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Local Inversion of Planar Maps with Nice Nondifferentiability Structure

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Abstract

When the plane is pie-sliced in $n \le 4$ parts (with nonempty interior and common vertex at the origin) our main result provides sufficient conditions for any map L, that is continuous and piecewise linear relatively to this slicing, to be invertible. This result cannot be plainly extended to a greater number of slices. Also, some examples show that the assumptions cannot be relaxed too much. Our result is proved by a combination of linear algebra and topological arguments.

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Introduction

Inspired by invertibility problems for PC^1 maps (see e.g., [7]) that naturally arise in Optimal Control (see e.g., [12]) we focus on the invertibility of continuous maps of the plane which are piecewise linear.

When the plane is pie-sliced in $n \le 4$ parts (with nonempty interior and common vertex at the origin) our main result, Theorem 4.2 below, provides sufficient conditions for any map L, that is continuous and piecewise linear relatively to this slicing, to be invertible. Some examples show that the assumptions of the theorem cannot be relaxed too much. For instance, convexity of the slices when n = 4 cannot be dropped altogether and, perhaps not surprisingly, our conditions cannot be plainly extended to a greater number of slices. This result is proved by a combination of linear algebra and topological arguments in which Theorems 4 and 5 of [10] (Theorems 2.1 and 2.2 below) play a crucial role. By contrast, an important tool of nonsmooth analysis, Clarke's Theorem [4, 5],

does not appear to be adequate for our purposes in the case n = 4, as we show by exhibiting an explicit example, Example 4.4, that cannot be satisfactorily treated by this theorem.

Our results depend on the particularly nice nondifferentiability structure that we assume throughout. In fact Example 2.1 in [7] shows that there exists a PC^1 function with 4 selection functions (which does not have such structure) which is not locally invertible at the origin despite being Fréchet differentiable at 0 with invertible differential.

As stated above, our interest in the invertibility of PC^1 maps stems from optimal control problems. Namely, if one considers a multiinput optimal control problem which is affine with respect to the control variable $u \in [-1, 1]^m$, then one cannot exclude the existence of bang-bang Pontryagin extremals. This gives rise to a PC^1 maximized Hamiltonian flow. In order to prove the optimality of the given Pontryagin extrema via Hamiltonian methods, one needs to prove the invertibility of the projection of such flow on the state space (see [1] for an introduction to Hamiltonian methods in control and [2, 13] for specific applications to bang-bang Pontryagin extremals). In particular, as in [11, 12] we are interested in what happens when two control components switch simultaneously just once. In this case the "interesting" part of the above-mentioned projection is 2-dimensional. This justifies our concern with the invertibility of planar maps. Moreover, a double switch gives rise to the "nice" nondifferentiability structure we consider in this paper with at most n = 5 pie-slices which reduce to 4 for the subsequent simple switches.

To the best of our knowledge, a comprehensive treatment of invertibility results in simple cases is not available in the literature. This has, perhaps, slowed down the study of bang–bang Pontryagin extremals with multiple switch behavior.

Some comments are in order concerning some of the illustrations included in this paper. Figures 1, 2 and 4 represent the piecewise linear maps contained in Examples 4.1, 4.2 and 4.4, respectively. In fact, they actually show the image of the unit circle S^1 under these maps. But, for the sake of clarity, we have altered the proportion between axes and, in order to enhance the view close to the origin, we logarithmically rescaled the radial distance from the origin. Notice that such transformations do not change the qualitative behavior of the maps (at least not the characteristics we are interested in).

2 Preliminaries and notation

2.1 Some notions of nonsmooth analysis

Following [7], a continuous function $f: U \subseteq \mathbb{R}^s \to \mathbb{R}^m$ is a *continuous selection of* C^1 *functions* if there exists a finite number of C^1 functions f_1, \ldots, f_ℓ , of U into \mathbb{R}^m such that the *active index set* $I := \{i : f(x) = f_i(x)\}$ is nonempty for each $x \in U$. The functions f_i 's are called *selection functions* of f. The function f is called a PC^1 *function* if at every point $x \in U$ there exists a neighborhood V such that the restriction of f to V is a continuous selection of C^1 functions.

Example 2.1 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \min \{y - x^2, 0, x\}, (x,y) \in \mathbb{R}^2.$$

The function f is a continuous selection of the three functions $f_1(x, y) = y - x^2$, $f_2(x, y) = 0$ and $f_3(x, y) = x$ and the active index set is summarized in the following table:

Active index set	Region of validity in \mathbb{R}^2
$\{1, 2, 3\}$	$\{(0,0)\}$
{2,3}	$\{(x, y) \in \mathbb{R}^2 : x = 0, y > 0\}$
{1,2}	$\{(x, y) \in \mathbb{R}^2 : x > 0, y = x^2\}$
{ 1 , 3}	$\{(x, y) \in \mathbb{R}^2 : x < 0, y = x^2 + x\}$
{1}	$\{(x, y) \in \mathbb{R}^2 : x > 0, y < x^2\} \cup \{(x, y) \in \mathbb{R}^2 : x < 0, y < x^2 + x\}$
{2}	$\{(x, y) \in \mathbb{R}^2 : x > 0, y > x^2\}$
{3}	$\{(x,y)\in\mathbb{R}^2: x<0, y>x^2+x\}$

The function f is indeed PC^1 .

A function $f: \mathbb{R}^s \to \mathbb{R}^m$ is said to be *piecewise linear* if it is a continuous selection of linear functions. We will actually focus on a much more restrictive class of piecewise linear functions, especially when m = s = 2.

Recall that a cone $C \subseteq \mathbb{R}^k$ with vertex at the origin is a positively homogeneous set, in the sense that if $v \in C$ then $\alpha v \in C$ for all $\alpha \ge 0$. Below we give more specialized notions.

