# MULTIPLICITY OF FORCED OSCILLATIONS ON MANIFOLDS AND APPLICATIONS TO MOTION PROBLEMS WITH ONE-DIMENSIONAL CONSTRAINTS

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Abstract. This paper illustrates through some pysical examples how the notion of *ejecting set* ([3]) can be used to get multiplicity results for forced oscillations. The motion problem of a mass point constrained to one-dimensional manifold and acted on by a periodic force is treated.

## 1. Introduction

In this paper we continue the research of [3], where we obtained qualitative results for forced oscillations on differentiable (boundaryless) manifolds that cannot be deduced via variational or implicit function methods. More precisely, in [3] we considered "small" periodic perturbations of autonomous second order differential equations on differentiable manifolds and, under suitable assumptions, we established the existence of multiple forced oscillations.

In [3] we framed the problem in an abstract topological setting, so that the results arose from a combination of analytical and topological tools as well as from local and global results on the set of the so-called $T$-pairs (see below for a precise definition). In that framework the key notion was that of *ejecting set*.

In this paper we focus on some applications of the results of [3] and illustrate, through some physical examples, how the notion of ejecting set can be used to get multiplicity results. We treat in some detail the motion problem of a mass point constrained to a 1-dimensional manifold $M$ and acted on by a periodic force. We consider therefore the two cases $M = S^1$ and $M = \mathbb{R}$, which are, up to a diffeomorphism, the only connected 1-dimensional boundaryless differentiable manifolds.

A particular attention is devoted to the second order scalar equation

$$\ddot{x} = g(x) - \mu\dot{x} + \lambda f(t, x, \dot{x}), \quad \lambda \geq 0,$$

where $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous, $f$ is $T$-periodic in $t$ ($T > 0$ is given), and $\mu \geq 0$. When the parameter $\lambda$ is small enough, we establish multiplicity results for the $T$-periodic solutions of the above equation in two cases: when the force $g$ vanishes and the frictional coefficient $\mu$ is arbitrary, and when $g$ has isolated zeros and $\mu$ is positive. The remaining case when $\mu = 0$ and $g$ does not vanish identically requires a more careful treatment and will be the subject of a forthcoming paper.

## 2. Ejecting sets and $T$-pairs

Let $M$ be a differentiable manifold embedded in $\mathbb{R}^k$. Given $T > 0$, we denote by $C^1_T(M)$ the metric subspace of the Banach space $C^1_T(\mathbb{R}^k)$ of all the $T$-periodic $C^1$ maps $x : \mathbb{R} \to M$ with the usual $C^1$ norm. Observe that $C^1_T(M)$ is not complete, unless $M$ is complete (i.e. closed in $\mathbb{R}^k$). Nevertheless, since $M$ is locally compact, $C^1_T(M)$ is always locally complete.

Given $q \in M$, $T_qM \subset \mathbb{R}^k$ denotes the tangent space to $M$ at $q$. By

$$TM = \left\{(q,v) \in \mathbb{R}^k \times \mathbb{R}^k : q \in M, v \in T_qM\right\}$$

we mean the tangent bundle of $M$.

We consider second order differential equations on $M$ of the form

$$\ddot{x}_\pi = h(x,\dot{x}) + \lambda f(t,x,\dot{x}), \quad \lambda \geq 0, \tag{2.1}$$

where $\lambda$ is a parameter, $h : TM \to \mathbb{R}^k$ and $f : \mathbb{R} \times TM \to \mathbb{R}^k$ are *tangent to* $M$, in the sense that $h(q,v)$ and $f(t,q,v)$ belong to $T_qM$ for all $(t,q,v) \in \mathbb{R} \times TM$. Here the map $f$ is assumed $T$-periodic in $t$. A solution of (2.1) is a $C^2$ map $x : J \to M$, defined on a nontrivial interval $J$, such that

$$\ddot{x}_\pi(t) = h\left(x(t),\dot{x}(t)\right) + \lambda f\left(t,x(t),\dot{x}(t)\right), \quad \forall t \in J,$$

where $\ddot{x}_\pi(t)$ denotes the orthogonal projection of $\ddot{x}(t) \in \mathbb{R}^k$ onto $T_{x(t)}M$. A solution of (2.1) is called a *forced oscillation* if it is periodic of the same period $T$ as that of the forcing term $f$.

For a more extensive treatment of second-order ODEs on manifolds from this embedded viewpoint see e.g. [1].

