

# BANG–BANG TRAJECTORIES WITH A DOUBLE SWITCHING TIME: SUFFICIENT STRONG LOCAL OPTIMALITY CONDITIONS

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## Abstract

This paper gives sufficient conditions for a class of bang-bang extremals with multiple switches to be locally optimal in the strong topology. The conditions are the natural generalizations of the ones considered in [5, 14] and [16]. We require both the *strict bang-bang Legendre condition*, and the second order conditions for the finite dimensional problem obtained by moving the switching times of the reference trajectory.

## 1 Introduction

We consider a Mayer problem, where the control functions are bounded and enter linearly in the dynamics.

$$\text{minimize } C(\xi, u) := c_0(\xi(0)) + c_f(\xi(T)) \quad (1.1a)$$

$$\text{subject to } \dot{\xi}(t) = f_0(\xi(t)) + \sum_{s=1}^m u_s f_s(\xi(t)) \quad (1.1b)$$

$$\xi(0) \in N_0, \quad \xi(T) \in N_f \quad (1.1c)$$

$$u = (u_1, \dots, u_m) \in L^\infty([0, T], [-1, 1]^m). \quad (1.1d)$$

Here  $T > 0$  is given, the state space is a  $n$ -dimensional manifold  $M$ ,  $N_0$  and  $N_f$  are smooth sub-manifolds of  $M$ . The vector fields  $f_0, f_1, \dots, f_m$  and the functions  $c_0, c_f$  are  $C^2$  on  $M, N_0$  and  $N_f$ , respectively.

We aim at giving second order sufficient conditions for a *reference bang-bang extremal couple*  $(\hat{\xi}, \hat{u})$  to be a *local optimizer in the strong topology*; the strong topology being the one induced by  $C([0, T], M)$  on the set of admissible trajectories, regardless of any distance of the associated controls. Therefore, optimality is with respect to neighboring trajectories, independently of the values of the associated controls. In particular, if the extremal is abnormal, we prove that  $\hat{\xi}$  is isolated among admissible trajectories.

We recall that a control  $\hat{u}$  (a trajectory  $\hat{\xi}$ ) is bang-bang if there is a finite number of switching times  $0 < \hat{t}_1 < \dots < \hat{t}_r < T$  such that each component  $\hat{u}_i$  of the reference control  $\hat{u}$  is constantly either  $-1$  or  $1$  on each interval  $(\hat{t}_k, \hat{t}_{k+1})$ . A switching time  $\hat{t}_k$  is called *simple* if only one control component changes value at  $\hat{t}_k$ , while it is called *multiple* if at least two control components change value.

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Second order conditions for the optimality of a bang-bang extremal with simple switches only are given in [5, 11, 14, 16] and references therein, while in [18] the author gives sufficient conditions, in the case of the minimum time problem, for  $L^1$ -local optimality - an intermediate condition between strong and local optimality - of a bang-bang extremal having both simple and multiple switches with the extra assumption that the Lie brackets of the switching vector fields is annihilated by the adjoint covector.

All the above cited papers require regularity assumptions on the switches (see the subsequent Assumptions 2.1, 2.2 and 2.3 which are the natural strengthening of necessary conditions) and the positivity of a suitable second variation.

Here we consider the problem of strong local optimality in the case of a Mayer problem, when at most one double switch occurs, but there are finitely many simple ones and no commutativity assumptions on the involved vector fields. More precisely we extend the conditions in [5, 14, 16] by requiring the sufficient second order conditions for the finite dimensional sub-problems that are obtained by allowing the switching times to move. The addition of a double switch is not a trivial extension of the known single-switch cases. In fact, as explained in Section 2.2, any perturbation of the switching time (of a double switch) of the components of  $\hat{u}$  generically creates two simple switches, that is it a *bang* arc is generated. On the contrary, the small perturbations of a single switch do not change the structure of the reference control.

We believe that the techniques employed here could be extended to the more general case when there are more than one double switch. However, such an extension may not be straightforward as the technical and notational complexities grow quickly with the number of double switches.

Preliminary results were given in [17], where the authors exploit a study case and in [15] that deals with a Bolza problem in the so-called *non-degenerate case*. Also stability analysis under parameter perturbations for this kind of bang-bang extremals was studied in [8].

We point out that, while in the case of simple switches the only variables are the switching times, each time a double switch occurs one has to consider the two possible combinations of the switching controls. The investigation of the invertibility of the involved Lipschitz continuous, piecewise  $C^1$  operators has been done via some topological methods described in the Appendix, or via Clarke's implicit function theorem (see [7, Thm 7.1.1.]) in some particular degenerate case.

The paper is organized as follows: Section 2.1 introduces the notation and the regularity hypotheses that are assumed through the paper. In Section 2.2, where our main result Theorem 2.3 is stated, we introduce a finite dimensional subproblem of (1.1) and its "second variations" (indeed this subproblem is  $C^{1,1}$  but not  $C^2$  so that the classical "second variation" is not well defined). The essence of the paper will be to show that the sufficient conditions for the optimality of an extremum of this subproblem are actually sufficient also for the optimality of the reference pair  $(\hat{\xi}, \hat{u})$  in problem (1.1). In Section 3 we briefly describe the Hamiltonian methods the proof is based upon. Section 4 contains the maximized Halmiltonian of the control system and its flow. In Section 5, we write the "second variations" of the finite-dimensional subproblem and study their sign on appropriate spaces. Section 6 is the heart of the paper and constitutes its more original

contribution; here we prove that the the projection onto a neighborhood of the graph of  $\hat{\xi}$  in  $\mathbb{R} \times M$  of the maximized flow defined in Section 4 is invertible (which is necessary for our Hamiltonian methods to work). Section 7 contains the conclusion of the proof of Theorem 2.3. In the Appendix we treat from an abstract viewpoint the problem, raised in Section 6, of local invertibility of a piecewise  $C^1$  function.

## 2 The result

The result is based on some regularity assumption on the vector fields associated to the problem and on a second order condition for a finite dimensional sub-problem. The regularity Assumptions 2.2 and 2.3 are natural, since we look for sufficient conditions. In fact Pontryagin Maximum Principle yields the necessity of the same inequalities but in weak form.

### 2.1 Notation and regularity

We assume we are given an admissible reference couple  $(\hat{\xi}, \hat{u})$  satisfying Pontryagin Maximum Principle (PMP) with adjoint covector  $\hat{\lambda}$  and that the reference control  $\hat{u}$  is bang-bang with switching times  $\hat{t}_1, \dots, \hat{t}_r$  such that only two kinds of switchings appear:

- $\hat{t}_i$  is a *simple switching time* i.e. only one of the control components  $\hat{u}_1, \dots, \hat{u}_m$  switches at time  $\hat{t}_i$ ;
- $\hat{t}_i$  is a *double switching time* i.e. exactly two of the control components  $\hat{u}_1, \dots, \hat{u}_m$  switch at time  $\hat{t}_i$ .

We assume that there is just one double switching time, which we denote by  $\hat{\tau}$ . Without loss of generality we may assume that the control components switching at time  $\hat{\tau}$  are  $\hat{u}_1$  and  $\hat{u}_2$  and that they both switch from the value  $-1$  to the value  $+1$ , i.e.

$$\lim_{t \rightarrow \hat{\tau}^-} \hat{u}_\nu = -1 \quad \lim_{t \rightarrow \hat{\tau}^+} \hat{u}_\nu = 1 \quad \nu = 1, 2.$$

In the interval  $(0, \hat{\tau})$ ,  $J_0$  simple switches occur (if no simple switch occurs in  $(0, \hat{\tau})$ , then  $J_0 = 0$ ), and  $J_1$  simple switches occur in the interval  $(\hat{\tau}, T)$  (if no simple switch occurs in  $(\hat{\tau}, T)$ , then  $J_1 = 0$ ). We denote the simple switching times occurring before the double one by  $\hat{\theta}_{0j}$ ,  $j = 1, \dots, J_0$ , and by  $\hat{\theta}_{1j}$ ,  $j = 1, \dots, J_1$  the simple switching times occurring afterwards. In order to simplify the notation, we also define  $\hat{\theta}_{00} := 0$ ,  $\hat{\theta}_{0, J_0+1} := \hat{\theta}_{10} := \hat{\tau}$ ,  $\hat{\theta}_{1, J_1+1} := T$ , i.e. we have

$$\hat{\theta}_{00} := 0 < \hat{\theta}_{01} < \dots < \hat{\theta}_{0, J_0} < \hat{\tau} := \hat{\theta}_{0, J_0+1} := \hat{\theta}_{10} < \hat{\theta}_{11} < \dots < \hat{\theta}_{1, J_1} < T := \hat{\theta}_{1, J_1+1}.$$

We shall use some basic tools and notation from differential geometry. For any submanifold  $N$  of  $M$ , and any  $x \in N$ ,  $T_x N$  and  $T_x^* N$  denote the tangent space to  $N$  at  $x$  and the cotangent space to  $N$  at  $x$ , respectively while  $T^* N$  denotes the cotangent bundle.

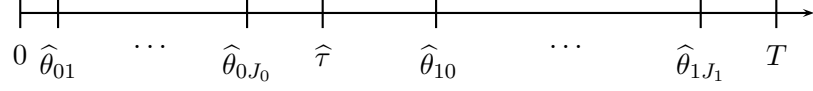


Figure 1: The sequence of switching times

For any  $w \in T_x^*M$  and any  $\delta x \in T_xM$ ,  $\langle w, \delta x \rangle$  denotes the duality product between a form and a tangent vector.

$\pi: T^*M \rightarrow M$  denotes the canonical projection from the tangent bundle onto the base manifold  $M$ . In coordinates  $\ell := (p, x)$ :

$$\pi: \ell = (p, x) \in T^*M \mapsto x \in M.$$

Throughout the paper, for any vector field  $f: x \in M \mapsto f(x) \in T_xM$ , we shall denote the associated Hamiltonian obtained by lifting  $f$  to  $T^*M$  by the corresponding capital letter, i.e.

$$F: \ell \in T^*M \mapsto \langle \ell, f(\pi\ell) \rangle \in \mathbb{R},$$

and  $\vec{F}$  will denote the Hamiltonian vector field associated to  $F$ . In particular for any  $s = 0, 1, \dots, m$   $F_s(\ell) := \langle \ell, f_s(\pi\ell) \rangle$  is the Hamiltonian associated to the drift ( $s = 0$ ) and to the controlled vector fields of system (1.1b).

If  $f, g: x \in M \mapsto f(x) \in T_xM$ , are differentiable vector fields, we denote their Lie bracket as  $[f, g]$ :

$$[f, g](x) := Dg(x)f(x) - Df(x)g(x)$$

The canonical symplectic two-form between two Hamiltonian vector fields  $\vec{F}$  and  $\vec{G}$  at a point  $\ell$  is denoted as  $\sigma(\vec{F}, \vec{G})(\ell)$ . In coordinates  $\ell := (p, x)$ :

$$\sigma(\vec{F}, \vec{G})(\ell) := -\langle p Dg(x), f(x) \rangle + \langle p Df(x), g(x) \rangle.$$

For any  $m$ -tuple  $u = (u_1, \dots, u_m) \in \mathbb{R}^m$  let us denote the control-dependent Hamiltonian

$$h_u: \ell \in T^*M \mapsto \langle \ell, f_0(\pi\ell) + \sum_{s=1}^m u_s f_s(\pi\ell) \rangle \in \mathbb{R}.$$

Let  $\hat{f}_t$  and  $\hat{F}_t$  be the reference vector field and the reference Hamiltonian, respectively:

$$\hat{f}_t(x) := f_0(x) + \sum_{s=1}^m \hat{u}_s(t) f_s(x), \quad \hat{F}_t(\ell) := \langle \ell, \hat{f}_t(\pi\ell) \rangle = h_{\hat{u}(t)}(\ell)$$

and let

$$H(\ell) := \max \{ h_u(\ell) : u \in [-1, 1]^m \}$$

be the maximized Hamiltonian of the control system. Also, let  $\hat{x}_0 := \hat{\xi}(0)$ ,  $\hat{x}_d := \hat{\xi}(\hat{\tau})$  and  $\hat{x}_f := \hat{\xi}(T)$ .

The reference flow, that is the flow associated to  $\widehat{f}_t$ , is defined on the whole interval  $[0, T]$  at least in a neighborhood of  $\widehat{x}_0$ . We denote it as

$$\widehat{S}: (t, x) \mapsto \widehat{S}_t(x).$$

Thus, in our situation PMP reads as follows:

There exist  $p_0 \in \{0, 1\}$  and an absolutely continuous function  $\widehat{\lambda}: [0, T] \rightarrow T^*M$  such that

$$(p_0, \widehat{\lambda}(0)) \neq (0, 0) \quad (2.1)$$

$$\pi \widehat{\lambda}(t) = \widehat{\xi}(t) \quad \forall t \in [0, T]$$

$$\dot{\widehat{\lambda}}(t) = \overrightarrow{F}_t(\widehat{\lambda}(t)) \quad \text{a.e. } t \in [0, T],$$

$$\widehat{\lambda}(0)|_{T_{\widehat{x}_0}N_0} = p_0 \text{dc}_0(\widehat{x}_0), \quad \widehat{\lambda}(T)|_{T_{\widehat{x}_f}N_f} = -p_0 \text{dc}_f(\widehat{x}_f) \quad (2.2)$$

$$\widehat{F}_t(\widehat{\lambda}(t)) = H(\widehat{\lambda}(t)) \quad \text{a.e. } t \in [0, T]. \quad (2.3)$$

We shall denote  $\widehat{\ell}_0 := \widehat{\lambda}(0)$  and  $\widehat{\ell}_f := \widehat{\lambda}(T)$ .

Maximality condition (2.3) implies  $\widehat{u}_s(t)F_s(\widehat{\lambda}(t)) = \widehat{u}_s(t)\langle \widehat{\lambda}(t), f_s(\widehat{\xi}(t)) \rangle \geq 0$  for any  $t \in [0, T]$  and any  $s = 1, \dots, m$ . We assume the following regularity condition holds:

**Assumption 2.1** (Regularity). *Let  $s \in \{1, \dots, m\}$ . If  $t$  is not a switching time for the control component  $\widehat{u}_s$ , then*

$$u_s(t)F_s(\widehat{\lambda}(t)) = \widehat{u}_s(t)\langle \widehat{\lambda}(t), f_s(\widehat{\xi}(t)) \rangle > 0. \quad (2.4)$$

In terms of the switching functions  $\sigma_s: t \in [0, T] \mapsto F_s \circ \widehat{\lambda}(t) \in \mathbb{R}$ ,  $s = 1, \dots, m$  Assumption 2.1 means  $\widehat{u}_s(t) = \text{sgn}(\sigma_s(t))$  whenever  $t$  is not a switching time of the reference control component  $\widehat{u}_s$ .

Notice that Assumption 2.1 implies that  $\text{argmax}\{h_u(\widehat{\lambda}(t)): u \in [-1, 1]^m\} = \widehat{u}(t)$  for any  $t$  that is not a switching time.

Let

$$k_{ij} := \widehat{f}_t|_{(\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+1})}, \quad j = 0, \dots, J_i, \quad i = 0, 1,$$

be the restrictions of  $\widehat{f}_t$  to each of the time intervals where the reference control  $\widehat{u}$  is constant and let  $K_{ij}(\ell) := \langle \ell, k_{ij}(\pi\ell) \rangle$  be the associated Hamiltonian. Then, from maximality condition (2.3) we get

$$\frac{d}{dt}(K_{ij} - K_{i,j-1}) \circ \widehat{\lambda}(t) \Big|_{t=\widehat{\theta}_{ij}} \geq 0$$

for any  $i = 0, 1$ ,  $j = 1, \dots, J_i$ , i.e. if  $\widehat{u}_{s(ij)}$  is the control component switching at time  $\widehat{\theta}_{ij}$  and  $\Delta_{ij} \in \{-2, 2\}$  is its jump, then

$$\frac{d}{dt}\Delta_{ij}\sigma_{s(ij)}(t) \Big|_{t=\widehat{\theta}_{ij}} \geq 0$$

We assume that the strong inequality holds at each simple switching time  $\widehat{\theta}_{ij}$ :

**Assumption 2.2.**

$$\left. \frac{d}{dt} (K_{ij} - K_{i,j-1}) \circ \widehat{\lambda}(t) \right|_{t=\widehat{\theta}_{ij}} > 0 \quad i = 0, 1, \quad j = 1, \dots, J_i. \quad (2.5)$$

Assumption 2.2 is known as the STRONG BANG-BANG LEGENDRE CONDITION FOR SIMPLE SWITCHING TIMES.

In geometric terms Assumption 2.2 means that at time  $t = \widehat{\theta}_{ij}$  the trajectory  $t \mapsto \widehat{\lambda}(t)$  crosses transversally the hypersurface of  $T^*M$  defined by  $K_{ij} = K_{i,j-1}$ , i.e. by the zero level set of  $F_{s(ij)}$ .

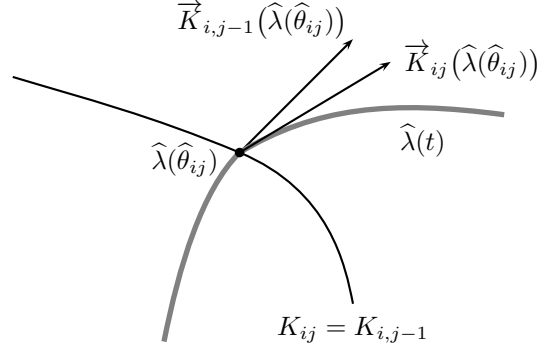


Figure 2: Behaviour at a simple switching time

As already said, without any loss of generality we can assume that the double switching time involves the first two components,  $\widehat{u}_1$  and  $\widehat{u}_2$  of the reference control  $\widehat{u}$  and that they both switch from  $-1$  to  $+1$ , so that

$$k_{10} = k_{0J_0} + 2f_1 + 2f_2.$$

Define the new vector fields

$$k_\nu := k_{0J_0} + 2f_\nu, \quad \nu = 1, 2,$$

with associated Hamiltonians  $K_\nu(\ell) := \langle \ell, k_\nu(\pi\ell) \rangle$ . Then, from maximality condition (2.3) we get

$$\begin{aligned} \left. \frac{d}{dt} 2\sigma_\nu(t) \right|_{t=\widehat{\tau}_-} = \left. \frac{d}{dt} 2F_\nu \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_-} = \left. \frac{d}{dt} (K_\nu - K_{0J_0}) \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_-} &\geq 0, \\ \left. \frac{d}{dt} 2\sigma_\nu(t) \right|_{t=\widehat{\tau}_+} = \left. \frac{d}{dt} 2F_\nu \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_+} = \left. \frac{d}{dt} (K_{10} - K_\nu) \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_+} &\geq 0, \end{aligned} \quad \nu = 1, 2.$$

We assume that the strict inequalities hold:

**Assumption 2.3.**

$$\left. \frac{d}{dt} (K_\nu - K_{0J_0}) \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_-} > 0, \quad \left. \frac{d}{dt} (K_{10} - K_\nu) \circ \widehat{\lambda}(t) \right|_{t=\widehat{\tau}_+} > 0, \quad \nu = 1, 2. \quad (2.6)$$

Assumption 2.3 means that at time  $\hat{\tau}$  the flow arrives the hypersurfaces  $F_1 = 0$  and  $F_2 = 0$  with transversal velocity  $\vec{K}_{0J_0}$  and leaves with velocity  $\vec{K}_{10}$  which is again transversal to both the hypersurfaces. We shall call Assumption 2.3 the STRONG BANG-BANG LEGENDRE CONDITION FOR DOUBLE SWITCHING TIMES.

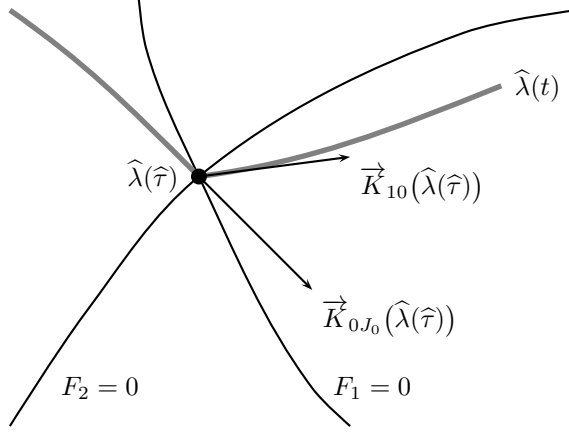


Figure 3: Behaviour at the double switching time

Equivalently, conditions (2.5) and (2.6) can be expressed in terms of the Lie brackets of vector fields or in terms of the canonical symplectic structure  $\sigma(\cdot, \cdot)$  on  $T^*M$ :

**Proposition 2.1.** *Assumption 2.2 is equivalent to*

$$\langle \hat{\lambda}(\hat{\theta}_{ij}), [k_{i,j-1}, k_{ij}](\hat{\xi}(\hat{\theta}_{ij})) \rangle = \sigma(\vec{K}_{i,j-1}, \vec{K}_{ij})(\hat{\lambda}(\hat{\theta}_{ij})) > 0 \quad (2.7)$$

for any  $i = 0, 1, \quad j = 1, \dots, J_i$ .