Definition 2.1 A cone $C \subseteq \mathbb{R}^k$ with nonempty and connected interior and vertex at the origin is called a *polyhedral cone* if it is the intersection of a finite number of close half-spaces.

Clearly, a polyhedral cone, as the intersection of convex sets, is convex. This is an excessively severe limitation for our purposes. By a *hyper-plane* in \mathbb{R}^k we mean a 1-codimensional linear subspace of \mathbb{R}^k .

Definition 2.2 Let $\pi_1, \pi_2 \subset \mathbb{R}^k$ be two half hyper-planes with common boundary $\partial \pi_1 = \partial \pi_2$ containing the origin. Thus $\mathbb{R}^k \setminus (\pi_1 \cup \pi_2)$ is an open set with two connected components A_1 and A_2 . We call each connected component an *open wedge* of \mathbb{R}^k . The closure of an open wedge of \mathbb{R}^k is called a *wedge* of \mathbb{R}^k .

Definition 2.3 A cone $C \subseteq \mathbb{R}^k$ with nonempty interior and vertex at the origin is called an *admissible cone* if it is the intersection of a finite number of wedges of \mathbb{R}^k .

Remark 2.1 Clearly, an admissible cone need not be convex, so it may not be poyhedral. Conversely, however, a polyhedral cone is always admissible. We also observe the following facts:

- 1. If $\pi_1 \cup \pi_2$ is an hyperplane in \mathbb{R}^k , then the wedges defined by π_1 and π_2 are two closed half-spaces. Otherwise only one of the wedges (the only convex one) is a polyhedral cone.
- 2. If k = 2 then any admissible cone is a wedge and is in fact an angle with vertex at the origin.

It is not difficult to prove that if $L : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible linear map and C is a polyhedral cone, then so is L(C) (it follows from linearity and the open mapping theorem). This is no longer true if we drop the invertibility assumption on L. In fact, although the linear image of a cone with vertex at the origin is a cone with the same vertex, if the map L is singular, a polyhedral cone could collapse under L into a cone with empty interior.

Definition 2.4 A finite collection of closed admissible cones of \mathbb{R}^k , C_1, \ldots, C_n , with pairwise disjoint interiors is called a *decomposition* of \mathbb{R}^k if $\mathbb{R}^k = \bigcup_{i=1}^n C_i$.

Observe that if $L : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible linear map, and C_1, \ldots, C_n is a decomposition of \mathbb{R}^k , then the sets $D_i = L(C_i)$, $i = 1, \ldots, n$, constitute a decomposition of \mathbb{R}^k as well. The same statement remains true if the linear map L is replaced by any bijective function that maps each cone C_i into an admissible cone D_i .

Definition 2.5 We say that a continuous map $G : \mathbb{R}^k \to \mathbb{R}^k$ is *strongly piecewise linear (at* 0) if there exist a decomposition C_1, \ldots, C_n of \mathbb{R}^k in admissible cones, and linear maps L_1, \ldots, L_n with

$$G(x) = L_i x$$
, for $x \in C_i$.

We also say that G is *nondegenerate* if sign(det L_i) is constant and nonzero for all i = 1, ..., n.

Notice that if G is a continuous strongly piecewise linear map as in Definition 2.5 above, then $L_i x = L_j x$ for any $x \in C_i \cap C_j$ and $i, j \in \{1, ..., n\}$. Moreover, G is positively homogeneous.

In this paper we are concerned with the global invertibility of continuous nondegenerate strongly piecewise linear maps. In this regard the following two simple observations are in order:

Lemma 2.1 Let $G : \mathbb{R}^k \to \mathbb{R}^k$ be strongly piecewise linear (at 0) as in Definition 2.5. Assume that G is injective, then the linear maps L_i 's are invertible for all i = 1, ..., n.

Proof. Let us use the notation of Definition 2.5. Assume by contradiction that there exists $i \in \{1, ..., n\}$ such that L_i is not invertible. Take $w \in \ker L_i \setminus \{0\}$. On one hand, if $w \in C_i \setminus \{0\}$ we have that $G(w) = L_i(w) = 0 = G(0)$ against the injectivity of G. On the other hand, if $w \notin C_i \setminus \{0\}$, take any $v \in int(C_i) \setminus \{0\}$. Since $int(C_i)$ is open, there exists $\alpha \neq 0$ such that $v + \alpha w \in C_i$. One has $v + \alpha w \neq v$ and

$$G(\nu + \alpha w) = L_i(\nu + \alpha w) = L_i(\nu) = G(\nu),$$

which, again, contradicts the injectivity of G.

In fact, in due course (see Proposition 4.1), we will see that when G in Lemma 2.1 is invertible, then it is nondegenerate.

Lemma 2.2 Let $G: \mathbb{R}^k \to \mathbb{R}^k$ be a continuous strongly piecewise linear map as in Definition 2.5, and let U be an open neighborhood of $0 \in \mathbb{R}^k$. Assume that the restriction $G|_U: U \to G(U)$ is invertible with continuous inverse, then G is globally invertible and its inverse is a continuous strongly piecewise linear map as well.

Proof. Let us first prove that G is injective. Let $x_1, x_2 \in \mathbb{R}^k$ be such that $G(x_1) = G(x_2)$. Since U is an open neighborhood of the origin, there exists $\rho > 0$ such that $\rho x_1 \in U$ and $\rho x_2 \in U$. Then

$$G(\rho x_1) = \rho G(x_1) = \rho G(x_2) = G(\rho x_2),$$

so that $\rho x_1 = \rho x_2$, i.e. $x_1 = x_2$.

