 13]), however, because of the Amann-Weiss result, with any of such methods, what is important is to prove the three Fundamental Properties (called, in this subsection, *Amann-Weiss axioms*): all the other classical properties will follow, as we have already shown in the previous subsection.

Let us show that from the three Amann-Weiss axioms one obtains an explicit formula for computing the degree of triples which are, in a sense to be made precise, dense in the family of the admissible triples. In particular, we will show that when an admissible triple (f, U, y) is actually regular, then

$$\deg(f, U, y) \;=\; \sum_{x \in f^{-1}(y) \cap U} \mathrm{i}(f, x).$$

The uniqueness of the degree will follow easily from the above formula and the Homotopy Invariance Property (see Theorem 3.17 below).

Recall first that, given a proper map $f : \mathbb{R}^k \to \mathbb{R}^k$, $\deg(f)$ stands for $\deg(f, \mathbb{R}^k, y)$, where y is any value in $\mathbb{R}^k$. This notation is justified by the Continuous Dependence Property, which, as all the other properties in the previous subsection, is a consequence of the axioms. Observe that, because of the Existence Property, if $\deg(f) \neq 0$, then f is surjective. We point out also that, as a consequence of the Homotopy Invariance axiom, if a homotopy $H : \mathbb{R}^k \times [0, 1] \to \mathbb{R}^k$ is proper, then $\deg(H(\cdot, \lambda))$ is well defined and independent of λ.

Let, as usual, $\mathrm{L}(\mathbb{R}^k)$ denote the normed space of linear endomorphisms of $\mathbb{R}^k$ and let $\mathrm{GL}(\mathbb{R}^k)$ stand for the open subset of $\mathrm{L}(\mathbb{R}^k)$ of the automorphisms; that is,

$$\mathrm{GL}(\mathbb{R}^k) = \Big\{ L \in \mathrm{L}(\mathbb{R}^k) : \det(L) \neq 0 \Big\}.$$

Now, let $L \in \mathrm{GL}(\mathbb{R}^k)$ be given. Since L is invertible, it is a proper map of $\mathbb{R}^k$ onto itself (notice that any homeomorphism is a proper map). Thus, $\deg(L)$ is well defined.

Let us show that the Amann-Weiss axioms imply

$$\deg(L) = \mathrm{sign}(\det(L)), \quad \forall L \in \mathrm{GL}(\mathbb{R}^k). \tag{3.2}$$

To this end, we recall that the open subset $\mathrm{GL}(\mathbb{R}^k)$ of $\mathrm{L}(\mathbb{R}^k)$ has exactly two connected components. Namely,

$$\mathrm{GL}_+(\mathbb{R}^k) = \Big\{ L \in \mathrm{L}(\mathbb{R}^k) : \det(L) > 0 \Big\}.$$

and

$$\mathrm{GL}_-(\mathbb{R}^k) = \Big\{ L \in \mathrm{L}(\mathbb{R}^k) : \det(L) < 0 \Big\}.$$

As a consequence of the Homotopy Invariance axiom it is easy to check that the map which assigns $\deg(L)$ to any $L \in \mathrm{GL}(\mathbb{R}^k)$ is locally constant. Indeed, if L_0 and L_1 are close one to the other, the homotopy $H(x,\lambda) = L_0 x + \lambda(L_1 - L_0)$ is proper. Consequently, $\deg(L)$ depends only on the component of $\mathrm{GL}(\mathbb{R}^k)$ containing L.

Since the identity I of $\mathbb{R}^k$ belongs to $\mathrm{GL}_+(\mathbb{R}^k)$, the Normalization axiom implies $\deg(L) = 1$, $\forall L \in \mathrm{GL}_+(\mathbb{R}^k)$.

Let us show that $\deg(L) = -1$, $\forall L \in \mathrm{GL}_-(\mathbb{R}^k)$. For this purpose consider the map $f : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$f(\xi_1, \dots, \xi_{k-1}, \xi_k) = (\xi_1, \dots, \xi_{k-1}, |\xi_k|).$$

This map is proper, since $\|f(x)\| = \|x\|$, $\forall x \in \mathbb{R}^k$. Thus $\deg(f)$ makes sense and is zero, because f is not surjective.

Let V_- and V_+ denote, respectively, the open half-spaces of the points in $\mathbb{R}^k$ with negative and positive last coordinate. Consider the two solutions

$$x_- = (0, \dots, 0, -1) \quad \text{and} \quad x_+ = (0, \dots, 0, 1)$$

of the equation $f(x) = y$, with $y = (0, \dots, 0, 1)$, and observe that $x_- \in V_-$, $x_+ \in V_+$.

By the Additivity axiom we get

$$0 = \deg(f) = \deg(f, V_-, y) + \deg(f, V_+, y).$$

Now, observe that in V_+ the map f coincides with the identity I of $\mathbb{R}^k$. Therefore, because of the Excision Property, one has

$$\deg(f, V_+, y) = \deg(I) = 1,$$

which implies $\deg(f, V_-, y) = -1$.

Since f in V_- coincides with the linear map $L_- \in \mathrm{GL}_-(\mathbb{R}^k)$ given by

$$(\xi_1, \dots, \xi_{k-1}, \xi_k) \mapsto (\xi_1, \dots, \xi_{k-1}, -\xi_k),$$

we obtain $\deg(L_-) = -1$. Thus, $\mathrm{GL}_-(\mathbb{R}^k)$ being connected, we finally get $\deg(L) = -1$ for all $L \in \mathrm{GL}_-(\mathbb{R}^k)$, as claimed.

Let us show how from the Amann-Weiss axioms one can deduce the formula (3.1) for computing the degree of a regular triple. More generally, we prove the following result.

Theorem 3.15 (Computation Formula). *Let (f, U, y) be an admissible triple. Assume that, at any $x \in f^{-1}(y) \cap U$, f is Fréchet differentiable with nonsingular derivative. Then $f^{-1}(y) \cap U$ is finite and*

$$\deg(f, U, y) \ = \sum_{x \in f^{-1}(y) \cap U} \mathrm{i}(f, x).$$

In order to prove Theorem 3.15 we need the following result.

Lemma 3.16. *Let (f, V, y_0) be an admissible triple. Assume that the equation $f(x) = y_0$ has a unique solution $x_0 \in V$. If f is Fréchet differentiable at x_0 and $f'(x_0)$ is invertible, then* $\deg(f, V, y) = \deg(f'(x_0))$.

Proof. Since f is differentiable at x_0, we have

$$f(x) = y_0 + f'(x_0)(x - x_0) + \|x - x_0\|\epsilon(x - x_0), \quad \forall x \in \overline{V},$$

where $\epsilon(h)$ is defined for $h \in -x_0 + \overline{V}$, is continuous, and such that $\epsilon(0) = 0$.