Assumption 2.3 is equivalent to

$$\begin{aligned} \langle \hat{\lambda}(\hat{\tau}), [k_{0J_0}, k_\nu](\hat{x}_d) \rangle &= \sigma(\vec{K}_{0J_0}, \vec{K}_\nu)(\hat{\lambda}(\hat{\tau})) > 0, \\ \langle \hat{\lambda}(\hat{\tau}), [k_\nu, k_{10}](\hat{x}_d) \rangle &= \sigma(\vec{K}_\nu, \vec{K}_{10})(\hat{\lambda}(\hat{\tau})) > 0 \end{aligned} \quad \nu = 1, 2. \quad (2.8)$$

In what follows we shall also need to reformulate Assumptions 2.2 and 2.3 in terms of the pull-backs along the reference flow of the vector fields  $k_{ij}$  and  $k_\nu$ . Define

$$g_{ij}(x) := \hat{S}_{\hat{\theta}_{ij}*}^{-1} k_{ij} \circ \hat{S}_{\hat{\theta}_{ij}}(x), \quad h_\nu(x) := \hat{S}_{\hat{\tau}*}^{-1} k_\nu \circ \hat{S}_{\hat{\tau}}(x)$$

and let  $G_{ij}, H_\nu$  be the associated Hamiltonians. We can restate Assumptions 2.2 and 2.3 as follows:

**Proposition 2.2.** *Assumption 2.2 is equivalent to*

$$\langle \hat{\ell}_0, [g_{i,j-1}, g_{ij}](\hat{x}_0) \rangle = \sigma(\vec{G}_{i,j-1}, \vec{G}_{ij})(\hat{\ell}_0) > 0 \quad (2.9)$$

for any  $i = 0, 1, \dots, J_i$ .

Assumption 2.3 is equivalent to

$$\begin{aligned} \langle \widehat{\ell}_0, [g_{0J_0}, h_\nu](\widehat{x}_0) \rangle &= \sigma \left( \vec{G}_{0J_0}, \vec{H}_\nu \right) (\widehat{\ell}_0) > 0, \\ \langle \widehat{\ell}_0, [h_\nu, g_{10}](\widehat{x}_0) \rangle &= \sigma \left( \vec{H}_\nu, \vec{G}_{10} \right) (\widehat{\ell}_0) > 0 \end{aligned} \quad \nu = 1, 2. \quad (2.10)$$

## 2.2 The finite dimensional sub-problem

By allowing the switching times of the reference control function to move we can define a finite dimensional sub-problem of the given one. In doing so we must distinguish between the simple switching times and the double switching time. Moving a simple switching time  $\widehat{\theta}_{ij}$  to time  $\theta_{ij} := \widehat{\theta}_{ij} + \delta_{ij}$  amounts to using the values  $\widehat{u}|_{(\widehat{\theta}_{i,j-1}, \widehat{\theta}_{ij})}$  and  $\widehat{u}|_{(\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+1})}$  of the reference control in the time intervals  $(\widehat{\theta}_{i,j-1}, \widehat{\theta}_{ij})$  and  $(\widehat{\theta}_{ij}, \widehat{\theta}_{i,j+1})$ , respectively. On the other hand, when we move the double switching time  $\widehat{\tau}$  we change the switching time of two different components of the reference control and we must allow for each of them to change its switching time independently of the other. This means that between the values of  $\widehat{u}|_{(\widehat{\theta}_{0J_0}, \widehat{\tau})}$  and  $\widehat{u}|_{(\widehat{\tau}, \widehat{\theta}_{01})}$  we introduce a value of the control which is not assumed by the reference one - at least in a neighborhood of  $\widehat{\tau}$  - and which may assume two different values according to which component switches first between the two available ones. Let  $\tau_\nu := \widehat{\tau} + \varepsilon_\nu$ ,  $\nu = 1, 2$ . We move the switching time of the first control component  $\widehat{u}_1$  from  $\widehat{\tau}$  to  $\tau_1 := \widehat{\tau} + \varepsilon_1$ , and the switching time of the second control component  $\widehat{u}_2$  from  $\widehat{\tau}$  to  $\tau_2 := \widehat{\tau} + \varepsilon_2$ .

Inspired by [5], let us introduce  $C^2$  functions  $\alpha: M \rightarrow \mathbb{R}$  and  $\beta: M \rightarrow \mathbb{R}$  such that  $\alpha|_{N_0} = p_0 c_0$ ,  $d\alpha(\widehat{x}_0) = \widehat{\ell}_0$  and  $\beta|_{N_f} = p_0 c_f$ ,  $d\beta(\widehat{x}_f) = -\widehat{\ell}_f$ .

Define  $\theta_{ij} := \widehat{\theta}_{ij} + \delta_{ij}$ ,  $j = 1, \dots, J_i$ ,  $i = 0, 1$ ;  $\theta_{0, J_0+1} := \min\{\tau_\nu, \nu = 1, 2\}$ ,  $\theta_{10} := \max\{\tau_\nu, \nu = 1, 2\}$ ,  $\theta_{00} := 0$  and  $\theta_{1, J_1+1} := T$ . We have a finite-dimensional sub-problem (FP) given by

$$\text{minimize } \alpha(\xi(0)) + \beta(\xi(T)) \quad (\text{FPa})$$

$$\text{subject to } \dot{\xi}(t) = \begin{cases} k_{0j}(\xi(t)) & t \in (\theta_{0j}, \theta_{0,j+1}) \quad j = 0, \dots, J_0, \\ k_\nu(\xi(t)) & t \in (\theta_{0, J_0+1}, \theta_{10}), \\ k_{1j}(\xi(t)) & t \in (\theta_{1j}, \theta_{1,j+1}) \quad j = 0, \dots, J_1 \end{cases} \quad (\text{FPb})$$

$$\text{and } \xi(0) \in N_0, \quad \xi(T) \in N_f. \quad (\text{FPc})$$

$$\text{where } \theta_{00} = 0, \quad \theta_{1, J_1+1} = T \quad (\text{FPd})$$

$$\theta_{ij} = \widehat{\theta}_{ij} + \delta_{ij}, \quad i = 0, 1, \quad j = 1, \dots, J_i, \quad (\text{FPe})$$

$$\theta_{0, J_0+1} := \widehat{\tau} + \min\{\varepsilon_1, \varepsilon_2\}, \quad \theta_{10} := \widehat{\tau} + \max\{\varepsilon_1, \varepsilon_2\} \quad (\text{FPf})$$

$$\text{and } \begin{cases} \nu = 1 & \text{if } \varepsilon_1 \leq \varepsilon_2, \\ \nu = 2 & \text{if } \varepsilon_2 \leq \varepsilon_1. \end{cases} \quad (\text{FPg})$$

We shall denote the solution, evaluated at time  $t$ , of (FPb) emanating from a point  $x \in M$  at time 0, as  $S_t(x, \delta, \varepsilon)$ . Observe that  $S_t(x, 0, 0) = \widehat{S}_t(x)$ .



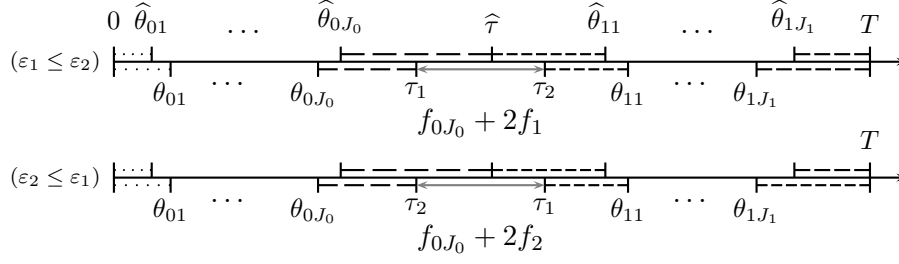


Figure 4: The different sequences of vector fields in the finite-dimensional sub-problem.

Notice that the reference control is achieved along  $\varepsilon_1 = \varepsilon_2$ , that is the reference flow is attained by (FP) on a point of non-differentiability of the functions

$$\theta_{0,J_0+1} := \hat{\tau} + \min\{\varepsilon_1, \varepsilon_2\}, \quad \theta_{10} := \hat{\tau} + \max\{\varepsilon_1, \varepsilon_2\}.$$

We are going to prove (see Remark 5.1 in Section 5) that despite this lack of differentiability of the switching times  $\theta_{0J_0}, \theta_{10}$ , (FP) is  $C^1$  (indeed  $C^{1,1}$ ) at  $\delta_{ij} = \varepsilon_1 = \varepsilon_2 = 0$

We can thus consider, on the kernel of the first variation of (FP), its second variation, piece-wisely defined as the second variation of the restrictions of (FP) to the half-spaces  $\{(\delta, \varepsilon): \varepsilon_1 \leq \varepsilon_2\}$  and  $\{(\delta, \varepsilon): \varepsilon_2 \leq \varepsilon_1\}$ . Because of the structure of (FP), this second variation is coercive if and only if both restrictions are positive-definite quadratic forms. In particular any of their convex combinations is positive-definite on the kernel of the first variation, i.e. Clarke's generalized Hessian at  $(x, \delta, \varepsilon) = (\hat{x}_0, 0, 0)$  is positive-definite on that kernel, see Remark 5.2 in Section 5.

In Section 5 we give explicit formulas both for the first and for the second variations. We shall ask for such second variations to be positive definite and prove the following theorem:

**Theorem 2.3.** *Let  $(\hat{\xi}, \hat{u})$  be a bang-bang regular extremal (in the sense of Assumption 2.1) for problem (1.1) with associated covector  $\hat{\lambda}$ . Assume all the switching times of  $(\hat{\xi}, \hat{u})$  but one are simple, while the only non-simple switching time is double.*

*Assume the strong Legendre conditions, Assumptions 2.2 and 2.3, hold. Assume also that the second variation of problem (FP) is positive definite on the kernel of the first variation. Then  $(\hat{\xi}, \hat{u})$  is a strict strong local optimizer for problem (1.1). If the extremal is abnormal ( $p_0 = 0$ ), then  $\hat{\xi}$  is an isolated admissible trajectory.*

### 3 Hamiltonian methods

The proof will be carried out by means of Hamiltonian methods, which allow us to reduce the problem to a finite dimensional one defined in a neighborhood of the final point of the reference trajectory. For a general introduction to such methods see e.g. [3]. We repeat here the argument for the sake of completeness.

In Section 4 we prove that the maximized Hamiltonian of the control system,  $H$ , is well defined and Lipschitz continuous on the whole cotangent bundle  $T^*M$ . Its Hamiltonian vector field  $\vec{H}$  is piecewise smooth in a neighborhood of the range of  $\hat{\lambda}$  and its flow, which

we denote as

$$\mathcal{H}: (t, \ell) \in [0, T] \times T^*M \mapsto \mathcal{H}_t(\ell) \in T^*M,$$

is well defined in a neighborhood of  $[0, T] \times \{\widehat{\ell}_0\}$  and  $\widehat{\lambda}$  is a trajectory of  $\vec{H}$ :  $\frac{d}{dt}\widehat{\lambda}(t) = \vec{H}(\widehat{\lambda}(t))$ , i.e.  $\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\ell}_0)$ .

In Sections 5-6 we prove that there exist a  $C^2$  function  $\alpha$  such that  $\alpha|_{N_0} = p_0 c_0$ ,  $d\alpha(x_0) = \widehat{\ell}_0$  and enjoying the following property: the map

$$\text{id} \times \pi \mathcal{H}: (t, \ell) \in [0, T] \times \Lambda \mapsto (t, \pi \mathcal{H}_t(\ell)) \in [0, T] \times M$$

is one-to-one onto a neighborhood of the graph of  $\widehat{\xi}$ , where  $\Lambda := \{d\alpha(x) : x \in \mathcal{O}(x_0)\}$ . Indeed the proof of this invertibility is the main core of the paper and its main novelty.

Under the above conditions the one-form  $\omega := \mathcal{H}^*(p dq - H dt)$  is exact on  $[0, T] \times \Lambda$ , hence there exists a  $C^1$  function

$$\chi: (t, \ell) \in [0, T] \times \Lambda \mapsto \chi_t(\ell) \in \mathbb{R}$$

such that  $d\chi = \omega$ . Also it may be shown (see, e.g. [5]) that  $d(\chi_t \circ (\pi \mathcal{H}_t)^{-1}) = \mathcal{H}_t \circ (\pi \mathcal{H}_t)^{-1}$  for any  $t \in [0, T]$ . Moreover we may assume  $\chi_0 = \alpha \circ \pi$

Observe that  $(t, \widehat{\xi}(t)) = (\text{id} \times \pi \mathcal{H})(t, \widehat{\ell}_0)$  and let us show how this construction leads to the reduction. Define

$$\mathcal{V} := (\text{id} \times \pi \mathcal{H})([0, T] \times \Lambda), \quad \psi := (\text{id} \times \pi \mathcal{H})^{-1}: \mathcal{V} \rightarrow [0, T] \times \Lambda$$

and let  $(\xi, u)$  be an admissible pair (i.e. a pair satisfying (1.1b)–(1.1c)–(1.1d)) such that the graph of  $\xi$  is in  $\mathcal{V}$ . We can obtain a closed path  $\Gamma$  in  $\mathcal{V}$  with a concatenation of the following paths:

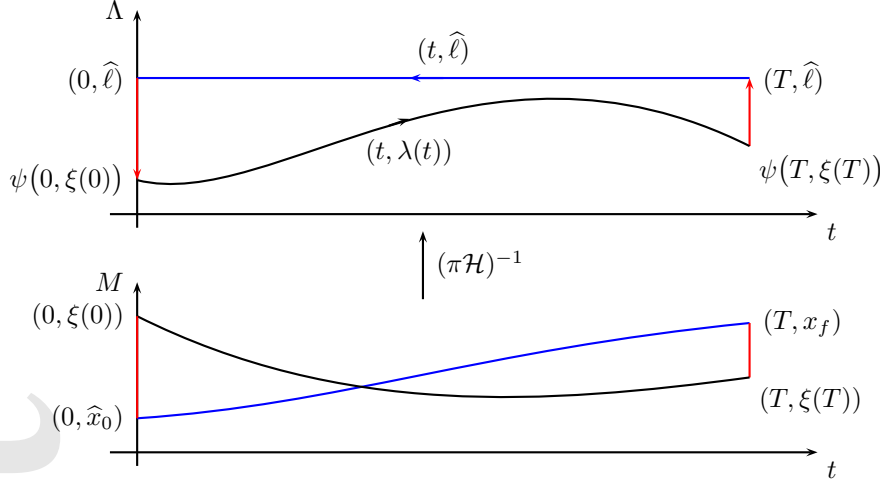
- $\Xi: t \in [0, T] \mapsto (t, \xi(t)) \in \mathcal{V}$ ,
- $\Phi_T: s \in [0, 1] \mapsto (T, \varphi_T(s)) \in \mathcal{V}$ , where  $\varphi_T: s \in [0, 1] \mapsto \varphi_T(s) \in M$  is such that  $\varphi_T(0) = \xi(T)$ ,  $\varphi_T(1) = \widehat{x}_f$ ,
- $\widehat{\Xi}: t \in [0, T] \mapsto (t, \widehat{\xi}(t)) \in \mathcal{V}$ , ran backward in time,
- $\Phi_0: s \in [0, 1] \mapsto (0, \varphi_0(s)) \in \mathcal{V}$ , where  $\varphi_0: s \in [0, 1] \mapsto \varphi_0(s) \in M$  is such that  $\varphi_0(0) = \widehat{x}_0$ ,  $\varphi_0(1) = \xi(0)$ .

Since the one-form  $\omega$  is exact we get

$$0 = \oint_{\Gamma} \omega = \int_{\psi(\Xi)} \omega + \int_{\psi(\Phi_T)} \omega - \int_{\psi(\widehat{\Xi})} \omega + \int_{\psi(\Phi_0)} \omega.$$

From the definition of  $\omega$  and the maximality properties of  $H$  we get

$$\int_{\psi(\widehat{\Xi})} \omega = 0, \quad \int_{\psi(\Xi)} \omega \leq 0 \tag{3.1}$$


 Figure 5: The closed path  $\Gamma$  and its preimage

so that

$$\int_{\psi(\Phi_T)} \omega + \int_{\psi(\Phi_0)} \omega \geq 0. \quad (3.2)$$

Since

$$\begin{aligned} \int_{\psi(\Phi_T)} \omega &= \int_{(\pi\mathcal{H}_T)^{-1} \circ \Phi_T} d(\chi_T \circ (\pi\mathcal{H}_T)^{-1}) = \chi_T \circ (\pi\mathcal{H}_T)^{-1}(\hat{x}_f) - \chi_T \circ (\pi\mathcal{H}_T)^{-1}(\xi(T)), \\ \int_{\psi(\Phi_0)} \omega &= \int_0^1 \langle d\alpha(\varphi_0(s)), \dot{\varphi}_0(s) \rangle ds = \alpha(\xi(0)) - \alpha(\hat{x}_0), \end{aligned}$$

inequality (3.2) yields

$$\alpha(\xi(0)) - \alpha(\hat{x}_0) + \chi_T \circ (\pi\mathcal{H}_T)^{-1}(\hat{x}_f) - \chi_T \circ (\pi\mathcal{H}_T)^{-1}(\xi(T)) \geq 0. \quad (3.3)$$

Thus

$$\begin{aligned} \alpha(\xi(0)) + \beta(\xi(T)) - \alpha(\hat{x}_0) - \beta(\hat{x}_f) &\geq \\ &\geq (\chi_T \circ (\pi\mathcal{H}_T)^{-1} + \beta)(\xi(T)) - (\chi_T \circ (\pi\mathcal{H}_T)^{-1} + \beta)(\hat{x}_f) \end{aligned} \quad (3.4)$$

that is: we only have to prove the local minimality at  $\hat{x}_f$  of the function

$$\mathcal{F}: x \in N_f \cap \mathcal{O}(\hat{x}_f) \mapsto (\chi_T \circ (\pi\mathcal{H}_T)^{-1} + \beta)(x) \in \mathbb{R}.$$

where  $\mathcal{O}(\hat{x}_f)$  is a small enough neighborhood of  $\hat{x}_f$ .

In proving both the invertibility of  $\text{id} \times \pi\mathcal{H}$  and the local minimality of  $\hat{x}_f$  for  $\mathcal{F}$  we shall exploit the positivity of the second variations of problem (FP). See [1, 2, 3] for a more general introduction to Hamiltonian methods.

#### 4 The maximized flow

We are now going to prove the properties of the maximized Hamiltonian  $H$  and of the flow - given by classical solutions - of the associated Hamiltonian vector field  $\vec{H}$ . Such flow will turn out to be Lipschitz continuous and piecewise- $C^1$ . In such construction we shall use only the regularity assumptions 2.1-2.2-2.3 and not the positivity of the second variations of problems (FP).

We shall proceed as follows:

*Step 1:* we first consider the simple switches occurring before the double one. We shall explain the procedure in details for the first simple switch. The others are treated iterating such procedure [5];

*Step 2:* we decouple the double switch obtaining two simple switches that might coincide and that give rise to as many flows;

*Step 3:* We consider the simple switches that occur after the double one. For each of the flows originating from the double switch we apply the same procedure of Step 1.