Let us now prove surjectivity by explicitly exhibiting the inverse. Since G is injective, the domain invariance theorem implies that G(U) is open in \mathbb{R}^k . Given $y \in \mathbb{R}^k$, define the map $H : \mathbb{R}^k \to \mathbb{R}^k$ as follows:

$$H(\mathbf{y}) = \frac{\|\mathbf{y}\|}{r} (\mathbf{G}|_{\mathbf{U}})^{-1} \left(r \frac{\mathbf{y}}{\|\mathbf{y}\|} \right)$$

where r > 0 is any positive number such that the sphere S_r of radius r and centered at the origin is contained in G(U). Clearly, H does not depend on the choice of r. One can directly verify that H is the inverse of G. Moreover, since by assumption $G|_U$ has continuous inverse, we get that H is continuous.

It only remains to prove that H is a strongly piecewise linear map. Let us use the notation of Definition 2.5. The first part of the proof together with Lemma 2.1 shows that, for i = 1, ..., n, all maps L_i are invertible. For all i = 1, ..., n set $D_i = L_i(C_i) = G(C_i)$. Observe that all the D_i 's

are admissible cones and that D_1, \ldots, D_n constitutes a decomposition of \mathbb{R}^k . It is not difficult to show that, in fact,

$$H(y) = L_i^{-1}y \qquad y \in D_i,$$

whence the assertion.

In this paper, we study the invertibility of continuous strongly piecewise linear maps. We will prove later (Proposition 4.1 below) that, if such a map is invertible, then it is necessarily nondegenerate. It is not difficult to see that the converse of this statement is not true (see for instance Examples 4.1 and 4.2 below). Our main concern will be finding simple sufficient conditions for the invertibility. Section 4 is devoted to this purpose. Before dealing with this problem, however, we need some preliminaries.

A classical notion which we need is that of Bouligand derivative. Let $U \subseteq \mathbb{R}^s$ be open and let $f: U \to \mathbb{R}^m$ be locally Lipschitz. We say that f is *Bouligand differentiable* at $x_0 \in U$ if there exists a positively homogeneous function, $f'(x_0, \cdot) \colon \mathbb{R}^s \to \mathbb{R}^m$ with the property that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - f'(x_0, x - x_0)\|}{\|x - x_0\|} = 0.$$
 (2.1)

This uniquely determined function $f'(x_0, \cdot)$ is called the *Bouligand derivative* of f at x_0 (see Examples 5.1 and 5.2). An important fact proved by Kuntz/Scholtes [7] is the following:

Proposition 2.1 (Prop. 2.1 in [7]) Let $U \subseteq \mathbb{R}^s$ be an open set. Any PC^1 function $f: U \to \mathbb{R}^m$ is locally Lipschitz and, at every $x_0 \in U$, has a piecewise linear Bouligand derivative $f'(x_0, \cdot)$ which is a continuous selection of the Fréchet derivatives of the selection functions of f at x_0 .

Following [10] we consider a generalization of the notion of Jacobian matrix $\nabla f(x)$ of a function $f: \mathbb{R}^k \to \mathbb{R}^k$ at a Fréchet differentiability point x. Let $f: \mathbb{R}^k \to \mathbb{R}^k$ be locally Lipschitz at x_0 . We define Jac(f, x) as the (nonempty) set of limit points of sequences $\{\nabla f(x_k)\}$ where $\{x_k\}$ is a sequence converging to x_0 and such that f is Fréchet differentiable at x_k with Jacobian $\nabla f(x_k)$. One can see ([10]), as a consequence of Rademacher's Theorem, that Jac (f, x_0) is nonempty. Moreover the convex hull of Jac (f, x_0) is equal to the Clarke generalized Jacobian $\partial f(x_0)$ of f at x_0 , see [4] or the book [5].

Let $f: U \subseteq \mathbb{R}^k \to \mathbb{R}^k$ be a PC¹ function (with selection functions f_i). The relation between the Bouligand derivative and the above generalized notion of Jacobian is clarified by the following formula [10, Lemma 2]:

$$\operatorname{Jac}\left(f'(x_0,\cdot),0\right) \subseteq \operatorname{Jac}(f,x_0) = \left\{\nabla f_{\mathfrak{i}}(x_0) : \mathfrak{i} \in \overline{I}(x_0)\right\},\tag{2.2}$$

where $\bar{I}(x_0) = \{i : x_0 \in cl \text{ int} \{x \in U : i \in I(x)\}\}$, see e.g. [7]. Notice that by Proposition 2.1 the map $f'(x_0, \cdot)$ is continuous and piecewise linear, hence it is locally Lipschitz. Thus $Jac(f'(x_0, \cdot), 0)$ is well defined.

The following two results of [10] play a crucial role in the following. Here, we slightly reformulate them to match our notation.

Theorem 2.1 (Thm. 4 of [10]) Let $f: U \subseteq \mathbb{R}^k \to \mathbb{R}^k$ be a PC¹ function. Then f is a Lipschitz local homeomorphism at $x_0 \in U$ if and only if $Jac(f, x_0)$ consists of matrices whose determinants have the same nonzero sign and, for a sufficiently small neighborhood U_0 of x_0 , deg (f, U_0, y_0) , $y_0 := f(x_0)$, is well-defined and has value ± 1 .

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Theorem 2.2 (Thm. 5 of [10]) Let $f: U \subseteq \mathbb{R}^k \to \mathbb{R}^k$ be a PC^1 function, and let $x_0 \in U$. Assume that

$$\operatorname{Jac}(f, x_0) = \operatorname{Jac}(f'(x_0, \cdot), 0),$$

then the following statements are equivalent:

- *1.* f *is a Lipschitz local homeomorphism at* $x_0 \in U$ *;*
- 2. $f'(x_0, \cdot)$ is bijective;
- 3. $f'(x_0, \cdot)$ is a Lipschitz (global) homeomorphism.

