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MULTIPLICITY OF FORCED OSCILLATIONS FOR SCALAR DIFFERENTIAL EQUATIONS

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Abstract. We give, via topological methods, multiplicity results for small periodic perturbations of scalar second order differential equations. In particular, we show that the equation

$$\ddot{x} = g(x) + \varepsilon f(t, x, \dot{x}),$$

where g is C^1 and f is continuous and periodic in t, has n forced oscillations, provided that g changes sign n times ($n > 1$).

1. Introduction

Despite the illusory simplicity of the equations considered, the problems of existence and multiplicity of periodic solutions for periodically forced second order scalar autonomous differential equations have been the subject of extensive studies. Classical references for this topic are e.g. the books [12, 19, 20]. Currently, although the research is now often pursued with different methods, the activity in this field is still vigorous and it is impossible to give here a complete account even of the most recent results in this field. We confine ourselves to mentioning some papers, as for instance [3, 2, 5, 15, 18, 22] along with their references, where various problems of existence and multiplicity of forced oscillations are investigated using different methods.

In this paper we consider parametrized second order differential equations on $\mathbb{R}$ of the form (the simpler case when damping is present was treated in [8]):

$$\ddot{x} = g(x) + \lambda f(t, x, \dot{x}), \quad \lambda \geq 0, \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous. We are interested in the *forced oscillations* of (1.1); that is, those periodic solutions with the same period as the forcing term f. We shall prove (Theorem 3.6) that if the function g changes sign n times, $n > 1$, then, for small values of λ, (1.1) admits *at least* n forced oscillations, provided that the uniqueness of solutions of the Cauchy problem for the unperturbed equation $\ddot{x} = g(x)$ holds.

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In the case $n = 1$ the result is true with an extra assumption on the unperturbed equation (Theorem 3.5). Namely, the non-T-isochronism of $\ddot{x} = g(x)$ is the key condition ensuring the existence of forced oscillations for the perturbed equation (1.1). Such a condition extends to the nonlinear case the non-T-resonance hypothesis for the linear equation

$$\ddot{x} = -ax + \lambda \sin \omega t\,.$$

Theorems 3.5 and 3.6 are obtained combining an analysis of the periodic orbits of the unperturbed equation $\ddot{x} = g(x)$ with a point-set topology result (Lemma 2.6) which gathers some previously known connectivity results (see e.g. [1, 11] and [6]), which are in the spirit of the so called Wyburn Lemma.

We note in passing that our results give only a *lower bound* which might be well below the actual number of forced oscillations of (1.1). In fact, it follows from a theorem of [4] that when $\lim_{x\to\pm\infty} g(x)/x = +\infty$, f depends only on t and has a minimal period $T > 0$, then (1.1) admits infinitely many T-periodic solutions for any $\lambda > 0$ (see also e.g. [16, 17] for the case $n = 1$).

The results described in this paper are not merely obtained by specializing to $\mathbb{R}$ the techniques that were exhibited in [7, 8] for the more general case of ODEs on manifolds. The spirit of Theorems 3.5 and 3.6 is indeed quite different from that of those papers which depended essentially on a different kind of connectivity result (Proposition 2.4).

2. Ejecting sets and T-pairs

We will denote by $C^1_T(\mathbb{R})$ the Banach space of all the T-periodic C^1 maps $x : \mathbb{R} \to \mathbb{R}$ with the usual C^1 norm.

A pair $(\lambda, x) \in [0,\infty) \times C^1_T(R)$ is called a *T-pair for the second-order equation* (1.1) if x is a solution of (1.1) corresponding to λ. In particular we will say that (λ, x) is *trivial* if $\lambda = 0$ and x is constant. Note that, in general, there may exist nontrivial T-pairs of (1.1) even for $\lambda = 0$.

One can show that the set of T-pairs of (1.1) is always a closed, locally compact subset of $[0,\infty) \times C^1_T(\mathbb{R})$ (see e.g. [6] or [9]). Moreover, any bounded closed subset of T-pairs is compact.