Observe that the linearized map of f at x_0, $g(x) := y_0 + f'(x_0)(x - x_0)$, is an affine map with linear part $f'(x_0) \in \mathrm{GL}(\mathbb{R}^k)$. Thus, g is proper and, because of the Translation Invariance Property, one has $\deg(g) = \deg(f'(x_0))$. Therefore, by the Excision Property, it is enough to show that

$$\deg(f, W, y_0) = \deg(g, W, y_0), \tag{3.3}$$

where W is a sufficiently small open neighborhood of x_0 contained in V.

For this purpose, define the homotopy $H : \overline{V} \times [0, 1] \to \mathbb{R}^k$ joining g with f by

$$H(x, \lambda) = y_0 + f'(x_0)(x - x_0) + \lambda\|x - x_0\|\epsilon(x - x_0).$$

We have

$$\|H(x, \lambda) - y_0\| \geq (m - \|\epsilon(x - x_0)\|)\,\|x - x_0\|,$$

where $m = \inf\{\|f'(x_0)v\| : \|v\| = 1\}$ is positive, $f'(x_0)$ being invertible. This shows that, in a convenient neighborhood W of x_0, the homotopy of triples (H, W, y_0) is admissible, and the equality 3.3 is established. □

Proof of Theorem 3.15. Since f is proper on $\overline{U}$, the set $f^{-1}(y) \cap \overline{U}$ is compact, and the condition $y \notin f(\partial U)$ ensures that it is contained in U. On the other hand, as in the proof of Lemma 3.16, the assumption that at any $x \in f^{-1}(y) \cap \overline{U}$ the derivative $f'(x)$ is injective ensures that this set is made up of isolated points. Therefore, it is actually a finite set. Let $V_1, V_2, \ldots, V_n$ be pairwise disjoint open subsets of U, each of them containing exactly one point of $f^{-1}(y)$. The Additivity axiom implies

$$\deg(f, U, y) = \sum_{i=1}^{n} \deg(f, V_i, y),$$

and the assertion follows from Lemma 3.16 and formula (3.2) for computing the degree of a linear automorphism. □

The following result shows, in particular, that the degree of an admissible triple coincides with the degree of any sufficiently close regular triple, and this implies the uniqueness of the degree.

Theorem 3.17. *Let (f, U, y) be an admissible triple. If (g, U, z) is a regular triple which can be joined to (f, U, y) via an admissible homotopy, then*

$$\deg(f, U, y) \;=\; \sum_{x \in g^{-1}(z) \cap U} \mathrm{i}(g, x). \tag{3.4}$$

In particular, this is true for any regular triple (g, U, z) such that:

1. *g is ϵ-close to f;*
2. *z is σ-close to y;*

3. $\epsilon + \sigma < \operatorname{dist}(y, f(\partial U))$.

Proof. Let (g, U, z) be a regular triple which can be joined to (f, U, y) via an admissible homotopy. From the Homotopy Invariance axiom one gets

$$\deg(f, U, y) = \deg(g, U, z),$$

and the equality (3.4) follows from Theorem 3.15.

Assume now that (g, U, z) is a regular triple which satisfies properties 1, 2 and 3. It is enough to show that the homotopy of triples (H, U, α), defined by

$$H(x, \lambda) = (1 - \lambda)f(x) + \lambda g(x), \quad \alpha(\lambda) = (1 - \lambda)y + \lambda z,$$

is admissible on U. This can be done as in the proof of Corollary 3.8. □

3.5 First topological applications

We give now some direct topological applications of the Brouwer degree in Euclidean spaces.

Let us show, first of all, that the topological degree of a (non-constant) polynomial is the same as its algebraic degree. This provides one of the many proofs of the Fundamental Theorem of Algebra (which is actually a result of topological nature) and justifies the expression "degree" used by Brouwer. In some sense, the Brouwer degree is an extension of the algebraic notion of degree to more general situations.

As before, if $f : \mathbb{R}^k \to \mathbb{R}^k$ is a proper map, by $\deg(f)$ we shall mean the integer $\deg(f, \mathbb{R}^k, y)$, where y is any point of $\mathbb{R}^k$. Because of the Continuous Dependence Property (Theorem 3.13), $\deg(f)$ is well defined.

Observe that if $H : \mathbb{R}^k \times [0, 1] \to \mathbb{R}^k$ is a proper map, then, because of the Homotopy Invariance Property, $\deg(H(\cdot, \lambda))$ is independent of $\lambda \in [0, 1]$.

Theorem 3.18. *Let $p_n : \mathbb{C} \to \mathbb{C}$ be a polynomial of algebraic degree $n > 0$. Then p_n, regarded as a map from $\mathbb{R}^2$ into itself, has topological degree n.*

Proof. Write $p_n(z) = az^n + q(z)$, with $a \neq 0$ and $q(z)$ a polynomial of degree less then n. Consider the homotopy

$$H(z, \lambda) = az^n + \lambda q(z)$$

and observe that

$$\lim_{|z| \to +\infty} |H(z, \lambda)| = +\infty,$$

uniformly with respect to $\lambda \in [0, 1]$. Thus H is a proper map and, consequently, the topological degrees of the two maps p_n and $f_n : z \mapsto az^n$ are equal. To conclude that the Brouwer degree of f_n is n, observe that the equation $az^n = a$ has exactly n solutions each of them with index one (see Exercise 2.1). □

With the same method as in the proof of Theorem 3.18 one can show that if $p_n : \mathbb{C} \to \mathbb{C}$ is a polynomial of algebraic degree $n > 0$, then the map $f_n : \mathbb{C} \to \mathbb{C}$ given by $f_n(z) = p_n(\bar{z})$, where $\bar{z}$ is the conjugate of z, has degree $-n$. Thus, we have examples of proper maps from $\mathbb{R}^2$ into $\mathbb{R}^2$ of arbitrary nonzero degree. A simple (proper) map of degree zero is given by $f_0(x, y) = (x, y^2)$.

It is easy to check that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is proper and I denotes the identity on $\mathbb{R}^s$, then the map

$$I \times f : \mathbb{R}^s \times \mathbb{R}^2 \to \mathbb{R}^s \times \mathbb{R}^2$$

is proper and $\deg(I \times f) = \deg(f)$. Thus, if $k > 1$, one can find maps from $\mathbb{R}^k$ into itself of arbitrary degree.

Exercise 3.19. Show that if $f : \mathbb{R} \to \mathbb{R}$ is proper, then $\deg(f)$ may assume only three values: $-1, 0, 1$.