A pair $(\lambda,x) \in [0,\infty) \times C^1_T(M)$ is called a *$T$-pair for the second-order equation* (2.1) if $x$ is a solution of (2.1) corresponding to $\lambda$. In particular we will say that $(\lambda,x)$ is *trivial* if $\lambda = 0$ and $x$ is constant. Note that, in general, there may exist nontrivial $T$-pairs of (2.1) even for $\lambda = 0$, as in the case of the inertial motion on $S^1$.

One can show that, no matter whether or not $M$ is closed in $\mathbb{R}^k$, the subset $X$ of $[0,\infty) \times C^1_T(M)$ consisting of all the $T$-pairs of (2.1) is always closed and locally compact (see e.g. [2] or [4]). Moreover, by Ascoli's theorem, when $M$ is closed in $\mathbb{R}^k$, any bounded closed set of $T$-pairs is compact.

As in [5], we tacitly assume some natural identifications. That is, we will regard every space as its image in the following diagram of closed embeddings:

$$\begin{array}{ccc} [0,\infty) \times M & \longrightarrow & [0,\infty) \times C^1_T(M) \\ \uparrow & & \uparrow \\ M & \longrightarrow & C^1_T(M), \end{array} \tag{2.2}$$

where the horizontal arrows are defined by regarding any point $q$ in $M$ as the constant map $\hat{q}(t) \equiv q$ in $C^1_T(M)$, and the two vertical arrows are the natural identifications $q \mapsto (0,q)$ and $x \mapsto (0,x)$.

According to these embeddings, if $\Omega$ is an open subset of $[0,\infty) \times C^1_T(M)$, by $\Omega \cap M$ we mean the open subset of $M$ given by all $q \in M$ such that the pair $(0,\hat{q})$ belongs to $\Omega$. If $U$ is an open subset of $[0,\infty) \times M$, then $U \cap M$ represents the open set $\{q \in M \mid (0,q) \in U\}$.

We need some basic facts about the topological degree of tangent vector fields on manifolds.

Let $w : M \to \mathbb{R}^k$ be a continuous tangent vector field on $M$, and let $U$ be an open subset of $M$ in which we assume $w$ admissible for the degree, that is $w^{-1}(0) \cap U$ compact. Then, one can associate to the pair $(w,U)$ an integer, $\deg(w,U)$, called the degree (or characteristic) of the vector field $w$ in $U$, which, roughly speaking, counts (algebraically) the number of zeros of $w$ in $U$ (see e.g. [6, 7] and references therein). When $M = \mathbb{R}^k$, $\deg(w, U)$ is just the classical Brouwer degree, $\deg(w, V, 0)$, of $w$ at 0 in any bounded open neighborhood $V$ of $w^{-1}(0) \cap U$ whose closure is in $U$. Moreover, when $M$ is a compact manifold, the celebrated Poincaré-Hopf Theorem states that $\deg(v, M)$ coincides with the Euler-Poincaré characteristic of $M$ and, therefore, is independent of $v$.

We recall that when $q$ is an isolated zero of $w$, the index $\mathrm{i}(w, q)$ of $w$ at $q$ is given by $\deg(w, U)$, where $U$ is any isolating open neighborhood of $q$. If $w$ is $C^1$ and $q$ is a non-degenerate zero of $w$ (i.e. the Fréchet derivative $w'(q) : T_qM \to \mathbb{R}^k$ is injective), then $q$ is an isolated zero of $w$, $w'(q)$ maps $T_qM$ into itself, and $\mathrm{i}(w, q) = \operatorname{sign} \det w'(q)$ (see e.g. [7]).

The following result of [5] concerns the global structure of the set of $T$-pairs of (2.1).

**Theorem 2.1.** *Let $\Omega$ be an open subset of $[0, \infty) \times C^1_T(M)$. Assume that $\deg\big(h(\cdot, 0), \Omega \cap M\big)$ is well defined and nonzero. Then $\Omega$ contains a connected set $\Gamma$ of nontrivial $T$-pairs for (2.1) whose closure in $\Omega$ meets $M$ in $h(\cdot, 0)^{-1}(0)$ and is not contained in any compact subset of $\Omega$. Consequently, if $M$ is closed in $\mathbb{R}^k$, then $\Gamma$ is not contained in any bounded and complete subset of $\Omega$.*