As in [10], we tacitly assume some natural identifications. That is, we will regard every space as its image in the following diagram of closed embeddings:

$$\begin{tikzcd} {[0,\infty) \times \mathbb{R}} \arrow[r] & {[0,\infty) \times C^1_T(\mathbb{R})} \\ \mathbb{R} \arrow[r] \arrow[u] & C^1_T(\mathbb{R}), \arrow[u] \end{tikzcd} \tag{2.1}$$

where the horizontal arrows are defined by regarding any point p in $\mathbb{R}$ as the constant map $\hat{p}(t) \equiv p$ in $C^1_T(\mathbb{R})$, and the two vertical arrows are the natural identifications $p \mapsto (0, p)$ and $x \mapsto (0, x)$.

According to these embeddings, we say that $Z \subset [0,\infty) \times C^1_T(\mathbb{R})$ meets $K \subset \mathbb{R}$ if there exists a point $p \in K$ such that the pair $(0, \hat{p})$ belongs to Z. In this case we will say that p belongs to Z.

We will make use of the following consequence of Corollary 4.4 in [10] regarding the global structure of the set of T-pairs of (1.1).

Theorem 2.1. *If g changes sign at some (isolated) zero $p \in \mathbb{R}$, then* (1.1) *admits a connected set of nontrivial T-pairs whose closure meets p and is either unbounded or intersects $g^{-1}(0) \setminus \{p\}$.*

According to the fact that a T-pair might have $\lambda = 0$, it turns out that the connected set in Theorem 2.1 might be entirely contained in the slice $\{0\} \times C^1_T(\mathbb{R})$. This happens for instance for the following equation (with $T = 2\pi$):

$$\ddot{x} = -x + \lambda \sin t, \quad \lambda \geq 0.$$

In order to find multiplicity results for the forced oscillations of (1.1) we need to avoid such a "degenerate" situation. We tackle this problem from an abstract viewpoint.

We need some notation. Let S be a metric space and let C be a subset of $[0, \infty) \times S$. Given $\lambda \geq 0$, we denote by C_λ the slice $\big\{y \in S \mid (\lambda, y) \in C\big\}$. In what follows, S will be identified with the subset $\{0\} \times S$ of $[0, \infty) \times S$.

Definition 2.2. *Let C be a subset of $[0, \infty) \times S$. We say that a subset A of C_0 is an* ejecting set for C *if it is relatively open in C_0 and there exists a connected subset of C which meets A and is not included in C_0.*

We shall simply say that $p \in C_0$ is an ejecting point if $\{p\}$ is an ejecting set. In this case p is clearly isolated in C_0.

Using compactness arguments, it is not difficult to show the following lemma which relates ejecting sets and multiplicity results.

Lemma 2.3. *Let S be a metric space and let C be a locally compact subset of $[0, \infty) \times S$. Assume that C_0 contains n pairwise disjoint compact ejecting sets for C. Then, there exists $\delta > 0$ such that the cardinality of C_λ is greater than or equal to n, for $\lambda \in [0, \delta)$.*

In fact, in [7] we proved the following stronger result.

Proposition 2.4. *Let S be a metric space and let C be a locally compact subset of $[0, \infty) \times S$. Assume that C_0 contains n pairwise disjoint ejecting sets, $n - 1$ of which are compact. Then, there exists $\delta > 0$ such that the cardinality of C_λ is greater than or equal to n, for $\lambda \in [0, \delta)$.*

In [7] we also provided examples showing that in the above proposition the compactness assumption on $n - 1$ ejecting sets cannot be dropped.

In the sequel, we shall need an extension (Lemma 2.6 below) of the following well-known connectivity result (see e.g. [1] and [11, chapter V]).

Lemma 2.5. *Let Y be a compact Hausdorff space and let Y_1 and Y_2 be disjoint closed subsets of Y. Then either there exists a connected subset of $Y \setminus (Y_1 \cup Y_2)$ whose closure intersects Y_1 and Y_2 or there exist disjoint compact and open sets K_1 and K_2 in Y such that $K_1 \supset Y_1$ and $K_2 \supset Y_2$.*

The following result reduces to Lemma 2.5 when $n = 2$ and Y is compact. Besides, when $n = 1$ one gets Lemma 1.4 of [6].