Moreover, if any of (1)–(3) holds, then f is a local PC^1 homeomorphism at x_0 .

We conclude this subsection recalling the classical notion of Bouligand tangent cone. Let $C \subseteq \mathbb{R}^k$ be a nonempty closed subset. Given $x \in C$, the *Bouligand tangent cone to* C *at* x is the set:

$$\{v \in \mathbb{R}^{\kappa} : \exists \alpha_j \to 0^+, \exists v_j \to v \text{ s.t. } x + \alpha_j v_j \in C\}.$$

2.2 Topological degree

In this section we briefly recall the notion of Brouwer degree of a map and summarize some of its properties that will be used in the rest of the paper. Major references for this topic are, for instance, Milnor [9], Deimling [6] and Lloyd [8]; see also [3] for a quick introduction.

A triple (f, U, p), with $p \in \mathbb{R}^k$ and f a proper map defined in some neighborhood of the open set $U \subseteq \mathbb{R}^k$, is said to be *admissible* if $f^{-1}(p) \cap U$ is compact. Given an admissible triple (f, U, p), it is defined an integer deg(f, U, p), called the *degree of* f *in* U *respect to* p, that in some sense counts (algebraically) the elements of $f^{-1}(p)$ which lie in U. In fact, when in addition to the admissibility of (f, U, p) we let f be C¹ in a neighborhood of $f^{-1}(p) \cap U$ and assume p is a regular value of f, the set $f^{-1}(p) \cap U$ is finite, and one has

$$\deg(\mathbf{f}, \mathbf{U}, \mathbf{p}) = \sum_{\mathbf{x} \in \mathbf{f}^{-1}(\mathbf{p}) \cap U} \operatorname{sign} \det(\mathbf{f}'(\mathbf{x})),$$
(2.3)

where f'(x) denotes the (Fréchet) derivative of f at x. See e.g. [9] for a broader definition in the case when (f, U, p) is just an admissible triple.

The Brouwer degree enjoys many known properties only a few of which are needed in this paper. We now remind some of them.

Homotopy Invariance. If (H, U, α) is a smooth admissible homotopy joining two admissible triples, then

$$\deg(\mathsf{H}(\cdot, 0), \mathsf{U}, \alpha(0)) = \deg(\mathsf{H}(\cdot, 1), \mathsf{U}, \alpha(1)).$$

Excision. If (f, U, y) is admissible and V is an open subset of U such that $f^{-1}(y) \cap U \subseteq V$, then (f, V, y) is admissible and

$$\deg(f, U, y) = \deg(f, V, y).$$

Boundary Dependence. Let $U \subseteq \mathbb{R}^k$ be open, and let f and g be \mathbb{R}^k -valued functions defined in a neighborhood of U such that f(x) = g(x) for all $x \in \partial U$. Assume that U is bounded or, more generally, that f and g are proper and the difference map $f - g : \overline{U} \to \mathbb{R}^k$ has bounded image. Then

$$\deg(f, U, y) = \deg(g, U, y)$$

for any $y \in \mathbb{R}^k \setminus f(\partial U)$.

Observe that if $f: \mathbb{R}^k \to \mathbb{R}^k$ is proper then deg (f, \mathbb{R}^k, p) is well-defined for any $p \in \mathbb{R}^k$, moreover, by the above property, it is actually independent of the choice of p. In this case we shall simply write deg(f) instead of the more cumbersome deg (f, \mathbb{R}^k, p) .

Finally, we mention a well-known integral formula for the computation of the degree of an admissible triple when the dimension of the space is k = 2 (see e.g. [6, 8]) which we present here in a simplified form.

Assume that $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a proper map, let $B_r \subseteq \mathbb{R}^2$ be a ball of radius r > 0 centered at the origin and let $S_r = rS^1 = \partial B_r$. If $0 \notin f(S_r)$, then the degree of f in B_r relative to 0 coincides with the winding number of the curve $\sigma: [0, 1] \to \mathbb{R}^2$ given by

$$\sigma(t) = f(r\cos(2\pi t), r\sin(2\pi t)).$$

In other words,

$$\deg(f, B_r, 0) = \frac{1}{2\pi} \int_{f(S_r)} \omega$$

where ω is the 1-form

$$\omega = \frac{x \, dy}{x^2 + y^2} - \frac{y \, dx}{x^2 + y^2}$$

In fact, if B_r is large enough to contain the compact set $f^{-1}(0)$, then

$$\deg(f) = \frac{1}{2\pi} \int_{f(S_{\tau})} \omega.$$
(2.4)

3 Piecewise continuous linear maps and topological degree

Observe that any nondegenerate continuous strongly piecewise linear map G is differentiable in $\mathbb{R}^k \setminus \bigcup_{i=1}^n \partial C_i$. It is easily shown that G is proper, and therefore deg(G, \mathbb{R}^k , p) is well-defined for any $p \in \mathbb{R}^k$. In fact, one immediately checks that $G^{-1}(0) = \{0\}$. So, as remarked above, we can write deg(G) in lieu of deg(G, \mathbb{R}^k , p).

The following linear algebra result plays an important role in the paper.

Proposition 3.1 Let A and B be linear automorphisms of \mathbb{R}^k . Assume that for some $v \in \mathbb{R}^k \setminus \{0\}$, A and B coincide on the hyperplane $\{v\}^{\perp}$. Then, the map \mathcal{L}_{AB} defined by $x \mapsto Ax$ if $\langle v, x \rangle \ge 0$, and by $x \mapsto Bx$ if $\langle v, x \rangle \le 0$, is a homeomorphism if and only if det $(A) \cdot det(B) > 0$.