**Corollary 2.2.** *Assume that $M$ is closed in $\mathbb{R}^k$. If $q \in M$ is an isolated zero of $h(\cdot, 0)$ with $\mathrm{i}\big(h(\cdot, 0), q\big) \neq 0$, then (2.1) admits a connected set $\Gamma$ of nontrivial $T$-pairs whose closure meets $q$ and is either unbounded or intersects $h(\cdot, 0)^{-1}(0) \setminus \{q\}$. The assertion is true, in particular, if $h$ is $C^1$ and the Fréchet derivative $h(\cdot, 0)'(q) : T_qM \to \mathbb{R}^k$ of $h(\cdot, 0)$ at $q$ is injective.*

*Proof.* Apply Theorem 2.1 taking as $\Omega$ the complement in $[0, \infty) \times C^1_T(M)$ of the closed set $h(\cdot, 0)^{-1}(0) \setminus \{q\}$, and observe that, being $M$ closed, any bounded and closed subset of $[0, \infty) \times C^1_T(M)$ is complete. □

We point out that the set $\Gamma$ might be completely "vertical". That is, contained in $\{0\} \times C^1_T(M)$, as it happens for the following differential equation in $M = \mathbb{R}$ (with $q = 0$ and $T = 2\pi$):

$$\ddot{x} = -x + \lambda \sin t, \quad \lambda \geq 0.$$

In order to find multiplicity results for the forced oscillations of (2.1) it is necessary to avoid such a "degenerate" situation. We tackle this problem from an abstract viewpoint.

We need some notation. Let $Y$ be a metric space and $C$ a subset of $[0, \infty) \times Y$. Given $\lambda \geq 0$, we denote by $C_\lambda$ the slice $\big\{y \in Y \mid (\lambda, y) \in C\big\}$. In what follows, $Y$ will be identified with the subset $\{0\} \times Y$ of $[0, \infty) \times Y$.

**Definition 2.3.** Let $C$ be a subset of $[0, \infty) \times Y$. We say that a subset $A$ of $C_0$ is an *ejecting set* (for $C$) if it is relatively open in $C_0$ and there exists a connected subset of $C$ which meets $A$ and is not included in $C_0$.

We shall simply say that $q \in C_0$ is an *ejecting point* if $\{q\}$ is an ejecting set. In this case, being $\{q\}$ open in $C_0$, $q$ is clearly isolated in $C_0$

In [3] we proved the following theorem which relates ejecting sets and multiplicity results.

**Theorem 2.4.** *Let $Y$ be a metric space and let $C$ be a locally compact subset of $[0,\infty)\times Y$. Assume that $C_0$ contains $n$ pairwise disjoint ejecting sets, $n-1$ of which are compact. Then, there exists $\delta > 0$ such that the cardinality of $C_\lambda$ is greater than or equal to $n$ for any $\lambda \in [0,\delta)$.*

In [3] we provided examples showing that in Theorem 2.4 the assumption that $n-1$ ejecting sets are compact cannot be dropped.

Let $q$ be a zero of $h(\cdot,0)$. If $h$ is $C^1$, we give a condition which ensures that $q$ (regarded as a trivial $T$-pair) is an ejecting point for the subset $X$ of $[0,\infty)\times C^1_T(M)$ consisting of the $T$-pairs of (2.1).

We say that a point $q \in h(\cdot,0)^{-1}(0)$ is *$T$-resonant* for the equation (2.1) if the linearized equation

$$\ddot{x} = D_1h(q,0)x + D_2h(q,0)\dot{x}\ , \tag{2.3}$$

which corresponds to $\lambda = 0$, admits nonzero $T$-periodic solutions. Here $D_1h(q,0)$ and $D_2h(q,0)$ denote the partial derivatives at $(q,0)$ of $h$ with respect to the first and the second variable. One can check that both $D_1h(q,0)$ and $D_2h(q,0)$ are endomorphisms of $T_qM$ (see e.g. [3]), thus (2.3) is a differential equation on the subspace $T_q(M)$ of $\mathbb{R}^k$.

If $q$ is non-$T$-resonant, then there is only one constant solution of (2.3). This implies $\det\left(D_1h(q,0)\right) \neq 0$. That is, $q$ is a non-degenerate zero of $h(\cdot,0)$. As a consequence of this fact and of Corollary 2.2 we get the following:

**Corollary 2.5** ([3])**.** *If $q \in h(\cdot,0)^{-1}(0)$ is non-$T$-resonant, then it is an ejecting point for $X$.*

When the unperturbed force $h$ reduces to a purely frictional force, it is convenient to substitute $X$ with a more significative subset. In this case we obtain other examples of ejecting sets. Consider the equation (2.1) with $h(q,v) = -\mu v$, $\mu \geq 0$. That is

$$\ddot{x}_\pi = -\mu\dot{x} + \lambda f(t,x,\dot{x}), \quad \lambda \geq 0. \tag{2.4}$$

Define the average force $w : M \to \mathbb{R}^k$ by

$$w(q) = \frac{1}{T}\int_0^T f(t,q,0)\,\mathrm{d}t, \tag{2.5}$$

and observe that $w$ is a tangent vector field on $M$.