Lemma 2.6. *Let Y be a locally compact Hausdorff topological space and let $Y_1, \dots, Y_n$, $n \geq 1$, be pairwise disjoint compact subsets of Y. Then the following alternative holds:*

(1) *there exists n pairwise disjoint compact open subsets $A_1, \dots, A_n$ of Y containing $Y_1, \dots, Y_n$ respectively;*

(2) *there exists a connected set of* $Y \setminus \bigcup_{i=1}^{n} Y_i$ *whose closure in* Y *meets* $\bigcup_{i=1}^{n} Y_i$ *and has one of the following properties:*
  (a) *it is not compact;*
  (b) *meets at least two different* Y_i*'s.*

Proof. We distinguish two cases.

Case 1 (Y *is compact*). In this case the proof is by induction on n.

For $n = 1$, the assertion holds clearly true since one can take $A_1 = Y$. If $n > 1$ assume the assertion true for $n - 1$. Apply Lemma 2.5 to the two disjoint closed subsets Y_n and $\bigcup_{i=1}^{n-1} Y_i$ of Y. Then, either there exists a connected set whose closure meets Y_n and one of the remaining Y_i's, or there exist disjoint compact open sets $K \supset Y_n$ and $H \supset \bigcup_{i=1}^{n-1} Y_i$. In the latter case, the assertion follows applying the inductive assumption to the $n - 1$ sets $Y_1, \dots, Y_{n-1}$ contained in the compact space H.

Case 2 (Y *is not compact*). Let $\widehat{Y} = Y \cup \{\infty\}$ be the one-point compactification of Y. Then, applying Case 1 to the Hausdorff space $\widehat{Y}$ we have that either there exist $n + 1$ pairwise disjoint compact open neighborhoods of $Y_1, \dots, Y_n, \{\infty\}$ (so that alternative (1) of the assertion holds) or there exists a connected subset C of

$$\widehat{Y} \setminus \Big(\bigcup_{i=1}^{n} Y_i \cup \{\infty\} \Big)$$

whose closure meets two different sets among $Y_1, \dots, Y_n, \{\infty\}$. In this case, alternative (2) of the assertion holds. In fact, there are two possibilities (not mutually exclusive) corresponding to (a) and (b) of (2):

- the closure of C meets $\{\infty\}$ and (at least) one of the Y_i's, and, thus, the closure of C in Y cannot be compact;
- the closure of C meets Y_i and Y_j for some $i \neq j$.

The proof is now complete. □

3. Application to second order scalar equations

In this section we will be concerned with the scalar equation

$$\ddot{x} = g(x) + \lambda f(t, x, \dot{x}), \quad \lambda \geq 0, \tag{3.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous and f is T-periodic in t (the period $T > 0$ is given).

Consider the unperturbed equation

$$\ddot{x} = g(x). \tag{3.2}$$

We shall always assume the uniqueness of the solutions of the Cauchy problems associated with (3.2).

The potential energy is any primitive V of $-g$, while the total mechanical energy, which is a first integral for (3.2), is

$$E(s, v) = \frac{v^2}{2} + V(s).$$

Lemma 3.1. *If $x(\cdot)$ is a periodic solution of (3.2), its image $[\alpha_x, \beta_x]$ is such that*
$$V(s) \leq V\big(\alpha_x\big) = V\big(\beta_x\big), \tag{3.3}$$
for any $s \in [\alpha_x, \beta_x]$. Consequently, if p is an isolated zero of g which is not a local minimum point of V, then there exists a neighborhood U of p with the property that there are no periodic solutions of (3.2) with image in U different from the constant $\hat{p}(t) \equiv p$.