*Step 1:* Regularity Assumption 2.1 implies that locally around  $\hat{\ell}_0$ , the maximized Hamiltonian is  $K_{00}$  and that  $\hat{\lambda}(t)$ , i.e. the flow of  $\vec{K}_{00}$  evaluated in  $\hat{\ell}_0$ , intersects the level set  $\{\ell \in T^*M : K_{01}(\ell) = K_{00}(\ell)\}$  at time  $\hat{\theta}_{01}$ . Assumption 2.2 yields that such intersection is transverse. This suggests us to define the switching function  $\theta_{01}(\ell)$  as the time when the flow of  $\vec{K}_{00}$ , emanating from  $\ell$ , intersects such level set and to switch to the flow of  $\vec{K}_{01}$  afterwards. To be more precise, we apply the implicit function theorem to the map

$$\Phi_{01}(t, \ell) := (K_{01} - K_{00}) \circ \exp t \vec{K}_{00}(\ell)$$

in a neighborhood of  $(t, \ell) := (\hat{\theta}_{01}, \hat{\ell}_0)$  in  $[0, T] \times T^*M$ , so that  $H(\ell) = K_{00}(\ell)$  for any  $t \in [0, \theta_{01}(\ell)]$ . We then iterate this procedure and obtain the switching surfaces  $\{(\theta_{0j}(\ell), \ell) : \ell \in \mathcal{O}(\hat{\ell}_0)\}$ ,  $j = 1, \dots, J_0$  where:

$$\theta_{00}(\ell) := 0 \quad \varphi_{00}(\ell) := \ell$$

and, for  $j = 1, \dots, J_0$ , we have

-  $\theta_{0j}(\ell)$  is the unique solution to

$$(K_{0j} - K_{0,j-1}) \circ \exp \theta_{0j}(\ell) \vec{K}_{0,j-1}(\varphi_{0,j-1}(\ell)) = 0$$

defined by the implicit function theorem in a neighborhood of  $(t, \ell) = (\hat{\theta}_{0j}, \hat{\ell}_0)$ ;

-  $\varphi_{0j}(\ell)$  is defined by

$$\varphi_{0j}(\ell) := \exp(-\theta_{0j}(\ell) \vec{K}_{0j}) \circ \exp \theta_{0j}(\ell) \vec{K}_{0,j-1}(\varphi_{0,j-1}(\ell)). \quad (4.1)$$

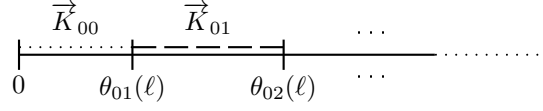


Figure 6: Construction of the maximized flow.

*Step 2:* Let us now show how to decouple the double switching time in order to define the maximized Hamiltonian  $H(\ell)$  in a neighborhood of  $(\hat{\tau}, \hat{\lambda}(\hat{\tau}))$ . In this we depart from [5] in that we introduce the new vector fields  $k_1, k_2$  in the sequence of values assumed by the reference vector field. We do this in five stages:

– for  $\nu = 1, 2$  let  $\tau_\nu(\ell)$  be the unique solution to

$$2F_\nu \circ \exp \tau_\nu(\ell) \vec{K}_{0J_0}(\varphi_{0J_0}(\ell)) = (K_\nu - K_{0J_0}) \circ \exp \tau_\nu(\ell) \vec{K}_{0J_0}(\varphi_{0J_0}(\ell)) = 0$$

defined by the implicit function theorem in a neighborhood of  $(\hat{\tau}, \hat{\ell}_0)$ ;

– choose

$$\theta_{0,J_0+1}(\ell) := \min \{ \tau_1(\ell), \tau_2(\ell) \};$$

– for  $\nu = 1, 2$ , define

$$\varphi'_{0,J_0+1}(\ell) := \exp(-\tau_\nu(\ell) \vec{K}_\nu) \circ \exp \tau_\nu(\ell) \vec{K}_{0J_0}(\varphi_{0J_0}(\ell)),$$

and let  $\theta'_{10}(\ell)$  be the unique solution to

$$\begin{aligned} 2F_{3-\nu} \circ \exp \theta'_{10}(\ell) \vec{K}_\nu(\varphi'_{0,J_0+1}(\ell)) &= \\ &= (K_{10} - K_\nu) \circ \exp \theta'_{10}(\ell) \vec{K}_\nu(\varphi'_{0,J_0+1}(\ell)) = 0 \end{aligned}$$

defined by the implicit function theorem in a neighborhood of  $(\hat{\tau}, \hat{\ell}_0)$ ;

– for  $\nu = 1, 2$  define

$$\varphi''_{10} := \exp(-\theta'_{10}(\ell) \vec{K}_{10}) \circ \exp \theta'_{10}(\ell) \vec{K}_\nu(\varphi'_{0,J_0+1}(\ell));$$

– choose

$$\theta_{10}(\ell) = \begin{cases} \theta_{10}^1(\ell) & \text{if } \tau_1(\ell) \leq \tau_2(\ell), \\ \theta_{10}^2(\ell) & \text{if } \tau_2(\ell) < \tau_1(\ell). \end{cases}$$

Notice that if  $\tau_1(\ell) = \tau_2(\ell)$ , then  $\theta_{10}^1(\ell) = \theta_{10}^2(\ell) = \tau_1(\ell) = \tau_2(\ell)$  so that  $\theta_{10}(\cdot)$  is continuous. To be more precise, the function  $\theta_{10}(\cdot)$  is Lipschitz continuous on its domain and is actually  $C^1$  on its domain but with the only possible exception of the set  $\{\ell \in T^*M : \tau_1(\ell) = \tau_2(\ell)\}$ .

*Step 3:* Finally we define analogous quantities for the simple switching times that follow the double one. For each  $j = 1, \dots, J_1$  we proceed in three stages:

– for  $\nu = 1, 2$  let  $\theta_{1j}^\nu(\ell)$  be the unique solution to

$$(K_{1j} - K_{1,j-1}) \circ \exp \theta_{1j}^\nu(\ell) \vec{K}_{1,j-1} (\varphi_{1,j-1}^\nu(\ell)) = 0$$

defined by the implicit function theorem in a neighborhood of  $(\hat{\theta}_{1j}^\nu, \hat{\ell}_0)$ ;

– define

$$\varphi_{1j}^\nu(\ell) := \exp(-\theta_{1j}^\nu(\ell) \vec{K}_{1j}) \circ \exp \theta_{1j}^\nu(\ell) \vec{K}_{1,j-1} (\varphi_{1,j-1}^\nu(\ell));$$

– choose

$$\theta_{1j}(\ell) = \begin{cases} \theta_{1j}^1(\ell) & \text{if } \tau_1(\ell) \leq \tau_2(\ell) \\ \theta_{1j}^2(\ell) & \text{if } \tau_2(\ell) < \tau_1(\ell). \end{cases}$$

We conclude the procedure defining  $\theta_{1,J_1+1}(\ell) = \theta_{1,J_1+1}^1(\ell) = \theta_{1,J_1+1}^2(\ell) := T$ .

To justify the previous procedure we have to show that we can actually apply the implicit function theorem to define the switching times  $\theta_{ij}(\ell)$  and that they are ordered as follows:

$$\theta_{0,j-1}(\ell) < \theta_{0j}(\ell) \dots < \theta_{0J_0}(\ell) < \theta_{0,J_0+1}(\ell) \leq \theta_{10}(\ell) < \theta_{11}(\ell) < \dots$$

We prove it with an induction argument. The functions  $\theta_{00}(\cdot)$  and  $\varphi_{00}(\cdot)$  are obviously well defined. Assume that  $\theta_{0j}, \varphi_{0j}$  are well defined for some  $j \geq 1$  and let

$$\Phi_{0,j+1}(t, \ell) = (K_{0,j+1} - K_{0j}) \circ \exp t \vec{K}_{0j} \circ \varphi_{0j}(\ell).$$

Then one can compute

$$\left. \frac{\partial \Phi_{0,j+1}}{\partial t} \right|_{(\hat{\theta}_{0,j+1}, \hat{\ell}_0)} = \sigma(\vec{K}_{0j}, \vec{K}_{0,j+1})(\hat{\lambda}(\hat{\theta}_{0,j+1}))$$

which is positive by Assumption 2.2, so that the implicit function theorem yields the  $C^1$  function  $\theta_{0,j+1}$ . Thus, we also get a  $C^1$  function  $\varphi_{0,j+1}$  by equation (4.1). By induction, the  $\theta_{0j}$ 's are well defined for any  $j = 1, \dots, J_0$  and, by continuity, the order is preserved for  $\ell$  in a neighborhood of  $\hat{\ell}_0$ . Also, the implicit function theorem yields a recursive formula for the linearizations of  $\theta_{0j}$  and  $\varphi_{0j}$  at  $\hat{\ell}_0$ :

$$\langle d\theta_{0j}(\hat{\ell}_0), \delta\ell \rangle = \frac{-\sigma(\exp(\hat{\theta}_{0j} \vec{K}_{0,j-1})_* \varphi_{0,j-1}(\delta\ell), (\vec{K}_{0j} - \vec{K}_{0,j-1})(\hat{\lambda}(\hat{\theta}_{0j})))}{\sigma(\vec{K}_{0,j-1}, \vec{K}_{0j})(\hat{\lambda}(\hat{\theta}_{0j}))} \quad (4.2)$$

$$\begin{aligned} \varphi_{0j}(\delta\ell) = \exp(-\hat{\theta}_{0j} \vec{K}_{0j})_* \left\{ -\langle d\theta_{0j}(\hat{\ell}_0), \delta\ell \rangle (\vec{K}_{0j} - \vec{K}_{0,j-1})(\hat{\lambda}(\hat{\theta}_{0j})) + \right. \\ \left. + \exp(\hat{\theta}_{0j} \vec{K}_{0,j-1})_* \varphi_{0,j-1}(\delta\ell) \right\}. \quad (4.3) \end{aligned}$$

Let us show that  $\theta_{0,J_0+1}$  and  $\theta_{10}$  are also well defined. Let

$$\Psi_\nu(t, \ell) = (K_\nu - K_{0J_0}) \circ \exp t \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell) \quad \nu = 1, 2.$$

Then

$$\frac{\partial \Psi_\nu}{\partial t} \Big|_{(\hat{\tau}, \hat{\ell}_0)} = \sigma \left( \vec{K}_{0J_0}, \vec{K}_\nu \right) (\hat{\lambda}(\hat{\tau})) \quad \nu = 1, 2$$

which are positive by Assumption 2.3, so that  $\tau_1(\cdot)$  and  $\tau_2(\cdot)$  are both well defined again by means of the implicit function theorem.

Now let

$$\Phi'_{10}(t, \ell) = (K_{10} - K_\nu) \circ \exp t \vec{K}'_\nu \circ \varphi'_{0,J_0+1}(\ell), \quad \nu = 1, 2$$

then

$$\frac{\partial \Phi'_{10}}{\partial t} \Big|_{(\hat{\tau}, \hat{\ell}_0)} = \sigma \left( \vec{K}'_\nu, \vec{K}'_{10} \right) (\hat{\lambda}(\hat{\tau})), \quad \nu = 1, 2$$

which are positive again by Assumption 2.3, and the same argument applies.

As already mentioned, by assumption  $\hat{\theta}_{0,j-1} < \hat{\theta}_{0j}$  and  $\hat{\theta}_{0J_0} < \hat{\tau}$  so that, by continuity,  $\theta_{0,j-1}(\ell) < \theta_{0j}(\ell)$  and  $\theta_{0J_0}(\ell) < \theta_{0,J_0+1}(\ell) = \min\{\tau_1(\ell), \tau_2(\ell)\}$  for any  $\ell$  in a sufficiently small neighborhood of  $\hat{\ell}_0$ .

Let us now show that  $\theta_{0,J_0+1}(\ell) \leq \theta_{10}(\ell)$ . We examine all the possibilities for  $\tau_1(\ell)$  and  $\tau_2(\ell)$ :

- assume  $\ell$  is such that  $\theta_{0,J_0+1}(\ell) = \tau_1(\ell) < \tau_2(\ell)$ . Since  $\Psi_2(\tau_2(\ell), \ell) = 0$  one has

$$\begin{aligned} \Psi_2(t, \ell) &= \frac{\partial \Psi_2}{\partial t}(\tau_2(\ell), \ell)(t - \tau_2(\ell)) + o(t - \tau_2(\ell)) = \\ &= (t - \tau_2(\ell)) \left( \sigma \left( \vec{K}_{0J_0}, \vec{K}_2 \right) \Big|_{\exp \tau_2(\ell) \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell)} + o(1) \right). \end{aligned}$$

In particular, choosing  $t = \theta_{0,J_0+1}(\ell) = \tau_1(\ell)$ , by Assumption 2.3 and by continuity, when  $\ell$  is sufficiently close to  $\hat{\ell}_0$ , we have  $\Upsilon_\ell(\theta_{0,J_0+1}(\ell)) < 0$ , that is:

$$\Psi_2(\theta_{0,J_0+1}(\ell), \ell) = (K_2 - K_{0J_0}) \circ \exp \theta_{0,J_0+1}(\ell) \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell) < 0. \quad (4.4)$$

Since  $K_2 - K_{0J_0} = 2F_2 = K_{10} - K_1$ , equation (4.4) can also be written as

$$0 > (K_{10} - K_1) \circ \exp 0 \vec{K}_1 \circ \exp \theta_{0,J_0+1}(\ell) \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell),$$

i.e. the switch of the component  $u_2$  has not yet occurred at time  $\tau_1(\ell)$ , so that  $\theta_{10}^1(\ell) - \tau_1(\ell) > 0$ .

- Analogous proof holds if  $\theta_{0,J_0+1}(\ell) = \tau_2(\ell) < \tau_1(\ell)$ ,
- If  $\ell$  is such that  $\tau_1(\ell) = \tau_2(\ell)$ , then  $\theta_{10}(\ell) = \theta_{0,J_0+1}(\ell)$ .

For the simple switches occurring after the double one, by continuity, we have:

$$\theta_{1j}(\ell) \leq \max\{\theta_{1j}^1(\ell), \theta_{1j}^2(\ell)\} < \min\{\theta_{1,j+1}^1(\ell), \theta_{1,j+1}^2(\ell)\} \leq \theta_{1,j+1}(\ell)$$

for  $\ell$  in a sufficiently small neighborhood of  $\widehat{\ell}_0$ .

For the purpose of future reference we report here the expression for the differentials of the  $\theta_{0j}$ 's,  $\tau_\nu$ 's and  $\theta_{1j}^\nu$ 's, and of the  $\varphi_{0j*}$ 's  $\varphi_*^\nu$ 's and  $\varphi_{1j*}^\nu$ 's. Such formulas can be proved with an induction argument.

**Lemma 4.1.** *For any  $j = 1, \dots, J_0$  consider the following endomorphism of  $T_{\widehat{\ell}_0}(T^*M)$ :*

$$\Delta_{0j}\delta\ell = \delta\ell - \sum_{s=1}^j \langle d\theta_{0s}(\widehat{\ell}_0), \delta\ell \rangle (\vec{G}_{0s} - \vec{G}_{0,s-1})(\widehat{\ell}_0). \quad (4.5)$$

Then

$$\langle d\theta_{0j}(\widehat{\ell}_0), \delta\ell \rangle = \frac{-\sigma(\Delta_{0,j-1}\delta\ell, (\vec{G}_{0j} - \vec{G}_{0,j-1})(\widehat{\ell}_0))}{\sigma(\vec{G}_{0,j-1}, \vec{G}_{0j})(\widehat{\ell}_0)}, \quad (4.6)$$

$$\varphi_{0j*}(\delta\ell) = \exp(-\hat{\theta}_{0j}\vec{K}_{0j})_* \widehat{\mathcal{H}}_{\hat{\theta}_{0j}} \Delta_{0j}\delta\ell, \quad (4.7)$$

$$\langle d\tau_\nu(\widehat{\ell}_0), \delta\ell \rangle = \frac{-\sigma(\Delta_{0J_0}\delta\ell, (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0))}{\sigma(\vec{G}_{0J_0}, \vec{H}_\nu)(\widehat{\ell}_0)}, \quad (4.8)$$

$$\begin{aligned} \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell \rangle &= \frac{-1}{\sigma(\vec{H}_\nu, \vec{G}_{10})(\widehat{\ell}_0)} \\ &\sigma(\Delta_{0J_0}\delta\ell - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell \rangle (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0), (\vec{G}_{10} - \vec{H}_\nu)(\widehat{\ell}_0)) \end{aligned} \quad (4.9)$$

and

$$\varphi_{0,J_0+1*}^\nu(\delta\ell) = \exp(-\hat{\tau}\vec{K}_\nu)_* \widehat{\mathcal{H}}_{\hat{\tau}} (\Delta_{0J_0}\delta\ell - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell \rangle (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0)). \quad (4.10)$$

Moreover

$$\begin{aligned} \langle d\theta_{10}^1(\widehat{\ell}_0), \delta\ell \rangle &= \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle - \langle d(\tau_1 - \tau_2)(\widehat{\ell}_0), \delta\ell \rangle \frac{\sigma(\vec{G}_{0J_0}, \vec{H}_2)(\widehat{\ell}_0)}{\sigma(\vec{H}_1, \vec{G}_{10})(\widehat{\ell}_0)}, \\ \langle d\theta_{10}^2(\widehat{\ell}_0), \delta\ell \rangle &= \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle - \langle d(\tau_2 - \tau_1)(\widehat{\ell}_0), \delta\ell \rangle \frac{\sigma(\vec{G}_{0J_0}, \vec{H}_1)(\widehat{\ell}_0)}{\sigma(\vec{H}_2, \vec{G}_{10})(\widehat{\ell}_0)}. \end{aligned} \quad (4.11)$$

Also, for  $\nu = 1, 2$  and  $j = 0, \dots, J_1$  consider the endomorphisms

$$\begin{aligned} \Delta_{1j}^\nu\delta\ell &= \Delta_{0J_0}\delta\ell - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell \rangle (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0) - \\ &- \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell \rangle (\vec{G}_{10} - \vec{H}_\nu)(\widehat{\ell}_0) - \sum_{s=1}^j \langle d\theta_{1s}^\nu(\widehat{\ell}_0), \delta\ell \rangle (\vec{G}_{1s} - \vec{G}_{1,s-1})(\widehat{\ell}_0) \end{aligned} \quad (4.12)$$



Then

$$\varphi_{10*}^\nu(\delta\ell) = \exp(-\hat{\theta}_{10}\vec{K}_{10})_* \widehat{\mathcal{H}}_{\hat{\theta}_{10}*} \Delta_{10}^\nu \delta\ell, \quad (4.13)$$

$$\langle d\theta_{1j}^\nu(\widehat{\ell}_0), \delta\ell \rangle = \frac{-\sigma\left(\Delta_{1,j-1}^\nu \delta\ell, (\vec{G}_{1j} - \vec{G}_{1,j-1})(\widehat{\ell}_0)\right)}{\sigma\left(\vec{G}_{1,j-1}, \vec{G}_{1j}\right)(\widehat{\ell}_0)}, \quad (4.14)$$

and

$$\varphi_{1j*}^\nu(\delta\ell) = \exp(-\hat{\theta}_{1j}\vec{K}_{1j})_* \widehat{\mathcal{H}}_{\hat{\theta}_{1j}*} \Delta_{1j}^\nu \delta\ell. \quad (4.15)$$

Thus we get that the flow of the maximized Hamiltonian coincides with the flow of the Hamiltonian  $H: (t, \ell) \in [0, T] \times T^*M \mapsto H_t(\ell) \in T^*M$ :

$$H: (t, \ell) \in [0, T] \times T^*M \mapsto H(\ell) \in T^*M \quad (4.16)$$

$$H_t(\ell) := \begin{cases} K_{0j}(\ell) & t \in (\theta_{0j}(\ell), \theta_{0,j+1}(\ell)], \quad j = 0, \dots, J_0 \\ K_\nu(\ell) & t \in (\theta_{0,J_0+1}(\ell), \theta_{10}(\ell)], \quad \theta_{0,J_0+1}(\ell) = \tau_\nu(\ell) \\ K_{1j}(\ell) & t \in (\theta_{1j}(\ell), \theta_{1,j+1}(\ell)], \quad j = 0, \dots, J_1. \end{cases}$$

## 5 The second variation

To choose an appropriate horizontal Lagrangian manifold  $\Lambda$  we need to write the second variations of sub-problem (FP) and exploit their positivity. To write an invariant second variation, as introduced in [4], we write the pull-back  $\zeta_t(x, \delta, \varepsilon)$  of the flows  $S_t$  along the reference flow  $\widehat{S}_t$ , which also permits us to analyze the influence of the double switch on the final point of trajectories.

For the sake of greater clarity we first clear the field of all the notational difficulties by performing our analysis in the case when only the double switch occurs. Only afterwards we will discuss the general case.