From Theorem 3.18 and the Existence Property of the degree follows immediately the

Fundamental Theorem of Algebra. *Any non-constant polynomial with complex coefficients admits at least one root.*

The following is another famous topological result that can be easily deduced from degree theory.

Brouwer Fixed Point Theorem. *Let U be the open unit ball in $\mathbb{R}^k$ and let $f : \overline{U} \to \mathbb{R}^k$ be continuous and such that $f(\overline{U}) \subseteq \overline{U}$ (or, more generally, $f(\partial U) \subseteq \overline{U}$). Then f has a fixed point in $\overline{U}$.*

Proof. If the triple $(I - f, U, 0)$ is not admissible, then f has a fixed point on ∂U, and we are done. Assume, therefore, this is not the case. Consider the homotopy $H : \overline{U} \times [0, 1] \to \mathbb{R}^k$ given by $H(x, \lambda) = x - \lambda f(x)$ and observe that $x \neq \lambda f(x)$ for all $x \in \partial U$ and $\lambda \in [0, 1]$. Thus, by the Homotopy Invariance Property, we have

$$\deg(I - f, U, 0) = \deg(I, U, 0).$$

On the other hand, the Excision Property implies

$$\deg(I, U, 0) = \deg(I, \mathbb{R}^k, 0) = 1.$$

The result now follows from the Existence Property applied to the equation $x - f(x) = 0$, $x \in U$. □

We recall that a subset A of a topological space X is a *retract* of X if there exists a continuous map $r : X \to A$, called *retraction*, whose restriction to A is the identity map. Clearly the boundary of an interval $[a, b] \subseteq \mathbb{R}$, being disconnected, is not a retract of $[a, b]$. The following easy consequence of the Boundary Dependence Property extends this elementary fact.

Theorem 3.20. *Let U be a bounded open subset of $\mathbb{R}^k$. Then ∂U is not a retract of $\overline{U}$.*

Proof. Assume there exists a map $r : \overline{U} \to \partial U$ such that $r(x) = x$, $\forall x \in \partial U$. Let $y \in U$. Since U is bounded, the Boundary Dependence Property (Theorem 3.14) implies

$$\deg(r, U, y) = \deg(I, U, y) = 1.$$

Hence the equation $r(x) = y$ has a solution in U, and this is a contradiction since r maps $\overline{U}$ onto ∂U. □

The fact that the boundary of the subset $(0,+\infty)$ of $\mathbb{R}$ is a retract of $[0,+\infty)$ shows that in Theorem 3.20 the assumption that U is bounded cannot be removed.

We give now some applications of degree theory to problems of existence and multiplicity of solutions for nonlinear equations in $\mathbb{R}^k$.

We start with the following result.

Proposition 3.21. *Let $f:\mathbb{R}^k\to\mathbb{R}^k$ be proper and let $g:\mathbb{R}^k\to\mathbb{R}^k$ be a bounded map. Then $f+g$ is proper and* $\deg(f+g)=\deg(f)$. *Consequently, if* $\deg(f)\neq 0$, *the equation $f(x)+g(x)=0$ has at least one solution.*

Proof. The map $f+g$ is proper (on the basis of Exercise 2.4), since the assumption

$$\lim_{\|x\|\to+\infty}\|f(x)\|=+\infty$$

implies

$$\lim_{\|x\|\to+\infty}\|f(x)+g(x)\|=+\infty.$$

For the same reason, also the map $(x,\lambda)\mapsto f(x)+\lambda g(x)$ is proper, and the equality $\deg(f+g)=\deg(f)$ follows immediately from the Homotopy Invariance Property. □

Let us show, with a simple example, how the above result can be applied to prove the existence of solutions of a nonlinear equation in $\mathbb{R}^k$.

Example 3.22. Consider the following nonlinear system of two equations in two unknowns:

$$\begin{cases} x^2-2y^2-\sin(xy) &= 0\\ xy+2\cos x+\frac{1}{1+y^2} &= 0\end{cases} \tag{3.5}$$

We claim that this system has at least one solution. It is not difficult to check that the map $f:\mathbb{R}^2\to\mathbb{R}^2$ given by $f(x,y)=(x^2-2y^2,xy)$ is proper. Indeed, let c be a positive constant and consider the inequalities

$$|x^2-2y^2|\le c \quad\text{and}\quad |xy|\le c.$$

The first one implies that when $|x|$ is large, so is $|y|$; but this is in contrast with the second one. Thus, if $f(x,y)$ belongs to a compact set, also (x,y) must stay in a compact set.

To compute the degree of f, which is well defined, observe that the system

$$\begin{cases} x^2-2y^2 &= 1\\ xy &= 0\end{cases}$$

has the following two solutions: $(1,0)$ and $(-1,0)$. One can check that these solutions are both regular with index 1. Thus $\deg(f)=2$, and Proposition 3.21 implies the existence of at least one solution of system (3.5), as claimed.

Actually, since $\deg(f)=2$, applying Sard's Lemma (and the definition of degree for a regular triple) one can say more: for almost all $(a,b)\in\mathbb{R}^2$ the system

$$\begin{cases} x^2-2y^2-\sin(xy) &= a\\ xy+2\cos x+\frac{1}{1+y^2} &= b\end{cases}$$

has at least two solutions (and, of course, at least one for all $(a,b)\in\mathbb{R}^2$).

From Proposition 3.21 it follows immediately that a nonlinear system of the type

$$Lx + g(x) = 0 \tag{3.6}$$

has at least one solution, provided that L is an invertible linear operator in $\mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}^k$ is a bounded map. This, on the other hand, can be shown also by means of the Brouwer Fixed Point Theorem, since (3.6) can be equivalently written as a fixed point equation in the form $x = -L^{-1}g(x)$, where the map $x \mapsto -L^{-1}g(x)$ sends the whole space $\mathbb{R}^k$ into a bounded set (and, in particular, some closed ball into itself).

An extension of this existence result is Corollary 3.24 below, which is a direct consequence of the following continuation principle (stated without any notion of degree theory).

Theorem 3.23 (Continuation Principle in Euclidean Spaces). *Let U be a bounded open subset of $\mathbb{R}^k$, $f : \overline{U} \to \mathbb{R}^k$ a continuous map of class C^1 in a neighborhood of $f^{-1}(0)$, and $h : \overline{U} \times [0,1] \to \mathbb{R}^k$ a continuous map. Assume that:*

1. *$h(x,0) = 0$ for all $x \in U$;*
2. *$f(x) + h(x,\lambda) \neq 0$ for all $(x,\lambda) \in \partial U \times [0,1]$;*
3. *$\det(f'(x)) \neq 0$ for any $x \in f^{-1}(0)$;*
4. *the integer*
$$\sum_{x \in f^{-1}(0)} \operatorname{sign}(\det(f'(x)))$$
is nonzero.