Proof. Let x be a periodic solution of period T and let $t_0, t_1 \in [0, T]$ be such that $x(t_0) = \alpha_x$ and $x(t_1) = \beta_x$. Clearly, one has $\dot{x}(t_0) = 0 = \dot{x}(t_1)$, therefore
$$V\big(\alpha_x\big) = E\big(x(t_0), \dot{x}(t_0)\big) = E\big(x(t_1), \dot{x}(t_1)\big) = V\big(\beta_x\big).$$
This yields the inequality (3.3), since for any $t \in [0, T]$
$$V\big(\alpha_x\big) = V\big(\beta_x\big) = \frac{\big(\dot{x}(t)\big)^2}{2} + V\big(x(t)\big) \geq V\big(x(t)\big).$$

Let us prove the last assertion. Let p be an isolated zero of g which is not a local minimum point for V. Since $V'(s) = -g(s)$, there exists a neighborhood U of p which does not contain minimum points of V. Then the inequality (3.3) yields the assertion. □

Lemma 3.2. *Let $\mathcal{G} \subset C^1_T(\mathbb{R})$ be a connected set of solutions of (3.2) containing a zero p of g in which g changes sign. Then one has $\alpha_x \leq p \leq \beta_x$ for any $x \in \mathcal{G}$, where α_x and β_x are as in Lemma 3.1. Moreover, if one of the two intervals*
$$A_p := \{\alpha_x \mid x \in \mathcal{G}\}, \quad B_p := \{\beta_x \mid x \in \mathcal{G}\}$$
is nontrivial, then the other is nontrivial as well, and V is decreasing on A_p and increasing on B_p. In addition, $A_p \cap B_p = \{p\}$.

Proof. If p is not a local minimum point for V, Lemma 3.1 implies that $\mathcal{G}$ reduces to $\{p\}$, and the assertions hold trivially. Assume, therefore, that p is a local minimum point for V.

Let us prove first that $\alpha_x \leq p$ for any $x \in \mathcal{G}$. Put $D = \{x \in \mathcal{G} \mid \alpha_x \leq p\}$. Clearly D is nonempty as it contains p, and is closed since $x \mapsto \alpha_x$ is continuous. Let us show that D is open in $\mathcal{G}$. It is enough to prove that, given $\bar{x} \in D$, if $\alpha_{\bar{x}} = p$ then $\bar{x}$ lays in the interior of D. Assume by contradiction $\bar{x}$ not in the relative interior of D in $\mathcal{G}$. Since p is a local minimum point of V, the inequality (3.3) of Lemma 3.1 yields $\beta_{\bar{x}} = p$, so that $\bar{x}(t) \equiv p$. Let $\delta > 0$ be such that $g(s) \neq 0$ in $(p, p + \delta)$. The fact that $\bar{x}$ is not in the relative interior of D implies the existence of $x \in \mathcal{G}$, sufficiently close to $\bar{x}$, such that $p < \alpha_x$. As $\bar{x}$ is constant, by the continuity of the map $x \mapsto \beta_x$, we may assume that $\beta_x < p + \delta$. Since V is strictly increasing in $(p, p + \delta)$, again by (3.3), one has $\alpha_x = \beta_x$. Therefore $x(t)$ is constantly equal to some constant $q \in (p, p + \delta)$. Consequently $g(q) = 0$, which is a contradiction. Thus D is open in $\mathcal{G}$, and $\alpha_x \leq p$ for any $x \in \mathcal{G}$, as claimed. Analogously $p \leq \beta_x$ for all $x \in \mathcal{G}$.

Suppose now that A_p does not reduce to $\{p\}$. Let us prove first that V is decreasing in A_p. Assume by contradiction that this is not the case. Then there exist a, $b \in A_p$, $a < b$, such that $V(a) < V(b)$. Since V is C^1, there exist $s_1, s_2 \in (a, b)$, with $s_1 < s_2$, such that $V'(s) > 0$ in $[s_1, s_2]$. Consequently, given $x \in \mathcal{G}$ such that $\alpha_x = s_1$, by Lemma 3.1 one has $\beta_x = s_1$ as well. Thus x is a constant solution of (3.2). This implies $V'(s_1) = 0$, which is a contradiction. The fact that if B_p is nontrivial then V is increasing in B_p can be proved in an analogous way.