Let  $\delta_{0,J_0+1} := \min\{\varepsilon_1, \varepsilon_2\}$ ,  $\delta_{10} := \max\{\varepsilon_1, \varepsilon_2\}$ . At time  $t = T$  we have

$$\begin{aligned} \zeta_T(x, \delta, \varepsilon) &= \widehat{S}_T^{-1} \circ S_T(x, \delta, \varepsilon) = \exp(-\delta_{10}) g_{10} \circ \\ &\quad \circ \exp(\delta_{10} - \delta_{01}) h_\nu \circ \exp(\delta_{01} - \delta_{00}) g_{0J_0}(x) \end{aligned}$$

where  $\nu = 1$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $\nu = 2$  otherwise. Let  $\widetilde{f}_1$  and  $\widetilde{f}_2$  be the pull-backs of  $f_1$  and  $f_2$  from time  $\widehat{t}$  to time  $t = 0$ , i.e.,

$$\widetilde{f}_\nu := \widehat{S}_{\widehat{t}*}^{-1} f_\nu \circ \widehat{S}_{\widehat{t}}, \quad \nu = 1, 2$$

so that

$$h_\nu = g_{0J_0} + 2\widetilde{f}_\nu, \quad \nu = 1, 2, \quad g_{10} = g_{0J_0} + 2\widetilde{f}_1 + 2\widetilde{f}_2.$$

The linearized flow at time  $T$  has the following form:

$$L(\delta x, \delta, \varepsilon) = \delta x + (\delta_{11} - \delta_{01})g_{01}(x) + 2(\delta_{11} - \varepsilon_1)\widetilde{f}_1(x) + 2(\delta_{11} - \varepsilon_2)\widetilde{f}_2(x),$$

which shows that the flow is  $C^1$ .

Let us now go back to the general case: at time  $t = T$  we have

$$\begin{aligned} \zeta_T(x, \delta, \varepsilon) &= \widehat{S}_T^{-1} \circ S_T(x, \delta, \varepsilon) = \exp(-\delta_{1J_1}) g_{1J_1} \circ \dots \circ \exp(\delta_{11} - \delta_{10}) g_{10} \circ \\ &\circ \exp(\delta_{10} - \delta_{0,J_0+1}) h_\nu \circ \exp(\delta_{0,J_0+1} - \delta_{0J_0}) g_{0J_0} \circ \dots \circ \exp \delta_{01} g_{00}(x) \end{aligned}$$

where  $\nu = 1$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $\nu = 2$  otherwise.

Define

$$\begin{aligned} a_{00} &:= \delta_{01}; \\ a_{0j} &:= \delta_{0,j+1} - \delta_{0j} \quad j = 1, \dots, J_0; \\ b &:= \delta_{10} - \delta_{0,J_0+1}; \\ a_{1j} &:= \delta_{1,j+1} - \delta_{1j} \quad j = 0, \dots, J_1 - 1; \\ a_{1J_1} &:= -\delta_{1J_1}. \end{aligned}$$

Then  $b + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} = 0$  and, with a slight abuse of notation, we may write

$$\begin{aligned} \zeta_T(x, a, b) &= \exp a_{1J_1} g_{1J_1} \circ \dots \circ \exp a_{11} g_{11} \circ \exp a_{10} g_{10} \\ &\circ \exp b h_\nu \circ \exp a_{0J_0} g_{0J_0} \circ \dots \circ \exp a_{01} g_{01} \circ \exp a_{00} g_{00}(x), \end{aligned}$$

- where  $\nu = 1$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $\nu = 2$  otherwise. Henceforward we will denote by  $a$  the  $(J_0 + J_1 + 2)$ -tuple  $(a_{00}, \dots, a_{0J_0}, a_{10}, \dots, a_{1J_1})$ .

The reference flow is the one associated to  $(a, b) = (0, 0)$ . The first order approximation of  $\zeta_T$  at a point  $(x, 0, 0)$  is given by

$$\begin{aligned} L(\delta x, a, b) &= \delta x + b h_\nu(x) + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} g_{ij}(x) = \\ &= \delta x + \sum_{j=0}^{J_0-1} a_{0j} g_{0j}(x) + (\delta_{0,J_0+1} - \delta_{0J_0}) g_{0J_0}(x) + \\ &\quad + (\delta_{10} - \delta_{0,J_0+1}) h_\nu(x) + (\delta_{11} - \delta_{10}) g_{10}(x) + \sum_{j=1}^{J_1} a_{1j} g_{1j}(x) \end{aligned}$$

where  $\nu = 1$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $\nu = 2$  otherwise. Introduce the pull-backs of  $f_1$  and  $f_2$  from time  $\hat{\tau}$  to time  $t = 0$ :

$$\tilde{f}_\nu := \widehat{S}_{\hat{\tau}}^{-1} f_\nu \circ \widehat{S}_{\hat{\tau}} \quad \nu = 1, 2.$$

Then  $h_\nu = g_{0J_0} + 2\tilde{f}_\nu$ ,  $\nu = 1, 2$ , and  $g_{10} = g_{0J_0} + 2\tilde{f}_1 + 2\tilde{f}_2$ . Thus

$$\begin{aligned} L(\delta x, a, b) &= \delta x + \sum_{j=0}^{J_0-1} a_{0j} g_{0j}(x) + (\delta_{0,J_0+1} - \delta_{0J_0}) g_{0J_0}(x) + \\ &\quad + (\delta_{10} - \delta_{0,J_0+1})(g_{0J_0} + 2\tilde{f}_\nu)(x) + (\delta_{11} - \delta_{10})(g_{0J_0} + 2\tilde{f}_1 + 2\tilde{f}_2)(x) + \sum_{j=1}^{J_1} a_{1j} g_{1j}(x) = \end{aligned}$$

$$\begin{aligned}
 &= \delta x + \sum_{j=0}^{J_0-1} a_{0j} g_{0j}(x) + (\delta_{11} - \delta_{0J_0}) g_{0J_0}(x) + 2(\delta_{11} - \varepsilon_1) \tilde{f}_1(x) + \\
 &+ 2(\delta_{11} - \varepsilon_2) \tilde{f}_2(x) + \sum_{j=1}^{J_1} a_{1j} g_{1j}(x).
 \end{aligned} \tag{5.1}$$

**Remark 5.1.** Equation 5.1 shows that in  $L(\delta x, a, b)$  we have the same first order expansion, whatever the sign of  $\varepsilon_2 - \varepsilon_1$ . This proves that the finite-dimensional problem (FP) is  $C^1$ .

Let  $\hat{\beta} := \beta \circ \hat{S}_T$  and  $\hat{\gamma} := \alpha + \hat{\beta}$ . Then the cost (FPa) can be written as

$$J(x, a, b) = \alpha(x) + \beta \circ S_T(x, a, b) = \alpha(x) + \hat{\beta} \circ \zeta_T(x, a, b)$$

By PMP  $d\hat{\gamma}(\hat{x}_0) = 0$ , thus the first variation of  $J$  at  $(x, a, b) = (\hat{x}_0, 0, 0)$  is given by

$$J'(\delta x, a, b) = \left( bh_\nu + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} g_{ij} \right) \cdot \hat{\beta}(\hat{x}_0)$$

which, by (5.1), does not depend on  $\nu$ , i.e. it does not depend on the sign of  $\varepsilon_2 - \varepsilon_1$ .

On the other hand, the second order expansion of  $\zeta_T^\nu(x, \cdot, \cdot)$  at  $(a, b) = (0, 0)$  is given by

$$\begin{aligned}
 &\exp \left( bh_\nu + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} g_{ij} + \frac{1}{2} \left\{ \sum_{j=0}^{J_0} a_{0j} \left[ g_{0j}, \sum_{s=j+1}^{J_0} a_{0s} g_{0s} + bh_\nu + \sum_{j=0}^{J_1} a_{1j} g_{1j} \right] + \right. \right. \\
 &\left. \left. + b \left[ h_\nu, \sum_{j=0}^{J_1} a_{1j} g_{1j} \right] + \sum_{j=0}^{J_1} a_{1j} \left[ g_{1j}, \sum_{s=j+1}^{J_1} a_{1s} g_{1s} \right] \right\} \right) (x).
 \end{aligned}$$

where  $\nu = 1$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $\nu = 2$  otherwise. Proceeding as in [5] we get for all  $(\delta x, a, b) \in \ker J'$ ,

$$\begin{aligned}
 J''_\nu[(\delta x, a, b)]^2 &= \frac{1}{2} \left\{ d^2\hat{\gamma}(\hat{x}_0)[\delta x]^2 + 2\delta x \cdot \left( \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} g_{ij} + bh_\nu \right) \cdot \hat{\beta}(\hat{x}_0) + \right. \\
 &+ \left( \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} g_{ij} + bh_\nu \right)^2 \cdot \hat{\beta}(\hat{x}_0) + \sum_{j=0}^{J_0} \sum_{i=0}^{j-1} a_{0i} a_{0j} [g_{0i}, g_{0j}] \cdot \hat{\beta}(\hat{x}_0) + \\
 &+ b \sum_{i=0}^{J_0} a_{0i} [g_{0i}, h_\nu] \cdot \hat{\beta}(\hat{x}_0) + \sum_{j=0}^{J_1} a_{1j} \left( \sum_{i=0}^{J_0} a_{0i} [g_{0i}, g_{1j}] + b[h_\nu, g_{1j}] + \right. \\
 &\left. + \sum_{i=0}^{j-1} a_{1i} [g_{1i}, g_{1j}] \right) \cdot \hat{\beta}(\hat{x}_0) \left. \right\}
 \end{aligned}$$

where, again,  $\nu = 1$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $\nu = 2$  otherwise.

**Remark 5.2.** The previous formula clearly shows that  $J_1'' = J_2''$  on  $\{(\delta x, a, b) : b = 0\}$ , i.e. on  $\{(\delta x, \delta, \varepsilon) : \varepsilon_1 = \varepsilon_2\}$ . The second variation is  $J_1''$  if  $\varepsilon_1 \leq \varepsilon_2$ ,  $J_2''$  otherwise. Its coercivity means that both  $J_1''$  and  $J_2''$  are coercive quadratic forms.

**Remark 5.3.** Isolating the addenda where  $a_{0J_0}$ ,  $b$ ,  $a_{10}$  appear, as in (5.1), one can easily see that  $J_1'' = J_2''$  if and only if  $[\tilde{f}_1, \tilde{f}_2] \cdot \widehat{\beta}(\widehat{x}_0) = 0$ , i.e. if and only if  $\langle \widehat{\lambda}(\hat{\tau}), [f_1, f_2](\widehat{x}_d) \rangle = 0$ . In other words: problem (FP) is twice differentiable at  $(x, \delta, \varepsilon) = (\widehat{x}_0, 0, 0)$  if and only if  $\langle \widehat{\lambda}(\hat{\tau}), [f_1, f_2](\widehat{x}_d) \rangle = 0$ .

The bilinear form associated to each  $J_\nu''$  is given by

$$\begin{aligned}
 J_\nu''((\delta x, a, b), (\delta y, c, d)) = & \frac{1}{2} \left\{ d^2 \widehat{\gamma}(\widehat{x}_0)(\delta x, \delta y) + \right. \\
 & + \delta y \cdot \left( \sum_{i=0}^{J_0} a_{0i} g_{0i} + b h_\nu + \sum_{i=0}^{J_1} a_{1i} g_{1i} \right) \cdot \widehat{\beta}(\widehat{x}_0) + \\
 & + \delta x \cdot \left( \sum_{i=0}^{J_0} c_{0i} g_{0i} + d h_\nu + \sum_{i=0}^{J_1} c_{1i} g_{1i} \right) \cdot \widehat{\beta}(\widehat{x}_0) + \\
 & + \left( \sum_{i=0}^{J_0} c_{0i} g_{0i} + d h_\nu + \sum_{i=0}^{J_1} c_{1i} g_{1i} \right) \cdot \left( \sum_{i=0}^{J_0} a_{0i} g_{0i} + b h_\nu + \sum_{i=0}^{J_1} a_{1i} g_{1i} \right) \cdot \widehat{\beta}(\widehat{x}_0) + \\
 & + \sum_{j=0}^{J_0} \sum_{i=0}^{j-1} a_{0i} c_{0j} [g_{0i}, g_{0j}] \cdot \widehat{\beta}(\widehat{x}_0) + d \sum_{i=0}^{J_0} a_{0i} [g_{0i}, h_\nu] \cdot \widehat{\beta}(\widehat{x}_0) + \\
 & \left. + \sum_{j=0}^{J_1} c_{1j} \left( \sum_{i=0}^{J_0} a_{0i} [g_{0i}, g_{1j}] + d [h_\nu, g_{1j}] + \sum_{i=0}^{j-1} a_{1i} [g_{1i}, g_{1j}] \right) \cdot \widehat{\beta}(\widehat{x}_0) \right\}
 \end{aligned} \tag{5.2}$$

By assumption, for each  $\nu = 1, 2$ ,  $J_\nu''$  is positive definite on

$$\begin{aligned}
 \mathcal{N}_0 := \left\{ (\delta x, a, b) \in T_{\widehat{x}_0} N_0 \times \mathbb{R}^{J_0+J_1+2} \times \mathbb{R} : \right. \\
 \left. b + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} = 0, \quad L(\delta x, a, b) \in T_{\widehat{x}_f} N_f \right\}.
 \end{aligned}$$

Again following the procedure of [5] we may redefine  $\alpha$  by adding a suitable second-order penalty at  $\widehat{x}_0$  (see e.g. [9], Theorem 13.2) and we may assume that each second variation  $J_\nu''$  is positive definite on

$$\begin{aligned}
 \mathcal{N} := \left\{ (\delta x, a, b) \in T_{\widehat{x}_0} M \times \mathbb{R}^{J_0+J_1+2} \times \mathbb{R} : \right. \\
 \left. b + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} = 0, \quad L(\delta x, a, b) \in T_{\widehat{x}_f} N_f \right\},
 \end{aligned}$$

i.e. we can remove the constraint on the initial point of admissible trajectories.

Let

$$\Lambda := \{d\alpha(x) : x \in M\}$$

and introduce the anti-symplectic isomorphism  $i$  as in [5],

$$i: (\delta p, \delta x) \in T_{\hat{x}_0}^* M \times T_{\hat{x}_0} M \mapsto -\delta p + d(-\hat{\beta})_* \delta x \in T(T^* M). \quad (5.3)$$

Define  $\vec{G}_{ij}'' = i^{-1}(\vec{G}_{ij}(\hat{\ell}_0))$ ,  $\vec{H}_\nu'' = i^{-1}(\vec{H}_\nu(\hat{\ell}_0))$ . The Hamiltonian fields  $\vec{G}_{ij}''$  and  $\vec{H}_\nu''$  are associated to the following linear Hamiltonians defined in  $T_{\hat{x}_0}^* M \times T_{\hat{x}_0} M$

$$G_{ij}''(\omega, \delta x) = \langle \omega, g_{ij}(\hat{x}_0) \rangle + \delta x \cdot g_{ij} \cdot \hat{\beta}(\hat{x}_0) \quad (5.4)$$

$$H_\nu''(\omega, \delta x) = \langle \omega, h_\nu(\hat{x}_0) \rangle + \delta x \cdot h_\nu \cdot \hat{\beta}(\hat{x}_0). \quad (5.5)$$

Moreover  $L_0'' := i^{-1}T_{\hat{\ell}_0} \Lambda = \left\{ \delta \ell \in T_{\hat{x}_0}^* M \times T_{\hat{x}_0} M : \delta \ell = (-D^2 \hat{\gamma}(\hat{x}_0)(\delta x, \cdot)) \right\}$ . With such notation, the bilinear form  $J_\nu''$  associated to the second variation can be written in a rather compact form, see, e.g. [5] or [14].

For any  $\delta e := (\delta x, a, b) \in \mathcal{N}$  let

$$\begin{aligned} \omega_0 &:= -D^2 \hat{\gamma}(\hat{x}_0)(\delta x, \cdot), \quad \delta \ell := (\omega_0, \delta x) = i^{-1}(d\alpha_* \delta x), \\ (\omega_\nu, \delta x_\nu) &:= \delta \ell + \sum_{i=0}^1 \sum_{j=0}^{J_i} a_{ij} \vec{G}_{ij}'' + b \vec{H}_\nu'' \quad \text{and} \quad \delta \ell_\nu := (\omega_\nu, \delta x_\nu). \end{aligned}$$

Then  $J_\nu''$  can be written as

$$\begin{aligned} J_\nu''((\delta x, a, b), (\delta y, c, d)) &= -\langle \omega_\nu, \delta y + \sum_{s=0}^{J_0} c_{0s} g_{0s} + d h_\nu + \sum_{s=0}^{J_1} c_{1s} g_{1s} \rangle \\ &+ \sum_{j=0}^{J_0} c_{0j} G_{0j}'' \left( \delta \ell + \sum_{s=0}^{j-1} a_{0s} \vec{G}_{0s}'' \right) + d H_\nu'' \left( \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'' \right) \\ &+ \sum_{j=0}^{J_1} c_{1j} G_{1j}'' \left( \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'' + b \vec{H}_\nu'' + \sum_{s=0}^{j-1} a_{1s} \vec{G}_{1s}'' \right) \end{aligned} \quad (5.6)$$

We shall study the positivity of  $J_\nu''$  as follows: consider

$$V := \left\{ (\delta x, a, b) \in \mathcal{N} : L(\delta x, a, b) = 0 \right\}$$

and the sequence

$$V_{01} \subset \dots \subset V_{0J_0} \subset V_{10} \subset \dots \subset V_{1J_1} = V$$

of sub-spaces of  $V$ , defined as follows

$$\begin{aligned} V_{0j} &:= \{ (\delta x, a, b) \in V : a_{0s} = 0 \quad \forall s = j+1, \dots, J_0, a_{1s} = 0 \} \\ V_{1j} &:= \{ (\delta x, a, b) \in V : a_{1s} = 0 \quad \forall s = j+1, \dots, J_1 \}. \end{aligned}$$

Observe that  $V_{0j}^1 = V_{0j}^2$  for any  $j = 0, \dots, J_0$ , so we denote these sets as  $V_{0j}$ . Moreover

$$\dim(V_{0j} \cap V_{0,j-1}^\perp) = \dim(V_{1k} \cap V_{1,k-1}^\perp) = 1, \quad \dim(V_{10} \cap V_{0J_0}^\perp) = 2$$

for any  $j = 2, \dots, J_0$ ,  $k = 0, \dots, J_1$  and  $\nu = 1, 2$  and  $J''_\nu$  is positive definite on  $\mathcal{N}$  if and only if it is positive definite on each  $V_{ij} \cap V_{i,j-1}^{\perp J''_\nu}$ ,  $V_{10} \cap V_{0J_0}^{\perp J''_\nu}$  and  $\mathcal{N} \cap V^{\perp J''_\nu}$ .

As in [5] one can prove a characterization, in terms of the maximized flow, of the intersections above. We state here such characterization without proofs which can be found in the aforementioned paper.