Proof. Let w_1, \ldots, w_{k-1} be a basis of the hyperplane $\{v\}^{\perp}$, then w_1, \ldots, w_{k-1}, v is a basis of \mathbb{R}^k . The matrix of $A^{-1}B$ in this basis is given by

$$\begin{pmatrix} & \gamma_1 \\ I_{k-1} & \vdots \\ & \gamma_{k-1} \\ \hline 0_{k-1}^t & \gamma_k \end{pmatrix}$$

where I_{k-1} is the (k-1)-unit matrix, O_{k-1} is the (k-1)-null vector and the γ_i 's are defined by

$$A^{-1}B\nu = \sum_{i=1}^{k-1} \gamma_i w_i + \gamma_k \nu.$$

Clearly γ_k is positive if and only if det(A) \cdot det(B) is positive.

Observe that if γ_k is negative then \mathcal{L}_{AB} is not one-to-one. In fact, being

$$Aw_{i} = Bw_{i}, \forall i = 1, \dots, k-1, \text{ and } \langle \sum_{i=1}^{k-1} -\frac{\gamma_{i}}{\gamma_{k}}w_{i} + \frac{1}{\gamma_{k}}\nu, \nu \rangle = \frac{\|\nu\|^{2}}{\gamma_{k}} < 0,$$

we get

$$\mathcal{L}_{AB}(\nu) = A\left(\sum_{i=1}^{k-1} -\frac{\gamma_i}{\gamma_k}w_i + \frac{1}{\gamma_k}A^{-1}B\nu\right) = \sum_{i=1}^{k-1} -\frac{\gamma_i}{\gamma_k}Aw_i + \frac{1}{\gamma_k}B\nu$$
$$= B\left(\sum_{i=1}^{k-1} -\frac{\gamma_i}{\gamma_k}w_i + \frac{1}{\gamma_k}\nu\right) = \mathcal{L}_{AB}\left(\sum_{i=1}^{k-1} -\frac{\gamma_i}{\gamma_k}w_i + \frac{1}{\gamma_k}\nu\right).$$

We now prove that \mathcal{L}_{AB} is injective if γ_k is positive. Assume this is not true. Since both A and B are invertible, there exist $z_A, z_B \in \mathbb{R}^k$ such that $\langle \nu, z_A \rangle > 0$, $\langle \nu, z_B \rangle < 0$ and $Az_A = Bz_B$ or, equivalently, $A^{-1}Bz_B = z_A$. Let

$$z_A = \sum_{i=1}^{k-1} c_A^i w_i + c_A v, \qquad z_B = \sum_{i=1}^{k-1} c_B^i w_i + c_B v,$$

so that $c_A > 0$, $c_B < 0$. The equality $A^{-1}Bz_B = z_A$ is equivalent to

$$\sum_{i=1}^{k-1} c_{B}^{i} w_{i} + c_{B} \sum_{i=1}^{k-1} \gamma_{i} w_{i} + c_{B} \gamma_{k} v = \sum_{i=1}^{k-1} c_{A}^{i} w_{i} + c_{A} v.$$

Consider the scalar product with v, we get $c_B \gamma_k ||v||^2 = c_A ||v||^2$, a contradiction.

We finally prove that, if γ_k is positive, then \mathcal{L}_{AB} is surjective. Let $z \in \mathbb{R}^k$. There exist y_A , $y_B \in \mathbb{R}^k$ such that $Ay_A = By_B = z$. If either $\langle v, y_A \rangle \ge 0$ or $\langle v, y_B \rangle \le 0$, there is nothing to prove. Let us assume $\langle v, y_A \rangle < 0$ and $\langle v, y_B \rangle > 0$. In this case $A^{-1}By_B = y_A$ and proceeding as above we get a contradiction.

Corollary 3.1 Let A, B and v be as in Proposition 3.1. Define \mathcal{L}_{AB} , as in Proposition 3.1, by

$$\mathcal{L}_{AB}(\mathbf{x}) = \begin{cases} A\mathbf{x} & \text{if } \langle \mathbf{v}, \mathbf{x} \rangle \ge \mathbf{0}, \\ B\mathbf{x} & \text{if } \langle \mathbf{v}, \mathbf{x} \rangle \le \mathbf{0}. \end{cases}$$

Assume that $det(A) \cdot det(B) > 0$. Then $deg(\mathcal{L}_{AB}) = sign det(A) = sign det(B)$.

Proof. The map \mathcal{L}_{AB} is invertible by Proposition 3.1. Take any $p \in \mathbb{R}^k$ such that the singleton $\{q\} = \mathcal{L}_{AB}^{-1}(p)$ does not belong to v^{\perp} . Then, Formula 2.3 yields the assertion.

Another useful tool for the computation of the topological degree of a strongly piecewise linear map is the following lemma:

Lemma 3.1 If G is a nondegenerate continuous strongly piecewise linear map as in Definition 2.5 with $det(L_i) > 0$, $\forall i = 1, ..., n$, then deg(G) > 0. In particular, if there exists $q \neq 0$ whose preimage $G^{-1}(q)$ is a singleton that belongs to at most two admissible cones C_i , then deg(G) = 1.

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Proof. Observe that the set $\cup_{i=1}^{n} G(\partial C_i)$ is nowhere dense hence $A := G(C_1) \setminus \bigcup_{i=1}^{n} G(\partial C_i)$ is non-empty. Take $x \in A$ and observe that if $y \in G^{-1}(x)$ then $y \notin \bigcup_{i=1}^{n} \partial C_i$. Thus, by (2.3),

$$\deg(G) = \sum_{y \in G^{-1}(x)} \text{sign } \det G'(y) = \#G^{-1}(x).$$
(3.1)

Since $G^{-1}(x) \neq \emptyset$, $\deg(G) > 0$.