Then, the equation $f(x) + h(x,1) = 0$ has at least one solution in U.

Proof. Define $H : \overline{U} \times [0,1] \to \mathbb{R}^k$ by $H(x,\lambda) = f(x) + h(x,\lambda)$ and observe that, because of assumption 2, the triple $(H, U, 0)$ is an admissible homotopy. Thus, from the Homotopy Invariance Property it follows that

$$\deg(H(\cdot,0), U, 0) = \deg(H(\cdot,1), U, 0).$$

On the other hand, because of condition 1, we have $H(\cdot,0) = f$, and the assertion now follows from assumptions 3 and 4, the Computation Formula (Theorem 3.15), and the Existence Property (Theorem 3.11). □

The following easy consequence of Theorem 3.23 extends the existence result related to equation (3.6), removing the assumption that the map g is bounded.

Corollary 3.24. *Let L be a linear operator in $\mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}^k$ a continuous map. If the set*

$$S = \left\{ x \in \mathbb{R}^k : Lx + \lambda g(x) = 0 \text{ for some } \lambda \in [0,1] \right\}$$

is bounded, then the equation $Lx + g(x) = 0$ has at least one solution.

Proof. Observe that the boundedness of S implies that L is injective and, consequently, $\det(L) \neq 0$. The assertion now follows from Theorem 3.23 with U any open ball containing S, $f = L$, and $h(x,\lambda) = \lambda g(x)$. □

Corollary 3.24 can be proved in a more elementary way: it can be deduced directly from the Brouwer Fixed Point Theorem (this is not so for the above continuation principle). Let us show, briefly, how this can be done.

Define $\sigma : \mathbb{R}^k \to [0,1]$ by $\sigma(x) = \max\{1 - \operatorname{dist}(x,S), 0\}$ and observe that any solution of the equation $Lx + \sigma(x)g(x) = 0$ lies in S. Therefore it is also a solution of $Lx + g(x) = 0$, since $\sigma(x) = 1$ for $x \in S$. To show that $Lx + \sigma(x)g(x) = 0$ has a solution, notice that L is invertible (since S is bounded) and apply the Brouwer Fixed Point Theorem to the equation $x = -\sigma(x)L^{-1}g(x)$.

Example 3.25. To illustrate how Corollary 3.24 applies, we prove that the system

$$\begin{cases} x + y + x^3 + \sin(xy) &= 0 \\ y + 2\cos(xy) + y^5 &= 0 \end{cases}$$

has at least one solution. For this purpose we need to show that all the possible solutions (x,y) of the system

$$\begin{cases} x + y + \lambda x^3 + \lambda \sin(xy) &= 0 \\ y + 2\lambda \cos(xy) + \lambda y^5 &= 0 \end{cases}$$

are *a priori* bounded when the parameter λ varies in $[0,1]$. In fact, the second equation implies that, if (x,y) is such a solution, then y must lie in the interval $[-2,2]$ and, as a consequence, from the first equation one gets $|x| \le 3$.

Degree can be useful to prove the existence of nontrivial solutions of an equation of the type $f(x) = 0$, where $f : \mathbb{R}^k \to \mathbb{R}^k$ satisfies the condition $f(0) = 0$. The following result is in this direction.

Theorem 3.26. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a proper map such that $f(0) = 0$. Assume that f is Fréchet differentiable at the origin. If $f'(0)$ is invertible and* $\deg(f) \neq \mathrm{i}(f,0)$*, then the equation $f(x) = 0$ has a nontrivial solution (i.e. a solution $x \neq 0$).*

Proof. If the equation $f(x) = 0$ had the unique solution $x = 0$, the Computation Formula would contradict the assumption $\deg(f) \neq \mathrm{i}(f,0)$. □

Example 3.27. Consider the system

$$\begin{cases} x - 2\sin(x + x^2 - y^2) &= 0 \\ 2x + y + 1 - \cos(xy) &= 0 \end{cases}$$

and observe that it admits the trivial solution $(0,0)$. Since the degree of the invertible linear operator $L : (x,y) \mapsto (x, 2x+y)$ is 1, by Proposition 3.21 the map

$$(L+g) : (x,y) \mapsto (x - 2\sin(x + x^2 - y^2), 2x + y + 1 - \cos(xy))$$

is proper with degree 1.

Now, notice that the linearized map of $(L+g)$ at the origin is given by $(x,y) \mapsto (-x, 2x+y)$, whose determinant is negative. Thus, Theorem 3.26 implies that the above system has at least one nontrivial solution (very likely, at least two, because of Sard's Lemma and the Computation Formula).

Degree theory has important applications in the study of bifurcation problems. Let us see its rôle in the finite dimensional context.

Let J be a real interval, U an open subset of $\mathbb{R}^k$ containing the origin $0 \in \mathbb{R}^k$, and $f : J \times U \to \mathbb{R}^k$ a continuous map satisfying the condition $f(\lambda, 0) = 0$ for any $\lambda \in J$. Consider the equation

$$f(\lambda, x) = 0. \tag{3.7}$$

Any pair $(\lambda, 0)$, with $\lambda \in J$, is called a *trivial solution* of the above equation and, consequently, any other solution is said to be *nontrivial*.

A *bifurcation point* of the equation (3.7) is a number $\lambda_0 \in J$ (or, equivalently, a trivial solution $(\lambda_0, 0) \in J \times U$) with the property that any neighborhood of $(\lambda_0, 0)$ contains nontrivial solutions.

For example, if f has the special form

$$f(\lambda, x) = \lambda x - Lx,$$

where L is a linear operator in $\mathbb{R}^k$, any eigenvalue of L is a bifurcation point. Therefore, in some sense, a bifurcation point is the nonlinear analogue of what in the liner case is the eigenvalue.

Assume now that f is continuously differentiable with respect to the second variable (at least in a neighborhood of the set $J \times \{0\}$ of trivial solutions) and let $\lambda_0 \in J$ be given. If the partial derivative $\partial_2 f(\lambda_0, 0)$ is nonsingular, then the Implicit Function Theorem implies that in a convenient neighborhood $I \times V$ of $(\lambda_0, 0)$ the set $f^{-1}(0)$ is the graph of a map from I to V. Consequently, the assumption $f(\lambda, 0) \equiv 0$ implies that λ_0 is not a bifurcation point. We have, therefore, the following result.