**Lemma 5.1.** *Let  $j = 1, \dots, J_0$  and  $\delta e = (\delta x, a, b) \in V_{0j}$ . Assume  $J''_\nu$  is positive definite on  $V_{0,j-1}$ . Then  $\delta e \in V_{0j} \cap V_{0,j-1}^{\perp J''_\nu}$  if and only if*

$$G''_{0s}(\delta\ell + \sum_{r=0}^{s-1} a_{0r} \vec{G}''_{0r}) = G''_{0,j-1}(\delta\ell + \sum_{s=0}^{j-2} a_{0s} \vec{G}''_{0s}), \quad \forall s = 0, \dots, j-2 \quad (5.7)$$

*i.e. if and only if*

$$a_{0s} = \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \quad \forall s = 0, \dots, j-2. \quad (5.8)$$

*In this case*

$$\begin{aligned} J''_\nu[\delta e]^2 &= a_{0j} (G''_{0j} - G''_{0,j-1})(\delta\ell + \sum_{s=0}^{j-1} a_{0s} \vec{G}''_{0s}) = \\ &= a_{0j} \sigma \left( \delta\ell + \sum_{s=0}^{j-1} a_{0s} \vec{G}''_{0s}, \vec{G}''_{0j} - \vec{G}''_{0,j-1} \right) \\ &= -a_{0j} \sigma \left( d\alpha_* \delta x + \sum_{s=0}^{j-1} a_{0s} \vec{G}''_{0s}(\widehat{\ell}_0), (\vec{G}''_{0j} - \vec{G}''_{0,j-1})(\widehat{\ell}_0) \right). \end{aligned} \quad (5.9)$$

**Lemma 5.2.** *Let  $\nu = 1, 2$  and  $\delta e = (\delta x, a, b) \in V_{10}$ . Assume  $J''_\nu$  is positive definite on  $V_{0,J_0}$ . Then  $\delta e \in V_{10} \cap V_{0J_0}^{\perp J''_\nu}$  if and only if*

$$G''_{0s}(\delta\ell + \sum_{\mu=0}^{s-1} a_{0\mu} \vec{G}''_{0\mu}) = G''_{0,J_0}(\delta\ell + \sum_{s=0}^{J_0-1} a_{0s} \vec{G}''_{0s}), \quad \forall s = 0, \dots, J_0 - 1 \quad (5.10)$$

*i.e. if and only if*

$$a_{0s} = \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \quad \forall s = 0, \dots, J_0 - 1. \quad (5.11)$$

In this case

$$\begin{aligned}
J_\nu''[\delta e]^2 &= b (H_\nu'' - G_{0,J_0}'') (\delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'') \\
&\quad + a_{10} (G_{10}'' - H_\nu'') (\delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'' + b \vec{H}_\nu'') = \\
&= b \sigma \left( \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'', \vec{H}_\nu'' - \vec{G}_{0,J_0}'' \right) + \\
&\quad + a_{10} \sigma \left( \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'' + b \vec{H}_\nu'', \vec{G}_{10}'' - \vec{H}_\nu'' \right) = \\
&= -b \sigma \left( d\alpha_* \delta x + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}(\widehat{\ell}_0), (\vec{H}_\nu - \vec{G}_{0,J_0})(\widehat{\ell}_0) \right) - \\
&\quad - a_{10} \sigma \left( d\alpha_* \delta x + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}(\widehat{\ell}_0) + b \vec{H}_\nu(\widehat{\ell}_0), (\vec{G}_{10} - \vec{H}_\nu)(\widehat{\ell}_0) \right).
\end{aligned} \tag{5.12}$$

**Lemma 5.3.** *Let  $\nu = 1, 2$ ,  $j = 1, \dots, J_1$  and  $\delta e = (\delta x, a, b) \in V_{1j}$ . Assume  $J_\nu''$  is positive definite on  $V_{1,j-1}$ . Then  $\delta e \in V_{1j} \cap V_{1,j-1}^{\perp J_\nu''}$  if and only if*

$$\begin{aligned}
G_{0s}''(\delta \ell + \sum_{i=0}^{s-1} a_{0i} \vec{G}_{0i}'') &= G_{1,j-1}''(\delta \ell + \sum_{i=0}^{J_0} a_{0i} \vec{G}_{0i}'' + b \vec{H}_\nu'' + \sum_{i=0}^{j-2} a_{1i} \vec{G}_{1i}'') = \\
&= H_\nu''(\delta \ell + \sum_{i=0}^{J_0} a_{0i} \vec{G}_{0i}'') = G_{1k}''(\delta \ell + \sum_{i=0}^{J_0} a_{0i} \vec{G}_{0i}'' + b \vec{H}_\nu'' + \sum_{i=0}^{k-1} a_{1i} \vec{G}_{1i}'') \\
&\quad \forall s = 0, \dots, J_0 \quad \forall k = 0, \dots, j-2
\end{aligned}$$

i.e. if and only if

$$\begin{aligned}
a_{0s} &= \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \quad \forall s = 0, \dots, J_0 \\
b &= \langle d(\theta_{10} - \theta_{0,J_0+1})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \\
a_{1s} &= \langle d(\theta_{1,s+1} - \theta_{1s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \quad \forall s = 0, \dots, j-2.
\end{aligned}$$

In this case

$$\begin{aligned}
J_\nu''[\delta e]^2 &= a_{1j} (G_{1j}'' - G_{1,j-1}'') (\delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'' + b \vec{H}_\nu'' + \sum_{i=0}^{j-1} a_{1i} \vec{G}_{1i}'') \\
&= a_{1j} \sigma \left( \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}'' + b \vec{H}_\nu'' + \sum_{i=0}^{j-1} a_{1i} \vec{G}_{1i}'', \vec{G}_{1j}'' - \vec{G}_{1,j-1}'' \right) \\
&= -a_{1j} \sigma \left( d\alpha_* \delta x + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}(\widehat{\ell}_0) + b \vec{H}_\nu(\widehat{\ell}_0) + \sum_{i=0}^{j-1} a_{1i} \vec{G}_{1i}(\widehat{\ell}_0), (\vec{G}_{1j} - \vec{G}_{1,j-1})(\widehat{\ell}_0) \right).
\end{aligned}$$

**Lemma 5.4.** *Let  $\nu = 1, 2$  and  $\delta e = (\delta x, a, b) \in \mathcal{N}$ . Assume  $J_\nu''$  is positive definite on  $V_{1J_1}$ . Then  $\delta e \in \mathcal{N} \cap V_{1J_1}^{\perp J_\nu''}$  if and only if*

$$\begin{aligned} G_{0s}''(\delta \ell + \sum_{i=0}^{s-1} a_{0i} \vec{G}_{0i}'') &= G_{1,J_1}''(\delta \ell + \sum_{i=0}^{J_0} a_{0i} \vec{G}_{0i}'' + b \vec{H}_\nu'' + \sum_{i=0}^{J_1-1} a_{1i} \vec{G}_{1i}'') = \\ &= H_\nu''(\delta \ell + \sum_{i=0}^{J_0} a_{0i} \vec{G}_{0i}'') = G_{1k}''(\delta \ell + \sum_{i=0}^{J_0} a_{0i} \vec{G}_{0i}'' + b \vec{H}_\nu'' + \sum_{i=0}^{k-1} a_{1i} \vec{G}_{1i}'') \\ &\qquad \qquad \qquad \forall s = 0, \dots, J_0 \quad \forall k = 0, \dots, J_1 \end{aligned}$$

i.e. if and only if  $\delta e \in \mathcal{N}$  and

$$\begin{aligned} a_{0s} &= \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \quad \forall s = 0, \dots, J_0 \\ b &= \langle d(\theta_{10} - \theta_{0,J_0+1})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \\ a_{1s} &= \langle d(\theta_{1,s+1} - \theta_{1s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \quad \forall s = 0, \dots, J_1 - 1. \end{aligned}$$

In this case

$$\begin{aligned} J_\nu''[\delta e]^2 &= -\langle \omega_\nu, \delta x + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} g_{is}(\widehat{x}_0) + b h_\nu(\widehat{x}_0) \rangle = \\ &= \sigma \left( (0, \delta x + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} g_{is}(\widehat{x}_0) + b h_\nu(\widehat{x}_0)), \right. \\ &\qquad \qquad \qquad \left. - D^2 \widehat{\gamma}(\widehat{x}_0)(\delta x, \cdot) + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} \vec{G}_{is}'' + b \vec{H}_\nu'' \right) = \\ &= -\sigma \left( d(-\widehat{\beta})_*(\delta x + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} g_{is}(\widehat{x}_0) + b h_\nu(\widehat{x}_0)), \right. \\ &\qquad \qquad \qquad \left. d\alpha_* \delta x + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} \vec{G}_{is}(\widehat{\ell}_0) + b \vec{H}_\nu(\widehat{\ell}_0) \right). \end{aligned}$$

## 6 The invertibility of the flow

We are now going to prove that the map

$$\text{id} \times \pi \mathcal{H}: (t, \ell) \in [0, T] \times \Lambda \mapsto (t, \pi \mathcal{H}_t(\ell)) \in [0, T] \times M$$

is one-to-one onto a neighborhood of the graph of  $\widehat{\xi}$ . Since the time interval  $[0, T]$  is compact and by the properties of flows, it suffices to show that  $\pi \mathcal{H}_{\widehat{\theta}_{ij}}, i = 1, 2, j = 1, \dots, J_i$  and  $\pi \mathcal{H}_{\widehat{\tau}}$  are one-to-one onto a neighborhood of  $\widehat{\xi}(\widehat{\theta}_{ij})$  and  $\widehat{\xi}(\widehat{\tau})$  in  $M$ , respectively.

The proof of the invertibility at the simple switching times  $\widehat{\theta}_{0j}, j = 1, \dots, J_0$  may be carried out either as in [5] or by means of Clarke's inverse function theorem (see [7, Thm



7.1.1.]), while the invertibility at the double switching time and at the simple switching times  $\hat{\theta}_{1j}$ ,  $j = 1, \dots, J_1$  will be proved by means of Clarke's inverse function theorem or by means of topological methods (see Theorem 7.6) according to the dimension of the kernel of  $d(\tau_1 - \tau_2)|_{T_{\hat{\ell}_0} \Lambda}$ .

For the sake of uniformity with the others switching times, for the simple switching times  $\hat{\theta}_{0j}$ ,  $j = 1, \dots, J_0$  and we give here the proof based on Clarke's inverse function theorem. Namely, we consider the expressions of  $\pi \mathcal{H}_{\hat{\theta}_{0j}}(\ell)$ , which are different according to whether  $\theta_{0j}(\ell)$  is greater than or smaller than  $\hat{\theta}_{0j}$ . We write the linearization of such expressions and their convex combinations. Finally, using the coercivity of the second variation on  $V_{0j}$  we prove that all their convex combinations are one-to-one.

The flow  $\mathcal{H}_{\hat{\theta}_{0j}}$  at time  $\hat{\theta}_{0j}$ , associated to the maximized Hamiltonian defined in equation (4.16), has the following expression:

$$\mathcal{H}_{\hat{\theta}_{0j}}(\ell) = \begin{cases} \exp \hat{\theta}_{0j} \vec{K}_{0,j-1}(\varphi_{0,j-1}(\ell)) & \text{if } \theta_{0j}(\ell) > \hat{\theta}_{0j} \\ \exp(\hat{\theta}_{0j} - \theta_{0j}(\ell)) \vec{K}_{0j} \circ \exp \theta_{0j}(\ell) \vec{K}_{0,j-1}(\varphi_{0,j-1}(\ell)) & \text{if } \theta_{0j}(\ell) < \hat{\theta}_{0j}. \end{cases}$$

**Lemma 6.1.** *Let  $j \in \{1, \dots, J_0\}$ . Define*

$$\begin{aligned} A_{0j} : \delta \ell \in T_{\hat{\ell}_0} \Lambda &\mapsto \pi_* \exp \hat{\theta}_{0j} \vec{K}_{0,j-1} \varphi_{0,j-1} \delta \ell \in T_{\hat{\xi}(\hat{\theta}_{0j})} M \\ B_{0j} : \delta \ell \in T_{\hat{\ell}_0} \Lambda &\mapsto A_{0j} \delta \ell - \langle d\theta_{0j}(\hat{\ell}_0), \delta \ell \rangle (k_{0j} - k_{0,j-1})|_{\hat{\xi}(\hat{\theta}_{0j})} \in T_{\hat{\xi}(\hat{\theta}_{0j})} M \end{aligned}$$

Then, for any  $t \in [0, 1]$ , the map

$$tA_{0j} + (1-t)B_{0j} : T_{\hat{\ell}_0} \Lambda \rightarrow T_{\hat{\xi}(\hat{\theta}_{0j})} M$$

is one-to-one.

*Proof.* Let  $t \in [0, 1]$  and let  $\delta \ell \in T_{\hat{\ell}_0} \Lambda$  such that  $(tA_{0j} + (1-t)B_{0j})(\delta \ell) = 0$ . We need to show that  $\delta \ell$  is null. From formula (4.3) it follows that  $\delta \ell$  is in  $\ker(tA_{0j} + (1-t)B_{0j})$  if and only if

$$\pi_* \hat{\mathcal{H}}_{\hat{\theta}_{0j}} \Delta_{0,j-1} \delta \ell = 0. \quad (6.1)$$

Let  $\delta x := \pi_* \delta \ell$ , so that  $\delta \ell = d\alpha_* \delta x$ . Equation (6.1) is equivalent to

$$\begin{aligned} \delta x + \sum_{s=1}^{j-2} \langle d(\theta_{0,s+1} - \theta_{0s})(\hat{\ell}_0), \delta \ell \rangle g_{0s}(\hat{x}_0) + \\ + (t \langle d\theta_{0j}(\hat{\ell}_0), \delta \ell \rangle - \langle d\theta_{0,j-1}(\hat{\ell}_0), \delta \ell \rangle) g_{0,j-1}(\hat{x}_0) - t \langle d\theta_{0j}(\hat{\ell}_0), \delta \ell \rangle g_{0j}(\hat{x}_0) = 0. \end{aligned} \quad (6.2)$$

Let  $\delta e := (\delta x, a, b)$  such that

$$\begin{aligned} a_{0s} &= \langle d(\theta_{0,s+1} - \theta_{0s})(\hat{\ell}_0), \delta \ell \rangle \quad s = 0, \dots, j-2 \\ a_{0,j-1} &= t \langle d\theta_{0j}(\hat{\ell}_0), \delta \ell \rangle - \langle d\theta_{0,j-1}(\hat{\ell}_0), \delta \ell \rangle \\ a_{0j} &= -t \langle d\theta_{0j}(\hat{\ell}_0), \delta \ell \rangle \\ a_{0s} &= b = a_{1r} = 0 \quad s = j+1, \dots, J_0, \quad r = 0, \dots, J_1. \end{aligned}$$

There are three possible cases:

- a) If  $t = 0$ , then  $\delta e \in V_{0,j-1} \cap V_{0,j-1}^{\perp J''_\nu} = \{0\}$ , because of the coercivity of  $J''_\nu$ .
- b) If  $t = 1$ , then  $\delta e \in V_{0j} \cap V_{0j}^{\perp J''_\nu} = \{0\}$ , because of the coercivity of  $J''_\nu$ . In both cases we thus have  $\delta x = 0$ , so that  $\delta \ell = d\alpha_* \delta x$  is also null.
- c) If  $t \in (0, 1)$ , then  $\delta e \in V_{0j} \cap V_{0,j-1}^{\perp J''_\nu}$ . Therefore, applying (5.9) we get

$$\begin{aligned}
 0 < J''_\nu[\delta e]^2 &= t \langle d\theta_{0j}(\widehat{\ell}_0), \delta \ell \rangle \sigma \left( \delta \ell + \sum_{s=0}^{j-2} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \delta \ell \rangle \vec{G}_{0s}(\widehat{\ell}_0) + \right. \\
 &+ (t \langle d\theta_{0j}(\widehat{\ell}_0), \delta \ell \rangle - \langle d\theta_{0,j-1}(\widehat{\ell}_0), \delta \ell \rangle) \vec{G}_{0,j-1}(\widehat{\ell}_0), (\vec{G}_{0j} - \vec{G}_{0,j-1})(\widehat{\ell}_0) \left. \right) = \\
 &= t \langle d\theta_{0j}(\widehat{\ell}_0), \delta \ell \rangle \sigma \left( \Delta_{0,j-1} \delta \ell + t \langle d\theta_{0j}(\widehat{\ell}_0), \delta \ell \rangle \vec{G}_{0,j-1}(\widehat{\ell}_0), (\vec{G}_{0j} - \vec{G}_{0,j-1})(\widehat{\ell}_0) \right) = \\
 &= -t(1-t) \langle d\theta_{0j}(\widehat{\ell}_0), \delta \ell \rangle^2 \sigma \left( \vec{G}_{0,j-1}, \vec{G}_{0j} \right) (\widehat{\ell}_0),
 \end{aligned}$$

a contradiction.  $\square$

Lemma 6.1 implies that Clarke's Generalized Jacobian of the map  $\pi \mathcal{H}_{\widehat{\theta}_{0j}}$  at  $\widehat{\ell}_0$  is of maximal rank. Therefore, by Clarke's inverse function theorem (see [7, Thm 7.1.1.]) the map  $\pi \mathcal{H}_{\widehat{\theta}_{0j}}$  is locally invertible about  $\widehat{\ell}_0$  with Lipschitz continuous inverse. Hence the map

$$\psi: (t, \ell) \in [0, T] \times \Lambda \mapsto (t, \pi \mathcal{H}_t(\ell)) \in [0, T] \times M \quad (6.3)$$

is locally invertible about  $[0, \widehat{\tau} - \varepsilon] \times \{\widehat{\ell}_0\}$ . In fact,  $\psi$  is locally one-to-one if and only if  $\pi \mathcal{H}_t$  is locally one-to-one in  $\widehat{\ell}_0$  for any  $t$ . On the other hand  $\pi \mathcal{H}_t$  is locally one-to-one for any  $t < \widehat{\tau}$  if and only if it is one-to-one at any  $\widehat{\theta}_{0j}$ .

We now show that such procedure can be carried out also on  $[\widehat{\tau} - \varepsilon, T] \times \{\widehat{\ell}_0\}$ , so that  $\psi$  will turn out to be locally invertible from a neighborhood  $[0, T] \times \mathcal{O} \subset [0, T] \times \Lambda$  of  $[0, T] \times \{\widehat{\ell}_0\}$  onto a neighborhood  $\mathcal{U} \subset [0, T] \times M$  of the graph  $\widehat{\Xi}$  of  $\widehat{\xi}$ . The first step will be proving the invertibility of  $\pi \mathcal{H}_{\widehat{\tau}}$  at  $\widehat{\ell}_0$ .

In a neighborhood of  $\widehat{\ell}_0$ ,  $\pi \mathcal{H}_{\widehat{\tau}}$  has the following piecewise representation:

1. if  $\min \{\tau_1(\ell), \tau_2(\ell)\} \geq \widehat{\tau}$ , then  $\pi \mathcal{H}_{\widehat{\tau}}(\ell) = \exp \widehat{\tau} \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell)$ ,
2. if  $\min \{\tau_1(\ell), \tau_2(\ell)\} = \tau_1(\ell) \leq \widehat{\tau} \leq \theta_{10}(\ell)$ , then

$$\pi \mathcal{H}_{\widehat{\tau}}(\ell) = \exp(\widehat{\tau} - \tau_1(\ell)) \vec{K}_1 \circ \exp \tau_1(\ell) \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell),$$

3. if  $\min \{\tau_1(\ell), \tau_2(\ell)\} = \tau_2(\ell) \leq \widehat{\tau} \leq \theta_{10}(\ell)$ , then

$$\pi \mathcal{H}_{\widehat{\tau}}(\ell) = \exp(\widehat{\tau} - \tau_2(\ell)) \vec{K}_2 \circ \exp \tau_2(\ell) \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell),$$

4. if  $\min \{\tau_1(\ell), \tau_2(\ell)\} = \tau_1(\ell) \leq \theta_{10}(\ell) \leq \widehat{\tau}$ , then

$$\begin{aligned}
 \pi \mathcal{H}_{\widehat{\tau}}(\ell) &= \exp(\widehat{\tau} \vec{K}_{10}) \circ \psi_{10}(\ell) = \exp(\widehat{\tau} - \theta_{10}(\ell)) \vec{K}_{10} \circ \\
 &\quad \circ \exp(\theta_{10}(\ell) - \tau_1(\ell)) \vec{K}_1 \circ \exp \tau_1(\ell) \vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell),
 \end{aligned}$$

5. if  $\min\{\tau_1(\ell), \tau_2(\ell)\} = \tau_2(\ell) \leq \theta_{10}(\ell) \leq \hat{\tau}$ , then

$$\begin{aligned} \pi\mathcal{H}_{\hat{\tau}}(\ell) &= \exp(\hat{\tau}\vec{K}_{10}) \circ \psi_{10}(\ell) = \exp(\hat{\tau} - \theta_{10}(\ell))\vec{K}_{10} \circ \\ &\quad \circ \exp(\theta_{10}(\ell) - \tau_2(\ell))\vec{K}_2 \circ \exp \tau_2(\ell)\vec{K}_{0J_0} \circ \varphi_{0J_0}(\ell). \end{aligned}$$

The invertibility of  $\pi\mathcal{H}_{\hat{\tau}}$  will be proved by means of two different arguments: in the generic case when  $d(\tau_1 - \tau_2)(\hat{\ell}_0): T_{\hat{\ell}_0}\Lambda \rightarrow \mathbb{R}$  is not identically zero, we will use the topological argument of Theorem 7.6 in the Appendix; whereas, in the opposite case we will apply Clarke's inverse function theorem [7, Thm 7.1.1.], as in the case of simple switches. In particular, in the special case when  $d\tau_1(\hat{\ell}_0)|_{T_{\hat{\ell}_0}\Lambda} \equiv d\tau_2(\hat{\ell}_0)|_{T_{\hat{\ell}_0}\Lambda} \equiv 0$  we will prove that  $\pi\mathcal{H}_{\hat{\tau}}$  is indeed differentiable at  $\hat{\ell}_0$ .