Consider the set $w^{-1}(0)$ regarded as a subset of $[0,\infty)\times C^1_T(M)$ according to the diagram (2.2), and denote by $\Xi$ the union of $w^{-1}(0)$ and of the set of the $T$-pairs of (2.4) with $\lambda > 0$. In other words,

$$\Xi = w^{-1}(0) \cup (X \setminus X_0),$$

where, we recall, $X$ denotes the set of $T$-pairs of (2.4).

In [2] it was shown that, when $\mu = 0$, the closure of $X \setminus X_0$ in $[0,\infty)\times C^1_T(M)$ is contained in $w^{-1}(0)$. This is true also when $\mu > 0$ since the same argument applies. Consequently $\Xi$, being a closed subset of $X$, is locally compact. As in Corollary 2.3 of [2] one obtains the following result.

**Theorem 2.6.** *Let $q$ be an isolated zero of $w$ such that $\mathrm{i}(w,q) \neq 0$. Then $q$ is an ejecting point for $\Xi$. This occurs, in particular, if $w$ is $C^1$ and $q$ is a non-degenerate zero of $w$.*

## 3. Application to multiplicity results

This section is devoted to illustrating how the notions and results previously discussed can be used to prove the existence of multiple forced oscillations. As before, $X$ will stand for the set of $T$-pairs of (2.1).

We begin with two physical examples.

**Example 3.1.** Consider the following forced pendulum equation:

$$\ddot{\theta} = -\sin\theta + \lambda f(t, \theta, \dot{\theta}), \tag{3.1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous, $2\pi$-periodic with respect to $\theta$ and $T$-periodic in $t$. Since the right hand side of (3.1) is $2\pi$-periodic in $\theta$, the above equation (which is in $\mathbb{R}$) can be regarded on the unit circle $M = S^1$ of $\mathbb{R}^2$ (the solutions from $\mathbb{R}$ to $S^1$ correspond under the transformation $\theta \mapsto (\sin\theta, -\cos\theta)$). In this way, the "north pole" $\mathtt{N} = \pi$ and the "south pole" $\mathtt{S} = 0$ are the unique zeros of the tangential component $-\sin\theta$ of the gravitational vector field.

We want to show that for $\lambda$ small enough equation (3.1), if regarded on $S^1$, admits at least two forced oscillations (observe that a solution of (3.1) on $S^1$ produces infinitely many solutions on $\mathbb{R}$). Corollary 2.5 implies that $\mathtt{N}$, being non-$T$-resonant, is ejecting (for $X$). Thus, our claim follows from Theorem 2.4 if we prove that $X_0 \setminus \{\mathtt{N}\}$ is an ejecting set, which means that there exists a connected subset of $T$-pairs intersecting the relatively open subset $X_0 \setminus \{\mathtt{N}\}$ of $X_0$ and not included in $X_0$.

Corollary 2.2 implies that there exists a connected set $\Gamma$ of nontrivial $T$-pairs whose closure $\overline{\Gamma}$ meets $\mathtt{S} \in X_0 \setminus \{\mathtt{N}\}$ and is either unbounded or contains $\mathtt{N}$. Let us show that $\overline{\Gamma} \not\subset \{0\} \times C^1_T(S^1)$. If this were not the case, then $\overline{\Gamma} = \{0\} \times \overline{\Gamma}_0$. Since $\overline{\Gamma}_0$ cannot meet the relatively open subset $\{\mathtt{N}\}$ of $X_0$, it would be unbounded. But this is false since, given any $x(\cdot) = \big(\sin\theta(\cdot), -\cos\theta(\cdot)\big) \in X_0$, the $T$-periodicity of $x(\cdot)$ implies

$$\|\dot{x}(t)\| = |\dot{\theta}(t)| \le T \quad \text{for any } t \in [0, T].$$

**Example 3.2.** Consider the so-called *parametrically excited pendulum.* That is, a pendulum moving in a vertical plane and whose pivot is subject to a vertical periodic driving. The motion equation can be written in the form

$$\ddot{\theta} + \mu\dot{\theta} + \big(1 + \lambda\omega(t)\big)\sin\theta = 0,$$

where $\omega$ is a $T$-periodic function and $\mu \ge 0$. As in the example above, this equation can be seen on $S^1$ and, from this viewpoint, we show that it admits at least two forced oscillations for small values of $\lambda \ge 0$. In fact, in the case when the frictional coefficient $\mu \ne 0$, both the north and the south poles are non-$T$-resonant and, consequently, ejecting points. When $\mu = 0$, the equation is of the form considered in the previous example.