Theorem 3.28 (Necessary Condition for Bifurcation). *Let f be as above, and consider the (continuous) real function $\varphi(\lambda) = \det(\partial_2 f(\lambda, 0))$, $\lambda \in J$. If $\lambda_0 \in J$ is a bifurcation point for the equation $f(\lambda, x) = 0$, then $\varphi(\lambda_0) = 0$.*

There are simple examples showing that the condition $\varphi(\lambda_0) = 0$ is not sufficient for λ_0 to be a bifurcation point. Perhaps, the simplest one is given by $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(\lambda, x) = (\lambda^2 + x^2)x$, in which $\lambda_0 = 0$.

A sufficient condition for λ_0 to be a bifurcation point is that $\varphi(\lambda)$ changes sign at λ_0. In fact, we have the following result.

Theorem 3.29 (Sufficient Condition for Bifurcation). *Let $[a, b]$ be a real interval, U an open subset of $\mathbb{R}^k$ containing the origin $0 \in \mathbb{R}^k$, and f a continuous map from $[a, b] \times U$ into $\mathbb{R}^k$ satisfying the condition $f(\lambda, 0) = 0$ for any λ in $[a, b]$. Assume that f is differentiable with respect to the second variable at any trivial solution $(\lambda, 0)$ of the equation $f(\lambda, x) = 0$, and define $\varphi : [a, b] \to \mathbb{R}$ by $\varphi(\lambda) = \det(\partial_2 f(\lambda, 0))$.*

If $\varphi(a)\varphi(b) < 0$, then the interval $[a, b]$ contains at least one bifurcation point. In particular, if $\varphi(\lambda)$ has a sign-jump at some $\lambda_0 \in [a, b]$, then λ_0 is a bifurcation point.

Proof. Assume the contrary. Then, by the compactness of $[a, b]$, there exists a *bounded* open neighborhood V of 0 such that $\overline{V} \subseteq U$ and that the solutions of the equation $f(\lambda, x) = 0$ which are in $[a, b] \times \overline{V}$ are all trivial. In particular

$$f(\lambda, x) \neq 0, \quad \forall (\lambda, x) \in [a, b] \times \partial V.$$

Thus, the Homotopy Invariance Property implies

$$\deg(f(a,\cdot),V,0)=\deg(f(b,\cdot),V,0).$$

On the other hand, by the Computation Formula (Theorem 3.15), we get

$$\deg(f(a,\cdot),V,0)=\operatorname{sign}(\varphi(a)) \quad \text{and} \quad \deg(f(b,\cdot),V,0)=\operatorname{sign}(\varphi(b)),$$

contradicting the assumption $\varphi(a)\varphi(b)<0$. □

The following simple example illustrates how Theorem 3.29 applies.

Example 3.30. The system

$$\left\{\begin{array}{lcl} x-\lambda\sin(x+x^2-y^2) & = & 0 \\ 2x+y+1-\cos xy & = & 0 \end{array}\right. \tag{3.8}$$

has a bifurcation point at $\lambda=1$. To see this consider the linearized problem

$$\left\{\begin{array}{lcl} x-\lambda x & = & 0 \\ 2x+y & = & 0 \end{array}\right.$$

(of (3.8) at the origin of $\mathbb{R}^2$) and observe that the function

$$\varphi(\lambda)=\det\begin{pmatrix}1-\lambda & 0\\ 2 & 1\end{pmatrix}$$

has a sign-jump at $\lambda=1$.

4 The extended case

In this section we extend the Brouwer degree to the class of *weakly admissible* triples; that is, triples of the type (f,U,y), where $U\subseteq\mathbb{R}^k$ is an open set, f is an $\mathbb{R}^k$-valued map defined (at least) on U, and $y\in\mathbb{R}^k$ is such that $f^{-1}(y)\cap U$ is compact.

The Excision Property of the degree for admissible triples (Theorem 3.10) shows that the following definition is well posed.

Definition 4.1. The *Brouwer degree* of a weakly admissible triple (f,U,y) is the integer

$$\deg(f,U,y) \;:= \deg(f,V,y),$$

where V is any bounded open neighborhood of $f^{-1}(y)\cap U$ such that $\overline{V}\subseteq U$.

Let, as above, U be an open subset of $\mathbb{R}^k$, H an $\mathbb{R}^k$-valued map defined (at least) on $U\times[0,1]$, and $\alpha:[0,1]\to\mathbb{R}^k$ a path. The triple (H,U,α) is said to be a *weakly admissible homotopy of triples* if the set

$$\Sigma=\Big\{(x,\lambda)\in U\times[0,1]: H(x,\lambda)=\alpha(\lambda)\Big\}$$

is compact.

The following result is a direct consequence of Theorem 3.9.

Theorem 4.2. *The Brouwer degree for weakly admissible triples satisfies the following three* fundamental properties*:*

(Normalization) $\deg(I, \mathbb{R}^k, 0) = 1$*, where* I *denotes the identity on* $\mathbb{R}^k$*;*

(Additivity) *if* (f, U, y) *is weakly admissible, and* U_1 *and* U_2 *are two disjoint open subsets of* U *such that* $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$*, then*

$$\deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y);$$

(Homotopy Invariance) *if* (H, U, α) *is a weakly admissible homotopy of triples, then*

$$\deg(H(\cdot, 0), U, \alpha(0)) = \deg(H(\cdot, 1), U, \alpha(1)).$$

Given an open subset U of $\mathbb{R}^k$ and an $\mathbb{R}^k$-valued map f defined (at least) on U, the integer $\deg(f, U, y)$ does not necessarily depend continuously on y. For instance, the triple $(\exp, \mathbb{R}, y)$ is weakly admissible for all $y \in \mathbb{R}$, but the map $y \mapsto \deg(\exp, \mathbb{R}, y)$ is discontinuous at $y = 0$. To avoid this inconvenience, given U and f as above, we weed out a subset of $\mathbb{R}^k$, called *boundary set of* f *in* U, with the property that the map $y \mapsto \deg(f, M, y)$ turns out to be well defined and continuous in the complement of this set. Moreover, when f is proper on $\overline{U}$, this set coincides with $f(\partial U)$.

Given $y \in \mathbb{R}^k$, we say that f is *y-proper in* U if there exists a neighborhood V of y such that $f^{-1}(K) \cap U$ is compact for any compact subset K of V (this means that the restriction of f from $U \cap f^{-1}(V)$ into V is proper). Clearly, the set

$$\big\{y \in \mathbb{R}^k : f \text{ is } y\text{-proper in } U\big\}$$

is open in $\mathbb{R}^k$. Consequently, its complement, called the *boundary set of* f *in* U and denoted by $\partial(f, U)$, is closed.

Clearly $\deg(f, U, y)$ is defined for any $y \in \mathbb{R}^k \setminus \partial(f, U)$ and, because of the homotopy property, depends continuously on y.