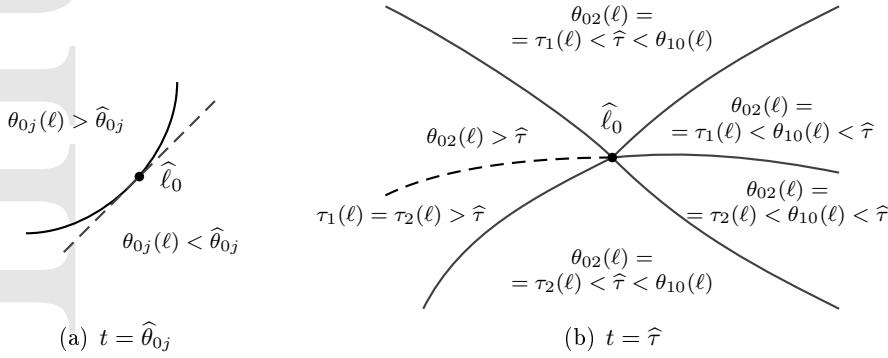


Figure 7: Local behaviour of  $\mathcal{H}_t$  near  $\hat{\ell}_0$  at a simple switching time and at the double one.

In all cases we need to write the piecewise linearized map  $(\pi\mathcal{H}_{\hat{\tau}})_*$ .

1. Let  $M^0 = \{\delta\ell \in T_{\hat{\ell}_0}\Lambda: \min\{\langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle, \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle\} \geq 0\}$ . Then

$$(\pi\mathcal{H}_{\hat{\tau}})_*\delta\ell = L^0\delta\ell := (\exp \hat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}\delta\ell \quad \forall \delta\ell \in M^0 \quad (6.4a)$$

2. Let  $M^{11} := \{\delta\ell \in T_{\hat{\ell}_0}\Lambda: \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle \leq 0 \leq \langle d\theta_{10}^1(\hat{\ell}_0), \delta\ell \rangle, \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle\}$ . Then

$$(\pi\mathcal{H}_{\hat{\tau}})_*\delta\ell = L^{11}\delta\ell := -2\langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle f_1(\hat{x}_{\hat{\tau}}) + \exp(\hat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}\delta\ell \quad \forall \delta\ell \in M^{11} \quad (6.4b)$$

3. Let  $M^{21} := \{\delta\ell \in T_{\hat{\ell}_0}\Lambda: \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle \leq 0 \leq \langle d\theta_{10}^2(\hat{\ell}_0), \delta\ell \rangle, \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle\}$ . Then

$$(\pi\mathcal{H}_{\hat{\tau}})_*\delta\ell = L^{21}\delta\ell := -2\langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle f_2(\hat{x}_{\hat{\tau}}) + \exp(\hat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}\delta\ell \quad \forall \delta\ell \in M^{21} \quad (6.4c)$$

4. Let  $M^{12} := \{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\theta_{10}^1(\widehat{\ell}_0), \delta\ell \rangle \leq 0, \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle\}$ . Then

$$\begin{aligned} (\pi\mathcal{H}_{\widehat{\tau}})_*\delta\ell &= L^{12}\delta\ell := -2\langle d\theta_{10}^1(\widehat{\ell}_0), \delta\ell \rangle f_2(\widehat{x}_{\widehat{\tau}}) - \\ &\quad - 2\langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle f_1(\widehat{x}_{\widehat{\tau}}) + \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}\delta\ell \quad \forall \delta\ell \in M^{12} \end{aligned} \quad (6.4d)$$

5. Let  $M^{22} := \{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\theta_{10}^2(\widehat{\ell}_0), \delta\ell \rangle \leq 0, \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle\}$ . Then

$$\begin{aligned} (\pi\mathcal{H}_{\widehat{\tau}})_*\delta\ell &= L^{22}\delta\ell := -2\langle d\theta_{10}^2(\widehat{\ell}_0), \delta\ell \rangle f_1(\widehat{x}_{\widehat{\tau}}) - \\ &\quad - 2\langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle f_2(\widehat{x}_{\widehat{\tau}}) + \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}\delta\ell \quad \forall \delta\ell \in M^{22} \end{aligned} \quad (6.4e)$$

**Lemma 6.2.** *The piecewise linearized maps (6.4) have the same orientation in the following sense: given any basis of  $T_{\widehat{\ell}_0} \Lambda_0$  and any basis of  $T_{\widehat{\xi}(\widehat{\tau})} M$ , the determinants of the matrices associated to the linear maps  $L^0, L^{\nu j}, \nu, j = 1, 2$ , in such bases, have the same sign.*

*Proof.* The proof is given by means of Lemma 7.1. We show that for any  $\delta\ell_1, \delta\ell_2 \in T_{\widehat{\ell}_0} \Lambda$  and  $\nu = 1, 2$  the following claims hold:

*Claim 1.* If  $\langle d\tau_\nu(\widehat{\ell}_0), \delta\ell \rangle < 0 < \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_1 \rangle$  then  $L^0(\delta\ell_1) \neq L^{\nu 1}(\delta\ell_2)$ , i.e.

$$\exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}(\delta\ell_1) \neq \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}(\delta\ell_2) - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_2 \rangle (k_\nu - k_{0J_0})(\widehat{x}_{\widehat{\tau}}).$$

*Claim 2.* If  $\langle d\theta_{01}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle < 0 < \langle d\theta_{01}^\nu(\widehat{\ell}_0), \delta\ell_1 \rangle$  then  $L^{\nu 1}(\delta\ell_1) \neq L^{\nu 2}(\delta\ell_2)$ , i.e.

$$\begin{aligned} \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}(\delta\ell_1) - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_1 \rangle (k_\nu - k_{0J_0})(\widehat{x}_{\widehat{\tau}}) &\neq \\ \neq \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}(\delta\ell_2) - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_2 \rangle (k_\nu - k_{0J_0})(\widehat{x}_{\widehat{\tau}}) - \\ - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle (k_{10} - k_\nu)(\widehat{x}_{\widehat{\tau}}) \end{aligned}$$

*Proof of Claim 1.* Fix  $\nu \in \{1, 2\}$  and assume, by contradiction, that there exist  $\delta\ell_1, \delta\ell_2 \in T_{\widehat{\ell}_0} \Lambda$  such that  $\langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_2 \rangle < 0 < \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_1 \rangle$  and

$$\begin{aligned} \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}(\delta\ell_1) &= \\ = \exp(\widehat{\tau}k_{0J_0})_*\pi_*\varphi_{0J_0*}(\delta\ell_2) - \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_2 \rangle (k_\nu - k_{0J_0})(\widehat{x}_{\widehat{\tau}}). \end{aligned} \quad (6.5)$$

Let  $\delta x_i := \pi_*\delta\ell_i, i = 1, 2$ . Taking the pull-back along the reference flow  $\widehat{S}_{\widehat{\tau}*}$  and using formula (4.7), equation (6.5) can be equivalently written as

$$\begin{aligned} \delta x_1 - \delta x_2 + \sum_{s=0}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \delta\ell_1 - \delta\ell_2 \rangle g_{0s}(\widehat{x}_0) + \\ + \left( -\langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_2 \rangle - \langle d\theta_{0J_0}(\widehat{\ell}_0), \delta\ell_1 - \delta\ell_2 \rangle \right) g_{0J_0}(\widehat{x}_0) + \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_2 \rangle h_\nu(\widehat{x}_0) = 0. \end{aligned}$$

That is, if we define  $\delta x := \delta x_1 - \delta x_2$ ,

$$a_{0s} := \begin{cases} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle & s = 0, \dots, J_0 - 1 \\ -\langle d\theta_{0J_0}(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle - \langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_2 \rangle & s = J_0 \end{cases}$$

$b := \langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_2 \rangle$ , and  $a_{1j} := 0$  for any  $j = 0, \dots, J_1$ , then  $\delta e := (\delta x, a, b) \in V_{10} \cap V_{0J_0}^\perp$ , so that by (5.12)

$$\begin{aligned} & -\langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_2 \rangle \sigma \left( d\alpha_* \delta x + \sum_{s=0}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_* \delta x \rangle \vec{G}_{0s}(\widehat{\ell}_0) \right. \\ & \left. + \langle d\theta_{0J_0}(\widehat{\ell}_0), d\alpha_* \delta x \rangle - \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_* \delta x_2 \rangle \vec{G}_{0J_0}(\widehat{\ell}_0), (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0) \right) > 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} & -\langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_2 \rangle \sigma \left( \Delta_{0J_0} d\alpha_* \delta x - \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_* \delta x_2 \rangle \vec{G}_{0J_0}(\widehat{\ell}_0), \right. \\ & \left. (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0) \right) > 0. \end{aligned}$$

Applying formula (4.8) we finally get

$$\langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_2 \rangle \langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_1 \rangle \sigma \left( \vec{G}_{0J_0}, \vec{H}_\nu \right) (\widehat{\ell}_0) > 0,$$

a contradiction.

*Proof of Claim 2.* Let us fix  $\nu \in \{1, 2\}$  and assume, by contradiction, that there exist  $\delta \ell_1, \delta \ell_2 \in T_{\widehat{\ell}_0} \Lambda$  such that  $\langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell_2 \rangle < 0 < \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell_1 \rangle$  and

$$\begin{aligned} & \exp(\hat{\tau} k_{0J_0})_* \pi_* \varphi_{0J_0}(\delta \ell_1) - \langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_1 \rangle (k_\nu - k_{0J_0})(\widehat{x}_{\hat{\tau}}) = \\ & = \exp(\hat{\tau} k_{0J_0})_* \pi_* \varphi_{0J_0}(\delta \ell_2) - \langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_2 \rangle (k_\nu - k_{0J_0})(\widehat{x}_{\hat{\tau}}) - \\ & \quad - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell_2 \rangle (k_{10} - k_\nu)(\widehat{x}_{\hat{\tau}}) \quad (6.6) \end{aligned}$$

Let  $\delta x_i := \pi_* \delta \ell_i$ ,  $i = 1, 2$ . Taking the pull-back along the reference flow and using formula (4.7), equation (6.6) can be equivalently written as

$$\begin{aligned} & \delta x_1 - \delta x_2 + \sum_{s=0}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle g_{0s}(\widehat{x}_0) + \\ & \quad + \langle d(\tau_\nu - \theta_{0J_0})(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle g_{0J_0}(\widehat{x}_0) + \\ & \quad + \left( -\langle d\tau_\nu(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell_2 \rangle \right) h_1(\widehat{x}_0) + \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell_2 \rangle g_{10}(\widehat{x}_0) = 0. \end{aligned}$$

That is, if we define  $\delta x := \delta x_1 - \delta x_2$ ,

$$a_{0s} := \begin{cases} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle & s = 0, \dots, J_0 - 1 \\ \langle d(\tau_\nu - \theta_{0J_0})(\widehat{\ell}_0), \delta \ell_1 - \delta \ell_2 \rangle & s = J_0 \end{cases}$$

$b := -\langle d\tau_\nu(\widehat{\ell}_0), \delta\ell_1 - \delta\ell_2 \rangle - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle$ , and

$$a_{1s} := \begin{cases} \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle & s = 0 \\ 0 & s = 1, \dots, J_1, \end{cases}$$

then  $\delta e := (\delta x, a, b) \in V_{10} \cap V_{0J_0}^\perp$  so that by Lemma 5.2,

$$\begin{aligned} & \left( \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle + \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \right) \sigma \left( d\alpha_*\delta x + \right. \\ & \quad \left. + \sum_{s=0}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_*\delta x \rangle \vec{G}_{0s}(\widehat{\ell}_0) + \right. \\ & \quad \left. + \langle d(\tau_\nu - \theta_{0J_0})(\widehat{\ell}_0), d\alpha_*\delta x \rangle \vec{G}_{0J_0}(\widehat{\ell}_0), (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0) \right) - \\ & \quad - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \sigma \left( d\alpha_*\delta x + \sum_{s=0}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), d\alpha_*\delta x \rangle \vec{G}_{0s}(\widehat{\ell}_0) + \right. \\ & \quad \left. + \langle d(\tau_\nu - \theta_{0J_0})(\widehat{\ell}_0), d\alpha_*\delta x \rangle \vec{G}_{0J_0}(\widehat{\ell}_0) - (\langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle + \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle) \vec{H}_\nu(\widehat{\ell}_0), \right. \\ & \quad \left. (\vec{G}_{10} - \vec{H}_\nu)(\widehat{\ell}_0) \right) > 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \left( \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle + \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \right) \sigma \left( \Delta_{0J_0} d\alpha_*\delta x + \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle \vec{G}_{0J_0}(\widehat{\ell}_0), \right. \\ & \quad \left. (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0) \right) - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \sigma \left( \Delta_{0J_0} d\alpha_*\delta x + \right. \\ & \quad \left. + \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle \vec{G}_{0J_0}(\widehat{\ell}_0) - (\langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle \right. \\ & \quad \left. + \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle) \vec{H}_\nu(\widehat{\ell}_0), (\vec{G}_{10} - \vec{H}_\nu)(\widehat{\ell}_0) \right) > 0 \end{aligned}$$

that is

$$\begin{aligned} & \left( \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle + \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \right) \left( -\langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle + \langle d\tau_\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle \right) \\ & \quad \sigma \left( \vec{G}_{0J_0}, \vec{H}_\nu \right) (\widehat{\ell}_0) - \\ & \quad - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \left( -\langle d\theta_{10}^\nu(\widehat{\ell}_0), d\alpha_*\delta x \rangle - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \right) \sigma \left( \vec{H}_\nu, \vec{G}_{10} \right) (\widehat{\ell}_0) > 0. \quad (6.7) \end{aligned}$$

Since  $d\alpha_*\delta x = \delta\ell_1 - \delta\ell_2$ , we get  $\langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_2 \rangle \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell_1 \rangle \sigma \left( \vec{H}_\nu, \vec{G}_{10} \right) (\widehat{\ell}_0) > 0$ , a contradiction.  $\square$

We can now complete the proof of the local invertibility of  $\pi\mathcal{H}_{\hat{\tau}}$ . Let us first consider the generic case when  $d(\tau_1 - \tau_2)(\widehat{\ell}_0)$  is not identically zero on  $T_{\widehat{\ell}_0}\Lambda$ .

We need to express the boundaries between the adjacent sectors  $M^0, M^{\nu j}$ .

- The boundary between  $M^0$  and  $M^{11}$  is given by

$$\{\delta\ell \in T_{\widehat{\ell}_0}\Lambda : 0 = \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle\};$$

- The boundary between  $M^0$  and  $M^{21}$  is given by

$$\{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : 0 = \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle\};$$

- The boundary between  $M^{11}$  and  $M^{12}$  is given by

$$\{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : \langle d\theta_{10}^1(\widehat{\ell}_0), \delta\ell \rangle = 0, \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle\};$$

- The boundary between  $M^{21}$  and  $M^{22}$  is given by

$$\{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : \langle d\theta_{10}^2(\widehat{\ell}_0), \delta\ell \rangle = 0, \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle\};$$

- The boundary between  $M^{12}$  and  $M^{22}$  is given by

$$\{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle = \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle \leq 0\};$$

According to Theorem 7.6, in order to prove the invertibility of our map it is sufficient to prove that both the map and its linearization are continuous in a neighborhood of  $\widehat{\ell}_0$  and of 0 respectively, that they maintain the orientation and that there exists a point  $\overline{\delta y}$  whose preimage is a singleton that belongs to at most two of the above defined sectors.

Notice that the continuity of  $\pi\mathcal{H}_{\widehat{\tau}}$  follows from the very definition of the maximized flow. Discontinuities of  $(\pi\mathcal{H}_{\widehat{\tau}})_*$  may occur only at the boundaries described above. A direct computation in formulas (6.4) shows that this is not the case. Let us now prove the last assertion.

For ‘‘symmetry’’ reasons it is convenient to look for  $\overline{\delta y}$  among those which belong to the image of the set  $\{\delta\ell \in T_{\widehat{\ell}_0} \Lambda : 0 < \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle = \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle\}$ . Observe that  $\langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle = \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle$  also implies  $\langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell \rangle = \langle d\tau_\nu(\widehat{\ell}_0), \delta\ell \rangle$ ,  $\nu = 1, 2$ , see formulas (4.11).

Let  $\overline{\delta\ell} \in T_{\widehat{\ell}_0} \Lambda$  such that  $0 < \langle d\tau_1(\widehat{\ell}_0), \overline{\delta\ell} \rangle = \langle d\tau_2(\widehat{\ell}_0), \overline{\delta\ell} \rangle$  and let  $\overline{\delta y} := L^0\overline{\delta\ell}$ .

Clearly  $\overline{\delta y}$  has at most one preimage per each of the above polyhedral cones. Let us prove that actually its preimage is the singleton  $\{\overline{\delta\ell}\}$ .

In fact we show that for  $\nu, j = 1, 2$ , there is no  $\delta\ell \in M^{\nu j}$  such that  $L^{\nu j}(\delta\ell) = \overline{\delta y}$ .

1. Fix  $\nu \in \{1, 2\}$  and assume, by contradiction, that there exists  $\delta\ell \in M^{1\nu}$  such that  $L^{1\nu}\delta\ell = \overline{\delta y}$ . The contradiction is shown exactly as in the proof of Claim 1 in Lemma 6.2.

2. Fix  $\nu \in \{1, 2\}$  and assume, by contradiction, that there exists  $\delta\ell \in M^{\nu 2}$  such that  $L^{\nu 2}\delta\ell = \overline{\delta y}$  that is: let  $\overline{\delta x} := \pi_*\overline{\delta\ell}$ , and  $\delta x := \pi_*\delta\ell$ . Taking the pull-back along the reference flow at time  $\widehat{\tau}$ , and recalling formula (4.7) we assume by contradiction that

$$\begin{aligned} \overline{\delta x} - \sum_{s=1}^{J_0} \langle d\theta_{0s}(\widehat{\ell}_0), \overline{\delta\ell} \rangle (g_{0s} - g_{0,s-1})(\widehat{x}_0) &= \delta x - \sum_{s=1}^{J_0} \langle d\theta_{0s}(\widehat{\ell}_0), \delta\ell \rangle (g_{0,s} - g_{0,s-1})(\widehat{x}_0) - \\ &\quad - \langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle (h_\nu - g_{0J_0})(\widehat{x}_0) - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta\ell \rangle (g_{10} - h_\nu)(\widehat{x}_0). \end{aligned}$$

or, equivalently,

$$\begin{aligned} \overline{\delta x} - \delta x + \sum_{s=1}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \overline{\delta \ell} - \delta \ell \rangle g_{0s}(\widehat{x}_0) - \\ - \left( \langle d\theta_{0J_0}(\widehat{\ell}_0), \overline{\delta \ell} - \delta \ell \rangle + \langle d\tau_1(\widehat{\ell}_0), \delta \ell \rangle \right) g_{0J_0}(\widehat{x}_0) - \\ - \langle d(\theta_{10}^\nu - \tau_\nu)(\widehat{\ell}_0), \delta \ell \rangle h_1(\widehat{x}_0) + \langle d\theta_{10}(\widehat{\ell}_0), \delta \ell \rangle g_{10}(\widehat{x}_0) = 0. \end{aligned}$$

Let  $\delta e := (\overline{\delta x} - \delta x, a, b)$ , where,

$$\begin{aligned} a_{0s} &:= \begin{cases} \langle d(\theta_{0,s+1} - \theta_{0s})(\widehat{\ell}_0), \overline{\delta \ell} - \delta \ell \rangle & s = 0, \dots, J_0 - 1, \\ \langle d\theta_{0J_0}(\widehat{\ell}_0), \overline{\delta \ell} - \delta \ell \rangle - \langle d\tau_1(\widehat{\ell}_0), \delta \ell \rangle & s = J_0, \end{cases} \\ b &:= -\langle d(\theta_{10}^\nu - \tau_\nu)(\widehat{\ell}_0), \delta \ell \rangle, \\ a_{1s} &:= \begin{cases} \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell \rangle & s = 0, \\ a_{1s} = 0 & s = 1, \dots, J_1. \end{cases} \end{aligned}$$

Then  $\delta e \in V_{10} \cap V_{0J_0}^{\perp J''}$  and Lemma 5.2 applies:

$$\begin{aligned} 0 < J''_\nu[\delta e]^2 &= -b \boldsymbol{\sigma} \left( \overline{\delta \ell} - \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}(\widehat{\ell}_0), (\vec{H}_\nu - \vec{G}_{0J_0})(\widehat{\ell}_0) \right) - \\ &- a_{10} \boldsymbol{\sigma} \left( \overline{\delta \ell} - \delta \ell + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}(\widehat{\ell}_0) + b \vec{H}_\nu(\widehat{\ell}_0), (\vec{G}_{10} - \vec{H}_1)(\widehat{\ell}_0) \right) = \\ &= \langle d(\theta_{10}^\nu - \tau_\nu)(\widehat{\ell}_0), \delta \ell \rangle \left( \langle d\tau_\nu(\widehat{\ell}_0), \overline{\delta \ell} - \delta \ell \rangle - \langle d\tau_\nu(\widehat{\ell}_0), \delta \ell \rangle \right) \boldsymbol{\sigma} \left( \vec{G}_{0J_0}, \vec{H}_\nu \right) (\widehat{\ell}_0) - \\ &- \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell \rangle \left( -\langle d\theta_{10}^\nu(\widehat{\ell}_0), \overline{\delta \ell} - \delta \ell \rangle - \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell \rangle \right) \boldsymbol{\sigma} \left( \vec{H}_\nu, \vec{G}_{10} \right) (\widehat{\ell}_0) + \\ &+ \langle d\tau_\nu(\widehat{\ell}_0), \overline{\delta \ell} \rangle \boldsymbol{\sigma} \left( \vec{G}_{0J_0}, \vec{H}_{3-\nu} \right) (\widehat{\ell}_0) = \\ &= \langle d(\theta_{10}^\nu - \tau_\nu)(\widehat{\ell}_0), \delta \ell \rangle \langle d\tau_\nu(\widehat{\ell}_0), \overline{\delta \ell} \rangle \boldsymbol{\sigma} \left( \vec{G}_{0J_0}, \vec{H}_\nu \right) (\widehat{\ell}_0) - \\ &- \langle d\theta_{10}^\nu(\widehat{\ell}_0), \delta \ell \rangle \left( \langle d\theta_{10}^\nu(\widehat{\ell}_0), \overline{\delta \ell} \rangle \boldsymbol{\sigma} \left( \vec{H}_\nu, \vec{G}_{10} \right) (\widehat{\ell}_0) + \right. \\ &\left. + \langle d\tau_\nu(\widehat{\ell}_0), \overline{\delta \ell} \rangle \boldsymbol{\sigma} \left( \vec{G}_{0J_0}, \vec{H}_{3-\nu} \right) (\widehat{\ell}_0) \right) \end{aligned}$$

which is a contradiction, since all the addenda are negative.