Let us show now that if A_p is nontrivial, then so is B_p. Since p is an isolated zero of g in which V attains a local minimum, there exists a left neighborhood of p where V is strictly decreasing. Consequently there exists $x \in \mathcal{G}$ such that $V(\alpha_x) = V(\beta_x) > V(p)$, and B_p is nontrivial. A symmetric argument shows that when B_p is nontrivial so is A_p.

It remains to show that the interval $A_p \cap B_p$, which clearly contains p, reduces to $\{p\}$. In fact, if this were not the case, the function V would be both decreasing and increasing in $A_p \cap B_p$, contradicting the assumption that g changes sign at p. □

We say that the equation (3.2) is T-*isochronous* if all its solutions are T-periodic.

Lemma 3.3. *Let $\mathcal{G}$, p, A_p and B_p be as in Lemma 3.2. Assume the equation* (3.2) *non-T-isochronous. Then $A_p \cup B_p \neq \mathbb{R}$.*

Proof. Assume by contradiction $A_p \cup B_p = \mathbb{R}$. By Lemma 3.2, V is decreasing on A_p and increasing on B_p. Therefore V attains its minimum at p. Let $\xi : J \to \mathbb{R}$ be a maximal non-T-periodic solution of (3.2) (recall that equation (3.2) is assumed non-T-isochronous). Since V is bounded from below, then $|\dot{\xi}(\cdot)|$ is bounded. This clearly implies that both the interval J and the image of ξ coincide with $\mathbb{R}$.

Denote by K the total energy of ξ, i.e.

$$E\left(\xi(t), \dot{\xi}(t)\right) = \frac{1}{2}[\dot{\xi}(t)]^2 + V\big(\xi(t)\big) = K, \quad \text{for all } t \in \mathbb{R}.$$

Let us show that the total energy of any $x \in \mathcal{G}$ is bounded above by K. To this end, take $x \in \mathcal{G}$ and let $t_0 \in \mathbb{R}$ be such that $\dot{x}(t_0) = 0$. Since $\xi(\mathbb{R}) = \mathbb{R}$, there exists $t_1 \in \mathbb{R}$ such that $\xi(t_1) = x(t_0)$. Therefore,

$$K = \frac{1}{2}[\dot{\xi}(t_1)]^2 + V\big(\xi(t_1)\big) \geq V\big(x(t_0)\big).$$

Since $\dot{x}(t_0) = 0$, $V\big(x(t_0)\big)$ represents the total energy of the solution x. Consequently,

$$E\left(x(t), \dot{x}(t)\right) = V\big(x(t_0)\big) \leq K, \quad \text{for all } t \in \mathbb{R}$$

as claimed.

Again the fact that V is bounded from below implies the existence of a constant H such that $|\dot{x}(t)| \leq H$ for all $t \in \mathbb{R}$. By integrating and recalling that, as proved in Lemma 3.2, p belongs to the image of x, one obtains

$$|x(t) - p| \leq HT \quad \text{for all } t \in \mathbb{R}.$$

Thus $A_p \cup B_p \subset [p - HT, p + HT]$, contradicting the assumption. □

Lemma 3.4. *Assume that the T-isochronism property does not hold for* (3.2)*. Let $\mathcal{G} \subset C^1_T(\mathbb{R})$ be a connected component of the set of T-periodic solutions of* (3.2) *containing a zero p of g in which g changes sign. Then $\mathcal{G}$ is compact and does not meet any zeros of g different from p.*

Proof. Let us prove first that $\mathcal{G}$ is compact. Define, as before, the intervals

$$A_p := \{\alpha_x \mid x \in \mathcal{G}\}, \quad B_p := \{\beta_x \mid x \in \mathcal{G}\}.$$

Since, by assumption, (3.2) is non-T-isochronous, by Lemma 3.3 we have $A_p \cup B_p \neq \mathbb{R}$. Since by Lemma 3.2 one has $A_p \cap B_p = \{p\}$, at least one of the two intervals, say B_p, is bounded. Given any $x \in \mathcal{G}$, one has

$$E\big(x(t), \dot{x}(t)\big) = E(\beta_x, 0) = V(\beta_x) \leq \sup V(B_p) < +\infty\,.$$