Consider now the second part of the assertion. Assume in addition that $q \notin \bigcup_{i=1}^{n} G(\partial C_i)$. Taking x = q in (3.1) we get deg(G) = 1.

Let us now remove the additional assumption. Let $\{p\} = G^{-1}(q)$ be such that p belongs to two cones. Without any loss of generality we can assume $p \in \partial C_1 \cap \partial C_2$. Observe that by assumption $p \neq 0$ does not belong to any ∂C_s for $s \geq 3$. Thus one can find a neighborhood V of p, with $V \subset$ int $(C_1 \cup C_2) \setminus \{0\}$. By the excision property of the topological degree deg(G) = deg(G, V, p). Let $\mathcal{L}_{L_1 L_2}$ be a map as in Proposition 3.1. Observe that, by Corollary 3.1, the assumption on the signs of the determinants of L_1 and L_2 imply that $deg(\mathcal{L}_{L_1 L_2}) = 1$. Also notice that $\mathcal{L}_{L_1 L_2}|_{\partial V} = G|_{\partial V}$. Hence, by the excision and boundary dependence properties of the degree we have

$$1 = \deg(\mathcal{L}_{L_1L_2}) = \deg(\mathcal{L}_{L_1L_2}, V, p) = \deg(G, V, p).$$

Thus, deg(G) = 1 as claimed.

Remark 3.1 One can show that if $det(L_i) < 0$, for all i = 1, ..., n, then

$$\deg(G) < 0.$$

In particular, if there exists $q \neq 0$ whose preimage $G^{-1}(q)$ is a singleton that belongs to at most two of the admissible cones C_i , then deg(G) = -1. To see this, it is enough to compose G with the permutation matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{k-2} \end{pmatrix}, \quad \mathbf{J} := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

and I_{k-2} is the $(k-2) \times (k-2)$ identity matrix.

We conclude this section by observing that if G is a nondegenerate continuous strongly piecewise linear map in \mathbb{R}^2 then, by (2.4),

$$\deg(\mathsf{G}) = \frac{1}{2\pi} \int_{\mathsf{G}(\mathsf{S}^1)} \omega.$$
(3.2)

(observe, in fact, that $G^{-1}(0) = \{0\}$). This formula plays an important role in what follows.

4 Main results: invertibility of piecewise linear maps

We now turn to our main scope that is invertibility of continuous strongly piecewise linear maps. We begin with a technical lemma concerning \mathbb{R}^k , whose proof we only sketch for the sake of completeness.

Lemma 4.1 Let $\Sigma_1, \ldots, \Sigma_n$ be subspaces of \mathbb{R}^k with $\operatorname{codim}(\Sigma_i) \ge 2$ for each $i = 1, \ldots, n$. Then $\mathbb{R}^k \setminus (\bigcup_{i=1}^n \Sigma_i)$ is path connected.

Proof. [Sketch of the proof.] Let $P_0 \neq P_1$ be points of $\mathbb{R}^k \setminus \bigcup_{i=1}^n \Sigma_i$, and let B be the complete metric space, with the usual supremum distance, of continuous curves of [0, 1] in \mathbb{R}^k joining P_0 and P_1 . For each i = 1, ..., n, denote by E_i the subset of B of the curves taking values in $\mathbb{R}^k \setminus \Sigma_i$. We claim that the following assertions hold for i = 1, ..., n:

- 1. The set E_i is open in the space B.
- 2. The set E_i is dense in the space B.

Assertion (1) is very straightforward and follows from the fact that any curve in E_i has a positive distance from the closed set Σ_i and that any curve closer to it than this distance lies necessarily in E_i . The proof of assertion (2) is in two parts: first one proves that $E_i \neq \emptyset$ by showing that the segment joining P_0 and P_1 can be approximated by a polygonal lying in a 2-dimensional affine space containing the P_0 and P_1 and intersecting Σ_i in exactly one point. Then, a similar argument can be used to prove that any curve in B can be approximated as closely as desired by an element of E_i .

Once the claim is established, the lemma follows from the fact that $\bigcap_{i=1}^{n} E_i$ is open and dense in B.

Remark 4.1 The argument of Lemma 4.1 and the Baire cathegory theorem can be combined to show that \mathbb{R}^k cannot be disconnected by the union Σ of a denumerable family of (at least) 2-codimensional subspaces. As a matter of fact, given a pair of points not lying in Σ , one could prove that the set of curves in $\mathbb{R}^k \setminus \Sigma$ joining these points is residual.

We are now in a position to prove the following relatively simple result concerning the invertibility of continuous strongly piecewise linear maps.

Proposition 4.1 Let G be a continuous strongly piecewise linear map from \mathbb{R}^k into itself. If G is invertible, then it is nondegenerate.

Proof. Let C_i , i = 1, ..., n be the admissible cones decomposition of \mathbb{R}^k relative to G and let $L_i = G|_{C_i}$. We need to show that $det(L_i) \neq 0$ for any i = 1, ..., n and that all these determinants have the same sign.

Observe first that Lemma 2.1, since G is injective, implies that no such determinant is null. We now show that all these determinants have the same sign. Let us introduce the following set:

$$S := \left\{ C_i \cap C_j : i, j \in \{1, \dots, n\}, \text{ codim}(\text{span}(C_i \cap C_j)) \ge 2 \right\}.$$

Notice that when the dimension k of the ambient space \mathbb{R}^k is 1, then $S = \emptyset$, if k = 2 then S is merely the origin whereas, for k = 3, it consists of a finite number of half-lines emanating from the origin. As the dimension k grows, S becomes more complicated. However, with the help of Lemma 4.1 one can prove that $\mathbb{R}^k \setminus S$ is path connected for all $k \ge 1$.