By Theorem 7.6 this proves the invertibility of  $\pi\mathcal{H}_{\hat{\tau}}$ , hence  $\psi$  is one-to-one in a neighborhood of  $[0, \hat{\theta}_{10} - \varepsilon] \times \{\widehat{\ell}_0\}$ .

Assume now that the non generic case  $T_{\widehat{\ell}_0} \Lambda \subset \ker d(\tau_1 - \tau_2)(\widehat{\ell}_0)$  holds. We are going to prove the Lipschitz invertibility of  $\pi\mathcal{H}_{\hat{\tau}}|_\Lambda$  by means of Clarke's inverse functions theorem, see [7]. The generalized Jacobian  $\partial(\pi\mathcal{H}_{\hat{\tau}})(\widehat{\ell}_0)$  (in the sense of Clarke) of  $\pi\mathcal{H}_{\hat{\tau}}: \Lambda \rightarrow M$  at  $\widehat{\ell}_0$  is the closed convex hull of the linear maps  $L^0, L^{\nu j}, \nu, j = 1, 2$  defined in (6.4).



We distinguish between two sub-cases:

1.  $\langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle = \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle = 0$  for any  $\delta\ell \in T_{\widehat{\ell}_0}\Lambda$

In this case we also have  $d\theta_{10}^1(\widehat{\ell}_0)|_{T_{\widehat{\ell}_0}\Lambda} \equiv d\theta_{10}^2(\widehat{\ell}_0)|_{T_{\widehat{\ell}_0}\Lambda} \equiv 0$ , see formulas (4.11), hence all the linear maps  $L^0, L^{\nu j}, \nu, j = 1, 2$  defined in (6.4) coincide with the map  $L^0$ , so that  $\pi\mathcal{H}_{\widehat{\tau}}$  is differentiable at  $\widehat{\ell}_0$ . The invertibility of  $L^0$  and Clarke's invertibility theorem yield the claim.

2.  $\langle d\tau_1(\widehat{\ell}_0), \delta\ell \rangle = \langle d\tau_2(\widehat{\ell}_0), \delta\ell \rangle$  for any  $\delta\ell \in T_{\widehat{\ell}_0}\Lambda$  but  $\ker(d\tau_1(\widehat{\ell}_0)|_{T_{\widehat{\ell}_0}\Lambda}) \neq T_{\widehat{\ell}_0}\Lambda$ . In this case we also have  $d\theta_{10}^1(\widehat{\ell}_0)|_{T_{\widehat{\ell}_0}\Lambda} \equiv d\theta_{10}^2(\widehat{\ell}_0)|_{T_{\widehat{\ell}_0}\Lambda} \equiv d\tau_1(\widehat{\ell}_0)|_{T_{\widehat{\ell}_0}\Lambda}$  (see formulas (4.11)) so that  $L^{12} \equiv L^{22}$ .

Let  $\{v_1, v_2, \dots, v_n\}$  be a basis of  $T_{\widehat{x}_0}M$  such that  $\langle d\tau_1(\widehat{\ell}_0), d\alpha_*v_1 \rangle = 1$  and  $\langle d\tau_1(\widehat{\ell}_0), d\alpha_*v_i \rangle = 0$  for  $i = 2, \dots, n$ . We will show that  $\partial(\pi\mathcal{H}_{\widehat{\tau}})(\widehat{\ell}_0)$  is made up of invertible matrices by showing that

$$(L^0)^{-1}(t_0L^0 + t_1L^{11} + t_2L^{21} + t_3L^{12} + t_4L^{22}) \circ d\alpha_*$$

is invertible for any  $t_0, \dots, t_4 \geq 0$  such that  $\sum_{i=0}^4 t_i = 1$ .

Let  $c_i^\nu, \nu = 1, 2, i = 1, \dots, n$  such that

$$(h_\nu - g_{0J_0})(\widehat{x}_0) = \sum_{i=1}^n c_i^\nu v_i.$$

We have

$$(L^0)^{-1}L^{\nu j} d\alpha_*v_i = v_i \quad i = 2, \dots, n \text{ and } \nu, j = 1, 2$$

and, for each  $\nu = 1, 2$ :

$$\begin{aligned} (L^0)^{-1}L^{\nu 1} d\alpha_*v_1 &= v_1 - (h_\nu - g_{0J_0})(\widehat{x}_0) = (1 - c_1^\nu)v_1 - \sum_{k=2}^n c_k^\nu v_k \\ (L^0)^{-1}L^{\nu 2} d\alpha_*v_1 &= v_1 - (h_\nu - g_{0J_0})(\widehat{x}_0) - (g_{10} - h_\nu)(\widehat{x}_0) = \\ &= (1 - c_1^1 - c_1^2)v_1 - \sum_{k=2}^n (c_k^1 + c_k^2)v_k. \end{aligned}$$

Thus the determinant of  $(L^0)^{-1}(t_0L^0 + t_1L^{11} + t_2L^{21} + t_3L^{12} + t_4L^{22}) \circ d\alpha_*$  is given by  $t_0 + t_1 \det(L^0)^{-1}L^{11} d\alpha_* + t_2 \det(L^0)^{-1}L^{21} d\alpha_* + (t_3 + t_4) \det(L^0)^{-1}L^{12} d\alpha_*$  which cannot be null since all the addenda are positive as it follows from Lemmata 6.2 and 7.1. This concludes the proof of the invertibility of  $\pi\mathcal{H}_{\widehat{\tau}}$ . Let us now turn to  $\pi\mathcal{H}_{\widehat{\theta}_{1j}}, j = 1, \dots, J_1$ .

For any  $j = 1, \dots, J_1$ , there are four regions in  $\Lambda$ , characterized by the following properties

$$\begin{aligned} \{\ell \in \Lambda : \theta_{1j}(\ell) \geq \widehat{\theta}_{1j} \text{ and } \theta_{0, J_0+1}(\ell) = \tau_1(\ell)\}, \\ \{\ell \in \Lambda : \theta_{1j}(\ell) \geq \widehat{\theta}_{1j} \text{ and } \theta_{0, J_0+1}(\ell) = \tau_2(\ell)\}, \\ \{\ell \in \Lambda : \theta_{1j}(\ell) < \widehat{\theta}_{1j} \text{ and } \theta_{0, J_0+1}(\ell) = \tau_1(\ell)\}, \\ \{\ell \in \Lambda : \theta_{1j}(\ell) < \widehat{\theta}_{1j} \text{ and } \theta_{0, J_0+1}(\ell) = \tau_2(\ell)\}. \end{aligned}$$

As for  $\pi\mathcal{H}_{\hat{\tau}}$ ,  $\pi\mathcal{H}_{\hat{\theta}_{1j}}$  turns out to be a Lipschitz continuous, piecewise  $C^1$  application. Its invertibility can be proved applying again Theorem 7.6. Let us write the piecewise linearized map  $(\pi\mathcal{H}_{\hat{\theta}_{1j}})_*$

- Let  $N_{1j}^{10} := \{\delta\ell \in T_{\hat{\ell}_0}\Lambda : \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle, \langle d\theta_{1j}^1(\hat{\ell}_0), \delta\ell \rangle \geq 0\}$ . Then

$$(\pi\mathcal{H}_{\hat{\theta}_{1j}})_*\delta\ell = A_{1j}^1\delta\ell := \exp(\hat{\theta}_{1j}k_{1,j-1*})\pi_*\varphi_{1,j-1}^1(\delta\ell)$$

- Let  $N_{1j}^{20} := \{\delta\ell \in T_{\hat{\ell}_0}\Lambda : \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle, \langle d\theta_{1j}^2(\hat{\ell}_0), \delta\ell \rangle \geq 0\}$ . Then

$$(\pi\mathcal{H}_{\hat{\theta}_{1j}})_*\delta\ell = A_{1j}^2\delta\ell := \exp(\hat{\theta}_{1j}k_{1,j-1*})\pi_*\varphi_{1,j-1}^2(\delta\ell)$$

- Let  $N_{1j}^{11} := \{\delta\ell \in T_{\hat{\ell}_0}\Lambda : \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle, \langle d\theta_{1j}^1(\hat{\ell}_0), \delta\ell \rangle \leq 0\}$ . Then

$$\begin{aligned} (\pi\mathcal{H}_{\hat{\theta}_{1j}})_*\delta\ell &= B_{1j}^1\delta\ell := \exp(\hat{\theta}_{1j}k_{1,j-1*})\pi_*\varphi_{1,j-1}^1(\delta\ell) - \\ &\quad - \langle d\theta_{1j}^1(\hat{\ell}_0), \delta\ell \rangle (k_{1j} - k_{1,j-1})(\hat{x}_{1j}) \end{aligned}$$

- Let  $N_{1j}^{21} := \{\delta\ell \in T_{\hat{\ell}_0}\Lambda : \langle d\tau_2(\hat{\ell}_0), \delta\ell \rangle \leq \langle d\tau_1(\hat{\ell}_0), \delta\ell \rangle, \langle d\theta_{1j}^2(\hat{\ell}_0), \delta\ell \rangle \leq 0\}$ . Then

$$\begin{aligned} (\pi\mathcal{H}_{\hat{\theta}_{1j}})_*\delta\ell &= B_{1j}^2\delta\ell := \exp(\hat{\theta}_{1j}\vec{K}_{1,j-1*})\pi_*\varphi_{1,j-1}^2(\delta\ell) - \\ &\quad - \langle d\theta_{1j}^2(\hat{\ell}_0), \delta\ell \rangle (k_{1j} - k_{1,j-1})(\hat{x}_{1j}) \end{aligned}$$

Analogously to what we did at time  $\hat{\tau}$ , let us first consider the non degenerate case  $\langle d(\tau_1 - \tau_2)(\hat{\ell}_0), \delta\ell \rangle \neq 0$  for some  $\delta\ell \in T_{\hat{\ell}_0}\Lambda$ : according to Theorem 7.6, we only have to prove that both the map and its piecewise linearization are continuous in a neighborhood of  $\hat{\ell}_0$  and of 0 respectively, that the linearized pieces are orientation preserving and that there exists a point  $\overline{\delta y}$  whose preimage is a singleton.

The only nontrivial part is the last statement which can be proved by picking  $\overline{\delta y} \in A_{1j}^1(N_{1j}^{10}) \cap A_{1j}^2(N_{1j}^{20})$ : let  $\overline{\delta\ell} \in T_{\hat{\ell}_0}\Lambda$  such that  $\langle d\tau_1(\hat{\ell}_0), \overline{\delta\ell} \rangle = \langle d\tau_2(\hat{\ell}_0), \overline{\delta\ell} \rangle > 0$  and let  $\overline{\delta y} := A_{1j}^1\overline{\delta\ell} = A_{1j}^2\overline{\delta\ell}$ .

Let  $\nu \in \{1, 2\}$  and assume, by contradiction, that there exists  $\delta\ell_\nu \in N_{1j}^{\nu 1}$  such that  $B_{1j}^\nu\delta\ell_1 = \overline{\delta y}$ , i.e.

$$\begin{aligned} \exp(\hat{\theta}_{1j}k_{1,j-1*})\pi_*\varphi_{1,j-1}^\nu(\overline{\delta\ell}) &= \\ &= \exp(\hat{\theta}_{1j}k_{1,j-1*})\pi_*\varphi_{1,j-1}^\nu(\overline{\delta\ell}) - \langle d\theta_{1j}^\nu(\hat{\ell}_0), \overline{\delta\ell} \rangle (k_{1j} - k_{1,j-1})(\hat{x}_{1j}). \end{aligned}$$

Taking the pull-back along the reference flow  $\hat{S}_{\hat{\theta}_{1j}}$  and defining  $\overline{\delta x} := \pi_*\overline{\delta\ell}$ ,  $\delta x_\nu := \pi_*\delta\ell_\nu$

we can equivalently write

$$\begin{aligned} \delta x - \bar{\delta x} - \sum_{s=1}^{J_0} \langle d\theta_{0s}(\hat{\ell}_0), \delta\ell - \bar{\delta\ell} \rangle (g_{0s} - g_{0,s-1})(\hat{x}_0) - \\ - \langle d\tau_\nu(\hat{\ell}_0), \delta\ell - \bar{\delta\ell} \rangle (h_\nu - g_{0J_0})(\hat{x}_0) - \langle d\theta_{10}^\nu(\hat{\ell}_0), \delta\ell - \bar{\delta\ell} \rangle (g_{10} - h_\nu)(\hat{x}_0) - \\ - \sum_{s=1}^{j-1} \langle d\theta_{1s}^\nu(\hat{\ell}_0), \delta\ell - \bar{\delta\ell} \rangle (g_{1s} - g_{1,s-1})(\hat{x}_0) - \langle d\theta_{1j}^\nu(\hat{\ell}_0), \hat{\ell}_0 \rangle (g_{1j} - g_{1,j-1})(\hat{x}_0) = 0 \end{aligned}$$

that is

$$\begin{aligned} \bar{\delta x} - \delta x + \sum_{s=0}^{J_0-1} \langle d(\theta_{0,s+1} - \theta_{0s})(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle g_{0s}(\hat{x}_0) + \\ + \langle d(\tau_\nu - \theta_{0J_0})(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle g_{0J_0}(\hat{x}_0) + \\ + \langle d(\theta_{10}^\nu - \tau_\nu)(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle h_\nu(\hat{x}_0) + \sum_{s=0}^{j-2} \langle d(\theta_{1,s+1}^\nu - \theta_{1s}^\nu)(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle g_{1s}(\hat{x}_0) + \\ + \left( \langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell \rangle - \langle d\theta_{1,j-1}^\nu(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle \right) g_{1,j-1}(\hat{x}_0) - \langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell \rangle g_{1j}(\hat{x}_0) = 0 \end{aligned}$$

Let  $\delta e := (\bar{\delta x} - \delta x, a, b)$ , where,

$$\begin{aligned} a_{0s} &:= \begin{cases} \langle d(\theta_{0,s+1} - \theta_{0s})(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle & s = 0, \dots, J_0 - 1, \\ \langle d(\tau_\nu - \theta_{0J_0})(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle & s = J_0, \end{cases} \\ b &:= \langle d(\theta_{10}^\nu - \tau_\nu)(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle \\ a_{1s} &:= \begin{cases} \langle d(\theta_{1,s+1}^\nu - \theta_{1s}^\nu)(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle & s = 0, \dots, j-2 \\ \langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell \rangle - \langle d\theta_{1,j-1}^\nu(\hat{\ell}_0), \bar{\delta\ell} - \delta\ell \rangle & s = j-1, \\ -\langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell \rangle & s = j, \\ 0 & s = j+1, \dots, J_1. \end{cases} \end{aligned}$$

Then  $\delta e \in V_{1j} \cap V_{1,j-1}^{\perp J_1''}$  and Lemma 5.3 applies:

$$\begin{aligned} 0 > a_{1j} \sigma \left( d\alpha_*(\delta x - \bar{\delta x}) + \sum_{s=0}^{J_0} a_{0s} \vec{G}_{0s}(\hat{\ell}_0) + \right. \\ \left. + b \vec{H}_\nu(\hat{\ell}_0) + \sum_{s=0}^{j-1} a_{1s} \vec{G}_{1s}(\hat{\ell}_0), (\vec{G}_{1j} - \vec{G}_{1,j-1})(\hat{\ell}_0) \right) = \\ = \langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell \rangle \left\{ \langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell - \bar{\delta\ell} \rangle \sigma \left( \vec{G}_{1,j-1}, \vec{G}_{1j} \right) (\hat{\ell}_0) - \right. \\ \left. - \langle d\theta_{1j}^\nu(\hat{\ell}_0), \delta\ell \rangle \sigma \left( \vec{G}_{1,j-1}, \vec{G}_{1j} \right) (\hat{\ell}_0) \right\} = \end{aligned}$$

$$= - \langle d\theta_{1j}^\nu(\widehat{\ell}_0), \delta\ell \rangle \langle d\theta_{1j}^\nu(\widehat{\ell}_0), \overline{\delta\ell} \rangle \sigma \left( \vec{G}_{1,j-1}, \vec{G}_{1j} \right) (\widehat{\ell}_0),$$

a contradiction.

Let us now turn to the degenerate case  $d\tau_1|_{T_{\widehat{\ell}_0}\Lambda} \equiv d\tau_2|_{T_{\widehat{\ell}_0}\Lambda}$ . From equations (4.14) one can recursively show that  $\langle d\theta_{1j}^1(\widehat{\ell}_0), \delta\ell \rangle|_{T_{\widehat{\ell}_0}\Lambda} = \langle d\theta_{1j}^2(\widehat{\ell}_0), \delta\ell \rangle|_{T_{\widehat{\ell}_0}\Lambda}$  for any  $\delta\ell \in T_{\widehat{\ell}_0}\Lambda$  and for any  $j = 1, \dots, J_1$ , so that  $A_{1j}^1 = A_{1j}^2$  and  $B_{1j}^1 = B_{1j}^2$  and the result can be proved repeating the proof of Lemma 6.1.

This proves the invertibility of  $\pi\mathcal{H}_{\widehat{\theta}_{1j}}$ ,  $j = 1, \dots, J_1$ . Thus the map

$$\text{id} \times \pi\mathcal{H}: [0, T] \times \Lambda \rightarrow M$$

is one-to-one from a neighborhood of  $[0, T] \times \{\widehat{\lambda}(0)\}$  in  $[0, T] \times \Lambda$  and we can apply the procedure described in Section 3.

### 6.1 Proof of Theorem 2.3

Let

$$\text{id} \times \pi\mathcal{H}: [0, T] \times \mathcal{O} \rightarrow \mathcal{V} = [0, T] \times \mathcal{U}$$

be one-to-one and let  $\xi: [0, T] \rightarrow M$  be an admissible trajectory whose graph is in  $\mathcal{V}$ .

Applying the Hamiltonian methods explained in Section 3 we have:

$$C(\xi, u) - C(\widehat{\xi}, \widehat{u}) \geq \mathcal{F}(\xi(T)) - \mathcal{F}(\widehat{x}_f).$$

Thus, to complete the proof of Theorem 2.3 it suffices to show that  $\mathcal{F}$  has a local minimum at  $\widehat{x}_f$ . In order to shorten the notation, let us denote  $\psi_T(\ell) := (\pi\mathcal{H}_T)^{-1}(\ell)$ .

**Theorem 6.3.**  *$\mathcal{F}$  has a strict local minimum at  $\widehat{x}_f$ .*

*Proof.* It suffices to prove that

$$d\mathcal{F}(\widehat{x}_f) = 0 \text{ and } D^2\mathcal{F}(\widehat{x}_f) > 0. \quad (6.8)$$

The first equality in (6.8) is an immediate consequence of the definition of  $\mathcal{F}$  and of PMP.