Exercise 4.3. Let $f : U \to \mathbb{R}^k$ be a map defined on an open subset U of $\mathbb{R}^k$. Prove that f is proper if and only if $\partial(f, U) = \emptyset$.

Exercise 4.4. Let f be an $\mathbb{R}^k$-valued map defined (at least) on an open set $U \subseteq \mathbb{R}^k$. Show that $f(\partial U) \subseteq \partial(f, U)$. If, in addition, f is proper on $\overline{U}$, prove that $f(\partial U) = \partial(f, U)$.

The following result is an useful extension of the above Homotopy Invariance Property.

Theorem 4.5 (General Homotopy Invariance Property). *Let* H *be an* $\mathbb{R}^k$*-valued map defined on an open subset* W *of* $\mathbb{R}^k \times [0, 1]$ *and* $\alpha : [0, 1] \to \mathbb{R}^k$ *a path. If the set*

$$\Sigma = \big\{(x, \lambda) \in W : H(x, \lambda) = \alpha(\lambda)\big\}$$

is compact, then

$$\deg(H_\lambda, W_\lambda, \alpha(\lambda))$$

does not depend on $\lambda \in [0, 1]$*, where* $H_\lambda : W_\lambda \to \mathbb{R}^k$ *denotes the partial map* $H(\cdot, \lambda)$ *defined on the slice* $W_\lambda = \{x \in \mathbb{R}^k : (x, \lambda) \in W\}$*.*

Proof. Clearly, given $\lambda \in [0,1]$, $\deg(H_\lambda, W_\lambda, \alpha(\lambda))$ is well defined, since the set $H_\lambda^{-1}(\alpha(\lambda))$ coincides with the λ-slice Σ_λ of Σ, which is compact. Therefore, it is enough to show that the function $\varphi : [0,1] \to \mathbb{Z}$ given by $\varphi(\lambda) = \deg(H_\lambda, W_\lambda, \alpha(\lambda))$ is locally constant. For this purpose, fix any $\mu \in [0,1]$ and consider any bounded open neighborhood V of Σ_μ such that $\overline{V} \subseteq W_\mu$. Since $\overline{V}$ is compact, there exists a closed neighborhood $I_\delta = [\mu - \delta, \mu + \delta] \cap [0,1]$ of μ in $[0,1]$ such that $\overline{V} \times I_\delta \subseteq W$.

We claim that, if δ is sufficiently small, then $\Sigma_\lambda \subseteq V$ for all $\lambda \in I_\delta$. Assume the contrary. Thus, there exists a sequence $\{(x_n, \lambda_n)\}$ in Σ such that $\lambda_n \to \mu$ and $x_n \notin V$ for all $n \in \mathbb{N}$. Because of the compactness of Σ (and the fact that V is open) we may assume that $x_n \to y \notin V$. Therefore $(y, \mu) \in \Sigma$, which implies $y \in \Sigma_\mu$; and this is a contradiction since, by assumption, $\Sigma_\mu \subseteq V$.

Assume, without loss of generality, $\Sigma_\lambda \subseteq V$ for all $\lambda \in I_\delta$. Since, in addition, $\overline{V}$ is compact and contained in W_λ for all $\lambda \in I_\delta$, by Definition 4.1 we get

$$\deg(H_\lambda, W_\lambda, \alpha(\lambda)) = \deg(H_\lambda, V, \alpha(\lambda)), \quad \forall \lambda \in I_\delta.$$

Now, observe that $H(x,\lambda) \neq \alpha(\lambda)$ for all $(x,\lambda) \in \partial V \times I_\delta$. Thus, the Homotopy Invariance Property of the degree (in Theorem 3.9) implies that $\varphi(\lambda)$ does not depend on $\lambda \in I_\delta$, and the assertion follows since μ is arbitrary. $\square$

5 Appendix: proof of the Homotopy Invariance Property for regular triples

Our purpose, here, is to prove a crucial result in the construction of the Brouwer degree: the Homotopy Invariance Property for regular triples (see Theorem 3.6).

Let U be open in $\mathbb{R}^k$ and $f : \overline{U} \to \mathbb{R}^k$ a proper smooth map (we recall that f is smooth on $\overline{U}$ if it admits a smooth extension on an open set containing $\overline{U}$). Consider the closed set

$$K = \{x \in \overline{U} : \det(f'(x)) = 0\}$$

of the critical points of f in $\overline{U}$. Recalling that proper maps are closed (see Exercise 2.3), the set

$$W = \mathbb{R}^k \setminus (f(\partial U) \cup f(K))$$

of the points y for which (f, U, y) is a regular triple is open. As already pointed out, for any $y \in W$, the set $f^{-1}(y)$ is compact and discrete, therefore finite.

We need the following lemma which cannot be considered as a particular case of Theorem 3.13 since the latter has been deduced from the three Fundamental Properties.

Lemma 5.1. *Let U, f and W be as above. Then the map*

$$\deg(f, U, \cdot) : W \to \mathbb{Z},$$

is locally constant.

Proof. Fix any $y \in W$ and let $f^{-1}(y) = \{x_1, x_2, \cdots, x_n\}$. Because of the Inverse Function Theorem, there exist n pairwise disjoint neighborhoods $U_1, U_2, \ldots, U_n$

of $x_1, x_2, \ldots, x_n$ which are mapped diffeomorphically onto neighborhoods V_1, V_2, ..., V_n of y. We may assume that each V_i is contained in W and that in each U_i the sign of $\det(f'(x))$ is constant. Therefore, if $z \in V := V_1 \cap V_2 \cap \cdots \cap V_n$ and $\Omega := U_1 \cup U_2 \cup \cdots \cup U_n$, the equation $f(x) = z$ has exactly n solutions in Ω and

$$\deg(f, U, y) = \deg(f, \Omega, z).$$

Observe now that, when $z \in \mathbb{R}^k$ is sufficiently close to y, the equation $f(x) = z$ has no solutions in $C = \overline{U} \setminus \Omega$. Indeed, this happens if z does not belong to $f(C)$, which is a closed subset of $\mathbb{R}^k$ not containing y. Thus, if z belongs to the neighborhood $V \setminus f(C)$ of y, we obtain

$$\deg(f, U, z) = \deg(f, \Omega, z) = \deg(f, U, y),$$

and the assertion is proved. □

Before proving the Homotopy Invariance Property for regular triples we recall some important facts regarding the family of (ordered) bases in a finite dimensional vector space.