Assume by contradiction $i, j \in \{1, ..., n\}$, $i \neq j$ are such that $det(L_i) det(L_j) < 0$. Since $\mathbb{R}^k \setminus S$ is path connected, it is not difficult to prove that there must exist two cones $C_{\overline{i}}$ and $C_{\overline{j}}$ such that $codim(span(C_{\overline{i}} \cap C_{\overline{j}})) = 1$ and $det(L_{\overline{i}}) det(L_{\overline{j}}) < 0$. Without any loss of generality we may assume $\overline{i} = 1, \overline{j} = 2$. Let $v \in \mathbb{R}^k$ such that $span(C_1 \cap C_2) = v^{\perp}$. Let $w_1, w_2, \ldots, w_{k-1} \in C_1 \cap C_2$ be a basis for $span(C_1 \cap C_2)$ such that

$$\left\{\sum_{i=1}^{k-1} c_i w_i \colon c_i \ge 0 \quad i=1,\ldots,k-1\right\} \subseteq (C_1 \cap C_2),$$

and let

$$L_1^{-1}L_2\nu = \gamma_k\nu + \sum_{i=1}^{k-1}\gamma_iw_i.$$

As in the proof of Proposition 3.1 one can show that $\gamma_k < 0$. Take $c_1, \ldots, c_{k-1} > 0$ and define

$$z_1 := v + \sum_{i=1}^{k-1} c_i w_i$$
 and $z_2 := \frac{1}{\gamma_k} v + \sum_{i=1}^{k-1} \left(c_i - \frac{\gamma_i}{\gamma_k} \right) w_i.$

An easy computation shows that $L_1z_1 = L_2z_2$. Choosing c_1, \ldots, c_{k-1} large enough, we can assume that $z_1 \in C_1$, $z_2 \in C_2$. Thus $G(z_1) = G(z_2)$, i.e. G is not injective, against the assumption. This contradiction shows that all determinants det (L_s) , $s \in \{1, \ldots, n\}$, share the same sign.

Simple considerations (e.g. Examples 4.1 and 4.2 below) show that the converse of Propositions 4.1 is not true in general. In order to partially invert this proposition, different situations must be considered. We begin with a simple consequence of Lemma 3.1.

Theorem 4.1 Let $G : \mathbb{R}^k \to \mathbb{R}^k$ be a continuous strongly piecewise linear map as in Definition 2.5 with det(L_i) of constant sign for all i = 1, ..., n. Assume also that there exists $q \in \mathbb{R}^k$ whose preimage $G^{-1}(q)$ is a singleton that belongs to at most two of the admissible cones C_i . Then G is a Lipschitz homeomorphism.

Proof. Lemma 3.1 and Remark 3.1 imply that $deg(G) = \pm 1$. The assertion follows from Theorem 2.1 and Lemma 2.2.

Remark 4.2 The condition in Theorem 4.1 concerning the existence of a point q whose preimage is a singleton belonging to at most two polyhedral cones, is equivalent to the existence of a half-line at the origin whose preimage is a single half-line. In fact, as a consequence of Theorem 4.1, one has that if the determinants $det(L_i)$ have constant sign for all i = 1, ..., n the existence of such a half-line implies that all the half-lines at the origin must have the same property.

Remark 4.3 Observe that the only nontrivial (i.e. such that are not reducible to linear maps) continuous strongly piecewise linear maps with n = 2 are those in which the cones are half-spaces. In fact, unless C_1 and C_2 are two half-spaces, then dim $(\text{span}(\partial C_1 \cap \partial C_2)) = k$ and two linear endomorphisms of \mathbb{R}^k that agree on k linearly independent vectors, necessarily coincide. Hence, when n = 2, it is sufficient to consider the case when the two nontrivial cones are half-spaces. This has already been done in Proposition 3.1.

The point q in Theorem 4.1 may be difficult to determine if the linear maps L_i 's are given in a complicate way. However, in some cases, invertibility of continuous nondegenerate strongly piecewise linear maps can be deduced merely from their nondifferentiability structure. The easiest nontrivial case, i.e. when n = 2, has already been treated (Proposition 3.1) in arbitrary dimension just by means of linear algebra. The other cases, n = 3 and n = 4, will be investigated in dimension k = 2 only.

We are now in a position to state our main result concerning the invertibility of continuous strongly piecewise linear maps in \mathbb{R}^2 .

Theorem 4.2 Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be as in Definition 2.5 and non-degenerate. If one of the following conditions holds:

1.
$$n \in \{1, 2, 3\};$$

2. n = 4 and all the admissible cones are convex;

then G has a continuous piecewise linear inverse.

Before providing the proof of this result, we show with two examples that the assumptions of Theorem 4.2 are, to some extent, sharp.



Figure 1: The image of S^1 under G in Example 4.1. For clarity's sake, the radial distance of $G(S^1)$ from (0, 0) has been rescaled.

Our first example shows that for n > 4 there are G's as above that are not invertible even if the cones are convex.