Let us prove that also the inequality holds.

Since  $d(\alpha \circ \pi\psi_T) = \mathcal{H}_T \circ \psi_T$ , we also have

$$d\mathcal{F} = \mathcal{H}_T \circ \psi_T + d\beta \quad (6.9)$$

$$\begin{aligned} D^2\mathcal{F}(\widehat{x}_f)[\delta x_f]^2 &= ((\mathcal{H}_T \circ \psi_T)_* + D^2\beta)(\widehat{x}_f)[\delta x_f]^2 \\ &= \sigma((\mathcal{H}_T \circ \psi_T)_*\delta x_f, d(-\beta)_*\delta x_f) \end{aligned} \quad (6.10)$$

From Lemma 5.4 we have

$$\begin{aligned} \sigma \left( d(-\widehat{\beta})_*(\delta x + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} g_{is}(\widehat{x}_0) + b h_\nu(\widehat{x}_0)), \right. \\ \left. d\alpha_*\delta x + \sum_{i=0}^1 \sum_{s=0}^{J_i} a_{is} \vec{G}_{is}(\widehat{\ell}_0) + b \vec{H}_\nu(\widehat{\ell}_0) \right) < 0. \end{aligned} \quad (6.11)$$

Applying  $\widehat{\mathcal{H}}_{T^*}$  to both arguments and using the anti-simmetry property of  $\sigma$  we get

$$\sigma(\mathcal{H}_{T^*} d\alpha_* \delta x, d(-\beta)_*((\pi \mathcal{H}_T)_* d\alpha_* \delta x)) > 0$$

which is exactly (6.10) with  $\delta x := \pi_* \psi_{T^*} \delta x_f$ .  $\square$

To conclude the proof of Theorem 2.3 we have to show that  $\widehat{\xi}$  is a strict minimizer. Assume  $C(\xi, u) = C(\widehat{\xi}, \widehat{u})$ . Since  $\widehat{x}_f$  is a strict minimizer for  $F$ , then  $\xi(T) = \widehat{x}_f$  and equality must hold in (3.1):

$$\langle \mathcal{H}_s(\psi_s^{-1}(\xi(s))), \dot{\xi}(s) \rangle = H_s(\mathcal{H}_s(\psi_s^{-1}(\xi(s)))).$$

By regularity assumption,  $u(s) = \widehat{u}(s)$  for any  $s$  at least in a left neighborhood of  $T$ , hence  $\xi(s) = \widehat{\xi}(s)$  and  $\psi_s^{-1}(\xi(s)) = \widehat{\ell}_0$  for any  $s$  in such neighborhood.  $u$  takes the value  $\widehat{u}|_{(\widehat{\theta}_{1,J_1}, T)}$  until  $\mathcal{H}_s \psi_s^{-1}(\xi(s)) = \mathcal{H}_s(\widehat{\ell}_0) = \widehat{\lambda}(s)$  hits the hyper-surface  $K_{1,J_1} = K_{1,J_1-1}$ , which happens at time  $s = \widehat{\theta}_{1,J_1}$ . At such time, again by regularity assumption,  $u$  must switch to  $\widehat{u}|_{(\widehat{\theta}_{1,J_1-1}, \widehat{\theta}_{1,J_1})}$ , so that  $\xi(s) = \widehat{\xi}(s)$  also for  $s$  in a left neighborhood of  $\widehat{\theta}_{1,J_1}$ . Proceeding backward in time, with an induction argument we finally get  $(\xi(s), u(s)) = (\widehat{\xi}(s), \widehat{u}(s))$  for any  $s \in [0, T]$ .

In the abnormal case the cost is zero, thus the existence of a strict local minimiser implies that the trajectory is isolated among admissible ones.

## 7 Appendix: Invertibility of piecewise $C^1$ maps

This Section is devoted to piecewise linear maps and to piecewise  $C^1$  maps. Our aim is to prove a sufficient condition, in terms of the ‘‘piecewise linearization’’, of piecewise  $C^1$  maps.

Some linear algebra preliminaries are needed.

**Lemma 7.1.** *Let  $A$  and  $B$  be linear automorphisms of  $\mathbb{R}^n$ . Assume that for some  $v \in (\mathbb{R}^n)^* \setminus \{0\}$ ,  $A$  and  $B$  coincide on the space  $\pi(v) := \{x \in \mathbb{R}^n : \langle v, x \rangle = 0\}$ . Then, the map  $\mathcal{L}_{AB}$  defined by  $x \mapsto Ax$  if  $\langle v, x \rangle \geq 0$ , and by  $x \mapsto Bx$  if  $\langle v, x \rangle \leq 0$ , is a homeomorphism if and only if  $\det(A) \cdot \det(B) > 0$ .*

*Proof.* Let  $w_1, \dots, w_{n-1}$  be a basis of the hyperplane  $\pi(v)$ . We complete it with  $v$  to obtain a basis of  $\mathbb{R}^n$ . The matrix of  $A^{-1}B$  in this basis is given by

$$\left( \begin{array}{c|c} \mathbf{I}_{n-1} & \begin{matrix} \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{matrix} \\ \hline \mathbf{0}_{n-1}^t & \gamma_n \end{array} \right)$$

where  $\mathbf{I}_{n-1}$  is the  $n-1$  unit matrix and  $\mathbf{0}_{n-1}$  is the  $n-1$  null vector and the  $\gamma_i$ 's are defined by

$$A^{-1}Bv = \sum_{i=1}^{n-1} \gamma_i w_i + \gamma_n v.$$

Thus  $\gamma_n$  is positive if and only if  $\det(A)\det(B)$  is positive and  $\gamma_n$  is zero if and only if either  $A$  or  $B$  is not invertible.

Observe that if  $\gamma_n$  is negative, then

$$\mathcal{L}_{AB}v = \mathcal{L}_{AB} \left( \sum_{i=1}^{n-1} -\frac{\gamma_i}{\gamma_n} w_i + \frac{1}{\gamma_n} v \right).$$

Thus, in this case  $\mathcal{L}_{AB}$  is not one-to-one.

We now prove that  $\mathcal{L}_{AB}$  is injective if  $\gamma_n$  is positive. Assume this is not true. Since both  $A$  and  $B$  are invertible, there exist  $z_A, z_B \in \mathbb{R}^n$  such that  $\langle v, z_A \rangle > 0$ ,  $\langle v, z_B \rangle < 0$  and  $Az_A = Bz_B$  or, equivalently,  $A^{-1}Bz_B = z_A$ . Let

$$z_A = \sum_{i=1}^{n-1} c_A^i w_i + c_A v, \quad z_B = \sum_{i=1}^{n-1} c_B^i w_i + c_B v.$$

Clearly  $c_A > 0$ ,  $c_B < 0$ . The equality  $A^{-1}Bz_B = z_A$  is equivalent to

$$\sum_{i=1}^{n-1} c_B^i w_i + c_B \sum_{i=1}^{n-1} \gamma_i w_i + c_B \gamma_n v = \sum_{i=1}^{n-1} c_A^i w_i + c_A v.$$

Consider the scalar product with  $v$ , we get  $c_B \gamma_n \|v\|^2 = c_A \|v\|^2$ , which is a contradiction.

We finally prove that, if  $\gamma_n$  is positive, then  $\mathcal{L}_{AB}$  is surjective. Let  $z \in \mathbb{R}^n$ . There exist  $y_A, y_B \in \mathbb{R}^n$  such that  $Ay_A = By_B = z$ . If either  $\langle v, y_A \rangle \geq 0$  or  $\langle v, y_B \rangle \leq 0$ , there is nothing to prove. Let us assume  $\langle v, y_A \rangle < 0$  and  $\langle v, y_B \rangle > 0$ . In this case  $A^{-1}By_B = y_A$  and proceeding as above we get a contradiction.  $\square$

**Definition 7.1.** Let  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous, piecewise linear map at 0, in the sense that  $G$  is continuous and there exists a decomposition  $S_1, \dots, S_k$  of  $\mathbb{R}^n$  in closed polyhedral cones (intersection of half spaces, hence convex) with nonempty interior and common vertex in the origin and such that  $\partial S_i \cap \partial S_j = S_i \cap S_j$ ,  $i \neq j$ , and linear maps  $L_1, \dots, L_k$  with

$$G(x) = L_i x, \quad x \in S_i,$$

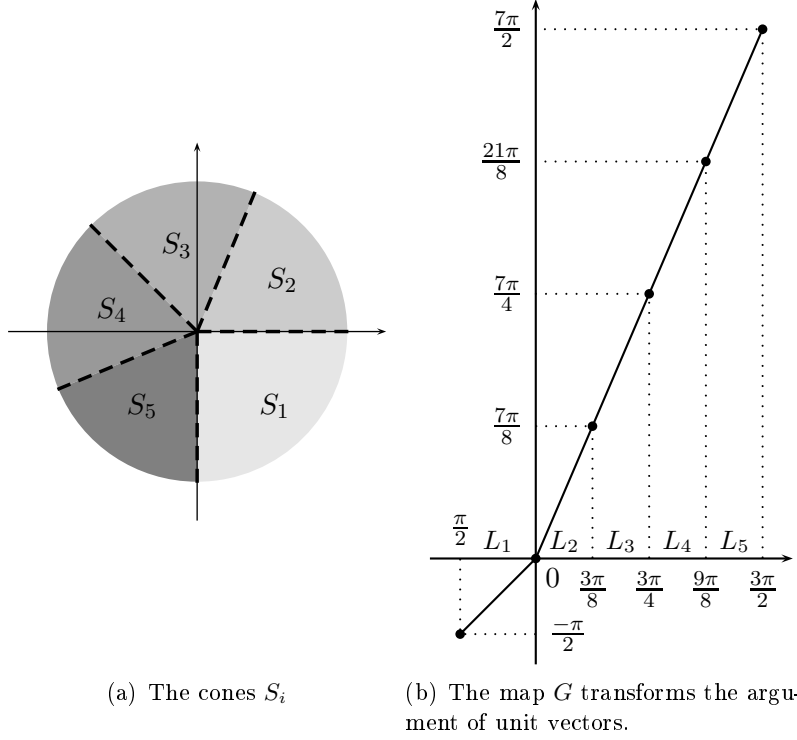
with  $L_i x = L_j x$  for any  $x \in S_i \cap S_j$ , and  $\det L_i \neq 0$ ,  $\forall i = 1, \dots, k$ .

**Example 7.1.** As an example of continuous piecewise linear map consider  $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad L_2 = \begin{pmatrix} 1 & -\sqrt{2} \\ 0 & \sqrt{2}-1 \end{pmatrix} \quad L_3 = \begin{pmatrix} -\sqrt{2} & -\sqrt{2}+1 \\ 1 & 0 \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 0 & 1 \\ -\sqrt{2}+1 & -\sqrt{2} \end{pmatrix} \quad L_5 = \begin{pmatrix} \sqrt{2}-1 & 0 \\ -\sqrt{2} & 1 \end{pmatrix}$$

where the  $L_i$ 's are applied in the corresponding cone  $S_i$  illustrated in picture 8


 Figure 8: Polyhedral cones and the transformation of unit vectors under  $G$ 

Observe that any continuous piecewise linear map  $G$  is differentiable in  $\mathbb{R}^n \setminus \cup_{i=1}^k \partial S_i$ . It is easily shown that  $G$  is proper, and therefore  $\deg(G, \mathbb{R}^n, p)$  is well-defined for any  $p \in \mathbb{R}^n$  (the construction in [12], Chapter 5 is still valid if the assumption on the compactness of the manifolds is replaced with the assumption that  $G$  is proper. Compare also [6]). Moreover  $\deg(G, \mathbb{R}^n, p)$  is constant with respect to  $p$ . So we shall denote it by  $\deg(G)$ .

We shall also assume that  $\det L_i > 0$  for any  $i = 1, \dots, k$ .

**Lemma 7.2.** *If  $G$  is as above, then  $\deg(G) > 0$ . In particular, if there exists  $q \neq 0$  such that its preimage  $G^{-1}(q)$  is a singleton that belongs to at most two of the convex polyhedral cones  $S_i$ , then  $\deg(G) = 1$ .*

*Proof.* Let us assume in addition that  $q \notin \cup_{i=1}^k G(\partial S_i)$ . Observe that the set  $\cup_{i=1}^k G(\partial S_i)$  is nowhere dense hence  $A := G(S_1) \setminus \cup_{i=1}^k G(\partial S_i)$  is non-empty.

Take  $x \in A$  and observe that if  $y \in G^{-1}(x)$  then  $y \notin \cup_{i=1}^k \partial S_i$ . Thus

$$\deg(G) = \sum_{y \in G^{-1}(x)} \text{sign det } dG(y) = \#G^{-1}(x). \quad (7.1)$$

Since  $G^{-1}(x) \neq \emptyset$ ,  $\deg(G) > 0$ . The second part of the assertion follows taking  $x = q$  in (7.1).

Let us now remove the additional assumption. Let  $\{p\} = G^{-1}(q)$  be such that  $p \in \partial S_i \cap \partial S_j$  for some  $i \neq j$ . Observe that by assumption  $p \neq 0$  does not belong to any cone

$\partial S_s$  for  $s \notin \{i, j\}$ . Thus one can find a neighborhood  $V$  of  $p$ , with  $V \subset \text{int}(S_i \cup S_j \setminus \{0\})$ . By the excision property of the topological degree  $\deg(G) = \deg(G, V, p)$ . Let  $\mathcal{L}_{L_i L_j}$  be a map as in Lemma 7.1. Observe that, the assumption on the signs of the determinants of  $L_i$  and  $L_j$  imply that  $\mathcal{L}_{L_i L_j}$  is orientation preserving. Also notice that  $\mathcal{L}_{L_i L_j}|_{\partial V} = G|_{\partial V}$ . The multiplicativity, excision and boundary dependence properties of the degree yield  $1 = \deg(\mathcal{L}_{L_i L_j}) = \deg(\mathcal{L}_{L_i L_j}, V, p) = \deg(G, V, p)$ . Thus,  $\deg(G) = 1$ , as claimed.  $\square$

### 7.1 Piecewise differentiable functions

**Lemma 7.3.** *Let  $A$  and  $B$  be linear endomorphisms of  $\mathbb{R}^n$ . Assume that for some  $v \in \mathbb{R}^n \setminus \{0\}$ ,  $A$  and  $B$  coincide on the space  $\{x \in \mathbb{R}^n : \langle x, v \rangle = 0\}$ . Then*

$$\det(tA + (1-t)B) = t \det A + (1-t) \det B \quad \forall t \in \mathbb{R}.$$

*Proof.* We can, without loss of generality, assume that  $|v| = 1$ . We can choose vectors  $w_2, \dots, w_n \in \mathbb{R}^n \setminus \{0\}$  such that  $v, w_2, \dots, w_n$  is an orthonormal basis of  $\mathbb{R}^n$ . In this basis, for  $t \in [0, 1]$  we can represent the operator  $tA + (1-t)B$  in the following matrix form:

$$\begin{pmatrix} ta_{11} + (1-t)b_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ta_{n1} + (1-t)b_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} ta_{11} + (1-t)b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & & \vdots \\ ta_{n1} + (1-t)b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$

Thus,

$$\begin{aligned} \det(tA + (1-t)B) &= \sum_{i=1}^n (-1)^{i+1} (ta_{i1} + (1-t)b_{i1}) \det A_{i1} \\ &= \sum_{i=1}^n (-1)^{i+1} (ta_{i1} + (1-t)b_{i1}) \det B_{i1} \end{aligned}$$

where  $A_{i1}$  and  $B_{i1}$  represent the  $(i1)$ -th cofactor of  $A$  and  $B$  respectively. Clearly,  $A_{i1} = B_{i1}$  for  $i = 1, \dots, n$ . Hence, we have

$$\begin{aligned} \det(tA + (1-t)B) &= \sum_{i=1}^n (-1)^{i+1} ta_{i1} \det A_{i1} + \\ &\quad + \sum_{i=1}^n (-1)^{i+1} (1-t)b_{i1} \det B_{i1} = t \det A + (1-t) \det B \end{aligned}$$

as claimed in the lemma.  $\square$

Lemmas 7.1 and 7.3 imply the following fact:



**Lemma 7.4.** *Let  $A$  and  $B$  be linear automorphisms of  $\mathbb{R}^n$ . Assume that for some  $v \in \mathbb{R}^n \setminus \{0\}$ ,  $A$  and  $B$  coincide on the space  $\{x \in \mathbb{R}^n : \langle x, v \rangle = 0\}$ . Assume that the map  $\mathcal{L}_{AB}$  defined by  $x \mapsto Ax$  if  $\langle x, v \rangle \geq 0$ , and by  $x \mapsto Bx$  if  $\langle x, v \rangle \leq 0$ , is a homeomorphism. Then,  $\det(A) \cdot \det(tA + (1-t)B) > 0$  for any  $t \in [0, 1]$ .*

Let  $\sigma_1, \dots, \sigma_r$  be a family of  $C^1$ -regular pairwise transversal hyper-surfaces in  $\mathbb{R}^n$  with  $x_0 \in \cap_{i=1}^r \sigma_i$  and let  $U \subset \mathbb{R}^n$  be an open and bounded neighborhood of  $x_0$ . Clearly, if  $U$  is sufficiently small,  $U \setminus \cup_{i=1}^r \sigma_i$  is partitioned into a finite number of open sets  $U_1, \dots, U_k$ .

Let  $f : \bar{U} \rightarrow \mathbb{R}^n$  be a continuous map such that there exist  $f_1, \dots, f_k \in C^1(\bar{U})$  with the property that

$$f(x) = f_i(x), \quad x \in \bar{U}_i, \quad (7.2)$$

with  $f_i(x) = f_j(x)$  for any  $x \in \bar{U}_i \cap \bar{U}_j$ . Notice that such a function is  $PC^1(\bar{U})$  (see e.g. [10] for a definition), and Lipschitz continuous in  $U$ .

Let  $S_1, \dots, S_k$  be the tangent cones (in the sense of Bouligand) at  $x_0$  to the sets  $U_1, \dots, U_k$ , (by the transversality assumption on the hyper-surfaces  $\sigma_i$  each  $S_i$  is a convex polyhedral cone with non empty interior) and assume  $df_i(x_0)x = df_j(x_0)x$  for any  $x \in S_i \cap S_j$ . Define

$$F(x) = df_i(x_0)x \quad x \in S_i. \quad (7.3)$$

so that  $F$  is a continuous piecewise linear map (compare [10]).

One can see that  $f$  is Bouligand differentiable and that its B-derivative is the map  $F$  (compare [10, 13]). Let  $y_0 := f(x_0)$ . There exists a continuous function  $\varepsilon$ , with  $\varepsilon(0) = 0$ , such that  $f(x) = y_0 + F(x - x_0) + |x - x_0|\varepsilon(x - x_0)$ .

**Lemma 7.5.** *Let  $f$  and  $F$  be as in (7.2)-(7.3), and assume that  $\det df_i(x_0) > 0$  for all  $i = 1, \dots, k$ . Then there exists  $\rho > 0$  such that  $\deg(f, B(x_0, \rho), y_0) = \deg(F, B(0, \rho), 0)$ . In particular,  $\deg(f, B(x_0, \rho), y_0) = \deg(F)$ .*

*Proof.* Consider the homotopy  $H(x, \lambda) = F(x - x_0) + \lambda|x - x_0|\varepsilon(x - x_0)$ ,  $\lambda \in [0, 1]$  and observe that

$$m := \inf\{|F(v)| : |v| = 1\} = \min_{i=1, \dots, k} \|df_i\| > 0.$$

Thus,

$$|H(x, \lambda)| \geq (m - |\varepsilon(x - x_0)|) |x - x_0|.$$

This shows that in a conveniently small ball centered at  $x_0$ , homotopy  $H$  is admissible. The assertion follows from the homotopy invariance property of the degree.  $\square$

**Theorem 7.6.** *Let  $f$  and  $F$  be as in (7.2)-(7.3) and assume  $\det df_i(x_0) > 0$ . Assume also that there exists  $p \in \mathbb{R}^n$  whose pre-image belongs to at most two of the convex polyhedral cones  $S_i$  and such that  $F^{-1}(p)$  is a singleton. Then  $f$  is a Lipschitzian homeomorphism in a sufficiently small neighborhood of  $x_0$ .*

*Proof.* From Lemmas 7.2-7.5, it follows that  $\deg(f, B(x_0, \rho), y_0) = 1$  for sufficiently small  $\rho > 0$ . By Theorem 4 in [13], we immediately obtain the assertion.  $\square$

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