Let $\Sigma_k = (e_1, e_2, \cdots, e_k)$ denote the standard basis of $\mathbb{R}^k$; that is,

$$e_1 = (1, 0, 0, \cdots, 0),\ e_2 = (0, 1, 0, \cdots, 0),\ \ldots,\ e_k = (0, \cdots, 0, 1).$$

A basis B of $\mathbb{R}^k$ is said to be *positively oriented* (in $\mathbb{R}^k$) if it is *equivalent* to Σ_k; meaning that the transition matrix from Σ_k to B has positive determinant. If this is not the case, B is *negatively oriented*. The *status* of a basis B of being positively or negatively oriented is obviously stable, since the transition matrix depends continuously on B in the topology of $(\mathbb{R}^k)^k$. Moreover, replacing just one vector of a basis B with its opposite makes B pass from one status to the other one (from positively oriented to negatively oriented or vice-versa). Finally, we point out that an ordered basis B of $\mathbb{R}^k$ is positively oriented if and only if so is the basis (B, e_{k+1}) of $\mathbb{R}^{k+1}$ obtained by adding to B (regarded as a basis of the subspace $\mathbb{R}^k \times \{0\}$ of $\mathbb{R}^{k+1}$) the last vector of the standard basis Σ_{k+1} of $\mathbb{R}^{k+1}$.

Given a k-dimensional vector space E and a linear isomorphism $L : E \to \mathbb{R}^k$, by $L^{-1}(\Sigma_k)$ we mean the preimage under L of the standard basis Σ_k. This, of course, is a basis of E. It is important to observe that, if L is an automorphism of $\mathbb{R}^k$, $\det(L) > 0$ if and only if $L^{-1}(\Sigma_k)$ is a positively oriented basis of $\mathbb{R}^k$. This elementary fact turns out to be crucial in the following proof.

Proof of the Homotopy Invariance Property for regular triples. Recall first that a triple (f, U, y) is regular if and only if so is $(f - y, U, 0)$, and in this case

$$\deg(f, U, y) = \deg(f - y, U, 0).$$

Therefore, putting $G(x, \lambda) = H(x, \lambda) - \alpha(\lambda)$, it is enough to show that the degree of the two regular triples $(G_0, U, 0)$ and $(G_1, U, 0)$ is the same, where, as usual, G_λ denotes the partial map $G(\cdot, \lambda)$.

Apply Lemma 5.1 to find an open neighborhood V of $0 \in \mathbb{R}^k$ made up of regular values for both G_0 and G_1 and such that

$$\deg(G_0, U, z) = \deg(G_0, U, 0) \quad \text{and} \quad \deg(G_1, U, z) = \deg(G_1, U, 0),$$

for all $z \in V$. Because of Sard's Lemma, there exists a regular value $y \in V$ for G in $U \times [0,1]$, and not only for the restriction of G to the boundary (in the sense of manifolds)

$$\delta(U \times [0,1]) = (U \times \{0\}) \cup (U \times \{1\}).$$

The assertion now follows if we show that

$$\deg(G_0 - y, U, 0) = \deg(G_1 - y, U, 0).$$

Therefore, we are reduced to proving that *if* $(F, U, 0)$ *is a smooth admissible homotopy of triples, and* 0 *is a regular value for* F *and for the partial maps* F_0 *and* F_1*, then* $\deg(F_0, U, 0) = \deg(F_1, U, 0)$.

Assume that F is such a homotopy. Since 0 is a regular value both for F and the restriction of F to $\delta(U \times [0,1])$, the Regularity Theorem (see e.g. [5, 6, 8]) for manifolds with boundary ensures that $F^{-1}(0)$ is a compact 1-dimensional manifold whose boundary is given by

$$\delta F^{-1}(0) = F^{-1}(0) \cap \delta(U \times [0,1]).$$

The points of $\delta F^{-1}(0)$ can be divided in two classes: $A_0 = F_0^{-1}(0) \times \{0\}$ and $A_1 = F_1^{-1}(0) \times \{1\}$, both finite since $0 \in \mathbb{R}^k$ is a regular value for the partial maps F_0 and F_1.

Any point in $\delta F^{-1}(0)$ can be given a sign $+1$ or -1 as follows: if $p = (x, \lambda) \in \delta F^{-1}(0)$, we put $\operatorname{sign}(p) = \operatorname{sign}(\det(F'_\lambda(x)))$. Thus, we need to prove that

$$\sum_{p \in A_0} \operatorname{sign}(p) = \sum_{p \in A_1} \operatorname{sign}(p).$$

This will be done by showing that any point $p \in \delta F^{-1}(0)$ has a unique companion $c(p) \in \delta F^{-1}(0)$ with the property that $\operatorname{sign}(p) = -\operatorname{sign}(c(p))$ if and only if both p and $c(p)$ belong to the same side (A_0 or A_1).

Recall that any smooth, compact, connected 1-dimensional real manifold with nonempty boundary (called an *arc*) is diffeomorphic to the interval $[0,1]$.[2] Therefore, any $p \in \delta F^{-1}(0)$ is an endpoint of an arc (the connected component of $\delta F^{-1}(0)$ containing p) having the other endpoint $c(p)$ still in $\delta F^{-1}(0)$. Incidentally, observe that the self-map c of $\delta F^{-1}(0)$ is a bijection (in fact, $c^{-1} = c$).

Consider, for example, the case when the endpoints p_0 and p_1 of an arc M contained in $F^{-1}(0)$ are both in A_0. We need to show that these two points have opposite sign. The other two cases (both the endpoints in A_1, or one in A_0 and the other in A_1) can be treated in a similar way, and their discussion will be omitted.

Roughly speaking, in order to prove that the two endpoints $p_0 = (x_0, 0)$ and $p_1 = (x_1, 0)$ of M have opposite sign we move, continuously, a basis B_t of $\mathbb{R}^{k+1}$ along M in such a way that at the departure (for $t = 0$) the basis coincides with

$$(F'_0(x_0)^{-1}(\Sigma_k), e_{k+1})$$

and at the arrival (for $t = 1$) coincides with

$$(F'_0(x_1)^{-1}(\Sigma_k), -e_{k+1}),$$

[2]This is a consequence of a well-known classification theorem for smooth 1-dimensional real manifolds with (possibly empty) boundary. See e.g., [5, 8].

where, we recall, F_0 stands for the partial map $F(\cdot, 0)$. Since B_t is a basis for all $t \in [0,1]$, the determinant of the transition matrix from B_t to Σ_{k+1} has constant sign. Thus, the two bases B_0 and B_1 turn out to be either both positively oriented or both negatively oriented. As a consequence of this, if, for example, the basis $F_0'(x_0)^{-1}(\Sigma_k)$ of $\mathbb{R}^k$ is positively oriented, the other basis $F_0'(x_1)^{-1}(\Sigma_k)$ must be negatively oriented, and this implies

$$\operatorname{sign}(\det(F_0'(x_0))) = -\operatorname{sign}(\det(F_0'(x_1))),$$

showing that the two endpoints of the arc M have opposite sign.