Therefore one has

$$|\dot{x}(t)| \leq k := \sqrt{2\big(\sup V(B_p) - \inf V(B_p)\big)},$$

and, since $p \in x\big([0,T]\big)$,

$$|x(t) - p| \leq Tk, \quad \text{for any } t \in [0,T].$$

Thus $\mathcal{G}$ is bounded in $C^1_T(\mathbb{R})$. The compactness of $\mathcal{G}$ now follows from the fact that it is a closed subset of the set of T-periodic solutions of (3.2).

Finally, if $q \in g^{-1}(0)$ belongs to $\mathcal{G}$, then $\alpha_q = \beta_q = q$. Consequently, by Lemma 3.2, one has $q = p$. $\square$

The following theorem clears the way to the main result of this section.

Theorem 3.5. *Assume that in equation* (3.1) *the force g satisfies the following conditions:*

- *g changes sign at n zeros;*
- *the T-isochronism property does not hold.*

Then there exists $\delta > 0$ such that (3.1) *has at least n forced oscillations for $\lambda \in [0, \delta)$.*

Proof. Let $p_1, \dots, p_n$ be zeros at which g changes sign. We shall prove that there exist n pairwise disjoint compact ejecting sets for the set X of T-pairs of (3.1) containing $p_1, \dots, p_n$. Then, the assertion will follow from Lemma 2.3.

By Lemma 3.4 we get that alternative (2) in Lemma 2.6 does not hold, therefore there exist n pairwise disjoint compact and open subsets $A_1, \dots, A_n$ of the slice

$$X_0 = \{x \in C^1_T(\mathbb{R}) \mid x \text{ is a solution of (3.2)}\}$$

containing $p_1, \dots, p_n$ respectively.

Since g changes sign at the points $p_1, \dots, p_n$, thus, by Theorem 2.1, for any $i = 1, \dots, n$, there exists a connected set Γ^i of nontrivial T-pairs for (3.1) whose closure $\overline{\Gamma^i}$ meets p_i and is either non-compact or intersects $g^{-1}(0) \setminus \{p_i\}$. By Lemma 3.4 we get $\overline{\Gamma^i} \not\subset X_0$. This implies that all the A_i's are ejecting sets for X. $\square$

We observe that the non-T-isochronism assumption in Theorem 3.5 is not very restrictive. In fact (see e.g. [13]), the unique odd continuous function g for which T-isochronism holds is

$$g(s) = -\left(\frac{2\pi}{T}\right)^2 s.$$

See also [21] along with the references therein, where a discussion on isochronal oscillations around a zero of g can be found.

If the function g changes sign at least two times, then the potential energy has a maximum point p. Thus, given a neighborhood U of p, the solutions of (3.2) starting close to p (at $t = 0$) remain in U for all $t \in [0, T]$ (recall we are assuming the uniqueness of the solutions of the Cauchy problem). Consequently, by Lemma

3.1, the equation (3.2) is not T-isochronous. We can therefore state our main multiplicity result:

Theorem 3.6. *Assume that in equation* (3.1) *the force g changes sign at $n > 1$ zeros. Then there exists $\delta > 0$ such that* (3.1) *has at least n forced oscillations for $\lambda \in [0, \delta)$.*

Observe that when, in equation (3.1), $g^{-1}(0)$ consists of isolated points and g changes sign at infinitely many zeros, Theorem 3.6 implies that, given $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that (3.1) admits at least n forced oscillations for any $\lambda \in [0, \delta_n)$. This does not mean that one necessarily has infinitely many T-periodic solutions for small $\lambda > 0$, as illustrated by the following equation where we set $T = 2\pi$

$$\ddot{x} = \sin x + \lambda(x - \dot{x}). \tag{3.4}$$

Clearly, due to the friction effect of the term $-\lambda\dot{x}$, the only possible T-periodic solutions are the zeros of $\sin x + \lambda x$. Thus (3.4) has a finite number of T-periodic solutions for any $\lambda > 0$.

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