In what follows we will be concerned with the scalar equation

$$\ddot{x} = g(x) - \mu\dot{x} + \lambda f(t, x, \dot{x}), \quad \lambda \ge 0, \tag{3.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous, $f$ is $T$-periodic in $t$, and $\mu \ge 0$. Observe that, as in the above examples, when the functions $g$ and $f$ are $2\pi$-periodic in $x$, the equation (3.2) can be interpreted on $S^1$.

In the case when $g$ vanishes we get the following multiplicity result.

**Theorem 3.3.** *Consider in* $\mathbb{R}$ *the equation*

$$\ddot{x} = -\mu\dot{x} + \lambda f(t, x, \dot{x}), \quad \lambda \geq 0. \tag{3.3}$$

*Assume that the average force* $w$*, defined as in* (2.5)*, changes sign in* $n$ *isolated zeros. Then there exists* $\delta > 0$ *such that* (3.3) *has at least* $n$ *forced oscillations for* $\lambda \in [0, \delta)$*.*

*Proof.* Let $q$ be an isolated zero in which $w$ changes sign. The homotopy property of the degree implies that $\mathrm{i}(w, q) = \pm 1$. The assertion follows from Theorems 2.4 and 2.6. □

In the case when $g$ does not vanish, the average force plays no role. Clearly, if the frictional coefficient $\mu$ is nonzero, $g$ is $C^1$ and changes sign in $n$ non-degenerate zeros, then it is clear that, for $\lambda$ sufficiently small, the equation (3.2) admits at least $n$ forced oscillations. In fact, all those zeros turn out to be non-$T$-resonant and, in particular, ejecting points.

Actually, still when the frictional coefficient is non-zero, a better result can be obtained.

**Theorem 3.4.** *Assume that in equation* (3.2) *the frictional coefficient* $\mu$ *is nonzero and the force* $g$ *changes sign in* $n$ *isolated zeros. Then there exists* $\delta > 0$ *such that* (3.2) *has at least* $n$ *forced oscillations for* $\lambda \in [0, \delta)$*.*

*Proof.* Let $q_1, \dots q_n$ be isolated zeros in which $g$ changes sign. For any $i \in \{1, \dots n\}$, the homotopy property of the degree yields $\mathrm{i}(g, q_i) = \pm 1$. Thus, by Corollary 2.2, for $i = 1, \dots n$, there exists a connected set $\Gamma^i$ of nontrivial $T$-pairs for (3.2) whose closure $\overline{\Gamma^i}$ meets $q_i$ and is either non-compact or intersects $g^{-1}(0) \setminus \{q_i\}$.

Clearly, due to the presence of friction, only constant periodic solution to (3.2) may exist for $\lambda = 0$. Therefore the connected component of $\left(\overline{\Gamma^i}\right)_0$ containing $q_i$ reduces to $\{q_i\}$. This means that, for $i = 1, \dots n$, the points $q_i$ are ejecting.

The assertion now follows from Theorem 2.4. □

## References


[1] Furi M., *Second order differential equations on manifolds and forced oscillations*, Proc. Conference on "Topological methods in differential equations and inclusions", Université de Montreal, 1994, NATO-ASI series C, Kluwer Acad. Publ., Dordrecht NL.

[2] Furi M. and Pera M. P., *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory,* Pacific J. of Math. **160** (1993) 219–244.

[3] Furi M., Pera M.P. and Spadini M., *Forced oscillations on manifolds and multiplicity results for periodically perturbed autonomous systems*, Journal of computational and applied mathematics **113** (2000), 241–254.

[4] Furi M. and Spadini M., *Multiplicity of forced oscillations for the spherical pendulum,* Topological Methods in Nonlinear Analysis **11** (1998), 147–157.

[5] Furi M. and Spadini M., *Branches of forced oscillations for periodically perturbed second-order autonomous ODE's on manifolds,* Journal of Differential Equations **154** (1999) 96–106.

[6] Hirsch M.W., *Differential Topology*, Graduate Texts in Math. Vol. 33, Springer-Verlag, Berlin, 1976.

[7] Milnor J.W., *Topology from the differentiable viewpoint,* Univ. press of Virginia, Charlottesville, 1965.


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