Let $\gamma(t) = (x(t), \lambda(t))$, $t \in [0,1]$, be a parametrization of the arc M. In other words, $\gamma : [0,1] \to M$ is a diffeomorphism from $[0,1]$ onto M, that we may assume to be oriented from p_0 to p_1 (i.e. $\gamma(0) = p_0$ and $\gamma(1) = p_1$).

Since $F(\gamma(t)) \equiv 0$, we have $F'(\gamma(t))\gamma'(t) \equiv 0$. Moreover, the assumption that γ is a diffeomorphism implies $\gamma'(t) \neq 0$ for all $t \in [0,1]$. Therefore, given t, the kernel of the surjective operator $F'(\gamma(t)) : \mathbb{R}^{k+1} \to \mathbb{R}^k$, which is 1-dimensional, is spanned by $\gamma'(t)$. This implies that the restriction of $F'(\gamma(t))$ to any k-dimensional subspace E of $\mathbb{R}^{k+1}$ not containing $\gamma'(t)$ is an isomorphism, and, consequently, the preimage of the standard basis Σ_k of $\mathbb{R}^k$ under this isomorphism is a basis of E.

To simplify the notation, given a point p in $U \times [0,1]$ and a subspace E of $\mathbb{R}^{k+1}$, if the restriction $F'(p)|_E$ of $F'(p)$ to E is an isomorphism, the preimage $(F'(p)|_E)^{-1}(\Sigma_k)$ of the standard basis Σ_k will be denoted by $\Sigma(p, E)$.

Since $\gamma'(t) = (x'(t), \lambda'(t))$ is a nonzero vector for any $t \in [0,1]$ and the partial derivatives $\partial_1 F(x(0), \lambda(0))$ and $\partial_1 F(x(1), \lambda(1))$ are invertible (recall that $\lambda(0) = \lambda(1) = 0$, and 0 is a regular value for the partial map $F(\cdot, 0)$), the identity

$$\partial_1 F(x(t), \lambda(t))x'(t) + \lambda'(t)\partial_2 F(x(t), \lambda(t)) \equiv 0$$

yields $\lambda'(0) \neq 0$ and $\lambda'(1) \neq 0$. The fact that $\lambda(t) \in [0,1]$ for all $t \in [0,1]$ actually implies $\lambda'(0) > 0$ and $\lambda'(1) < 0$. In other words, denoting by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^{k+1}$, we have $\langle \gamma'(0), e_{k+1} \rangle > 0$ and $\langle \gamma'(1), e_{k+1} \rangle < 0$.

Define a point p_t moving along the arc M by

$$p_t = \begin{cases} p_0 & \text{if } t \in [0, 1/3] \\ \gamma(3t-1) & \text{if } t \in [1/3, 2/3] \\ p_1 & \text{if } t \in [2/3, 1] \end{cases}$$

For any $t \in [0,1]$, define the vector $v_t \in \mathbb{R}^{k+1}$ by

$$v_t = \begin{cases} (1-3t)e_{k+1} + 3t\gamma'(0) & \text{if } t \in [0, 1/3] \\ \gamma'(3t-1) & \text{if } t \in [1/3, 2/3] \\ (3-3t)\gamma'(1) - (3t-2)e_{k+1} & \text{if } t \in [2/3, 1] \end{cases}$$

Observe that $v_t \neq 0$ for any $t \in [0,1]$. Thus the orthogonal space $v_t^\perp$ to v_t is always k-dimensional. Let us prove that the restriction of the derivative $F'(p_t)$ to $v_t^\perp$ is an isomorphism for all $t \in [0,1]$. For this purpose, given $t \in [0,1]$, we need to show that $v_t^\perp$ does not contain the one dimensional kernel of $F'(p_t)$. Since $0 \in \mathbb{R}^k$ is a regular value for F, this kernel coincides with the tangent space to M at p_t, which is spanned by $\gamma'(0)$ if $t \in [0,1/3]$, by $\gamma'(3t-1)$ if $t \in [1/3, 2/3]$ and by $\gamma'(1)$ if $t \in [2/3, 1]$.

Consider first the case of $t \in [0,1/3]$. We need to show that $\gamma'(0)$ does not belong to $v_t^\perp$, which means $\langle v_t, \gamma'(0)\rangle \neq 0$. In fact, since $\lambda'(0) > 0$, we have

$$\langle v_t, \gamma'(0)\rangle = (1-3t)\lambda'(0) + 3t\|\gamma'(0)\|^2 > 0.$$

If $t \in [1/3, 2/3]$,

$$\langle v_t, \gamma'(3t-1)\rangle = \|\gamma'(3t-1)\|^2 > 0.$$

Finally, it $t \in [2/3, 1]$, $\lambda'(1)$ being negative, one gets

$$\langle v_t, \gamma'(1)\rangle = (3-3t)\|\gamma'(1)\|^2 - (3t-2)\lambda'(1) > 0.$$

Let $t \in [0,1]$. Since, as claimed, the restriction of $F'(p_t)$ to $v_t^\perp$ is an isomorphism, it makes sense to define the following basis of $\mathbb{R}^{k+1}$:

$$B_t = (\Sigma(p_t, v_t^\perp), v_t).$$

Clearly B_t depends continuously on $t \in [0,1]$, as a map into $\left(\mathbb{R}^{k+1}\right)^{k+1}$. Observe also that the spaces $v_0^\perp$ and $v_1^\perp$ coincide with $\mathbb{R}^k \times \{0\}$. Therefore, identifying $\mathbb{R}^k$ with the subspace $\mathbb{R}^k \times \{0\}$ of $\mathbb{R}^{k+1}$, we have

$$B_0 = (F_0'(x_0)^{-1}(\Sigma_k), e_{k+1}) \quad \text{and} \quad B_1 = (F_0'(x_1)^{-1}(\Sigma_k), -e_{k+1}).$$

Now, as already pointed out, the fact that B_t is always a basis for $\mathbb{R}^{k+1}$ implies that B_0 and B_1 are either both positively oriented or both negatively oriented. Consequently, B_0 and

$$B_1^- = (F_0'(x_1)^{-1}(\Sigma_k), e_{k+1})$$

have opposite orientation, which implies that also the two bases

$$F_0'(x_0)^{-1}(\Sigma_k) \quad \text{and} \quad F_0'(x_1)^{-1}(\Sigma_k)$$

have opposite orientation. Thus,

$$\operatorname{sign}(\det(F_0'(x_0))) = -\operatorname{sign}(\det(F_0'(x_1))),$$

and the two endpoints p_0 and p_1 of the arc M have opposite sign, as claimed. $\square$

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