Example 4.1 Consider a nondegenerate continuous piecewise linear map $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as in Definition 2.5 by

$$L_{1} = \begin{pmatrix} 1 & -\sqrt{2} \\ 0 & \sqrt{2} - 1 \end{pmatrix} \quad L_{2} = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} + 1 \\ 1 & 0 \end{pmatrix}$$
$$L_{3} = \begin{pmatrix} 0 & 1 \\ -\sqrt{2} + 1 & -\sqrt{2} \end{pmatrix} \quad L_{4} = \begin{pmatrix} \sqrt{2} - 1 & 0 \\ -\sqrt{2} & 1 \end{pmatrix} \quad L_{5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the corresponding cones are given, in polar coordinates, by the pairs (ρ, θ) with arbitrary ρ 's and θ chosen as in the following table:

C ₁	C ₂	C ₃	C ₄	C ₅
$0 \le \theta \le \frac{3}{8}\pi$	$\tfrac{3}{8}\pi \le \theta \le \tfrac{3}{4}\pi$	$\tfrac{3}{4}\pi \le \theta \le \tfrac{9}{8}\pi$	$\frac{9}{8}\pi \le \theta \le \frac{3}{2}\pi$	$\tfrac{3}{2}\pi \le \theta \le 2\pi$

This map is illustrated in Figure 1. As the picture suggests, the above defined map G is not invertible because it is not injective.

Our second example shows an instance of non invertible G with n = 4 and one nonconvex cone.

Local inversion of planar maps with nice nondifferentiability structure

Example 4.2 Consider $G: \mathbb{R}^2 \to \mathbb{R}^2$ as in Definition 2.5, with

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 \\ 2\sqrt{3} & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} -2 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 2\sqrt{3} \\ 0 & 1 \end{pmatrix},$$

and the cones are given, in polar coordinates, by the pairs (ρ, θ) with arbitrary ρ 's and θ chosen as in the following table:

C ₁	C ₂	C ₃	C ₄
$0 \le \theta \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \theta \le \frac{2}{3}\pi$	$\frac{2}{3}\pi \le \theta \le \frac{11}{6}\pi$	$\frac{11}{6}\pi \le \theta \le 2\pi$

Figure 2 illustrates this map. As the picture suggests, G defined as above is not injective and therefore it is not invertible.



Figure 2: The image of S^1 under G in Example 4.2. For clarity's sake, the radial distance of $G(S^1)$ from (0,0) has been rescaled.

Let us now turn to the task of proving Theorem 4.2. The proof is done differently according to the number of nontrivial cones in which the plane is pie-sliced. The proof, in the cases of n = 2, boils down to Proposition 3.1 whereas the cases n = 3 and n = 4 will be treated with the help of Theorem 2.2. In order to apply this theorem it is necessary to estimate the topological degree of our map G. This will be done by means of geometric considerations. The proof of the following lemma is based on an elementary linear algebra argument and is left to the reader.

Lemma 4.2 Let $A: \mathbb{R}^2 \to \mathbb{R}^2$ be linear and nonsingular and let $C \subset \mathbb{R}^2$ be an admissible cone. Then $A(C) \subseteq \mathbb{R}^2$ is an admissible cone and the following statements hold:

1. If C does not contain a half-plane, then A(C) does not contain a half-plane.

2. If $C \subseteq \mathbb{R}^2$ contains a half-plane, then so does $A(C) \subseteq \mathbb{R}^2$.

This lemma has an useful consequence:

Lemma 4.3 Let $A: \mathbb{R}^2 \to \mathbb{R}^2$ be linear and nonsingular and let $C \subsetneq \mathbb{R}^2$ be a cone with vertex at the origin. Let Γ be the image of the arc $S^1 \cap C$. Then,

$$\left| \int_{\Gamma} \omega \right| \in \begin{cases} [0,\pi) & \text{if } C \text{ does not contain a half-plane,} \\ [\pi, 2\pi) & \text{otherwise.} \end{cases}$$
(4.1)

In particular, we have that
$$\left|\int_{\Gamma} \omega\right| < 2\pi$$
 and $\left|\int_{\Gamma} \omega\right| < \pi$ when C is strictly convex

Proof. Observe first that by Lemma 4.2 there exists a half-line s starting at the origin that does not intersect A(C). Clearly the differential form ω is exact in $\mathbb{R}^2 \setminus s$. Let P_1 and P_2 be the intersections of ∂C with S^1 . The path integral that appears in (4.1) does not depend on the chosen path connecting $A(P_1)$ and $A(P_2)$. With the choice of an appropriate path, for instance, the concatenation of a circular arc of radius $|A(P_2)|$ from $A(P_2)$ with the radial segment through $A(P_1)$ (see Figure 3), it is not difficult to show that $|\int_{\Gamma} \omega|$ is merely the angular distance (we consider the angle that does not contain the half-line s) between $A(p_1)$ and $A(p_2)$ as seen from the origin. The assertion now follows from Lemma 4.2.



Figure 3: The integration path in Lemma 4.3

Lemma 4.4 Let G be as in Theorem 4.2 with n = 3 and det $L_i > 0$, $\forall i = 1, 2, 3$. Then, deg(G) = 1.

Proof. We consider the two possible cases: when all the cones are strictly convex and when there is one cone containing a half-plane. For i = 1, 2, 3, let Γ_i , be the image $G(S^1 \cap C_i)$. In the first case, by Lemma 4.3 we have that $\int_{\Gamma_i} \omega < \pi$ for i = 1, 2, 3. Hence, by Lemma 3.1 and formula (3.2),

$$0 < \deg(\mathsf{G}) < \frac{\pi + \pi + \pi}{2\pi} = \frac{3}{2}.$$
(4.2)

Which, the degree being an integer, implies deg(G) = 1.

In the second case, only one of the cones, say C_1 , may contain a half-plane. Thus, by Lemma 4.3, we have that $\int_{\Gamma_1} \omega < 2\pi$ and $\int_{\Gamma_1} \omega < \pi$ for i = 2, 3. Hence, inequality (4.2) becomes

$$0 < \deg(\mathsf{G}) < \frac{2\pi + \pi + \pi}{2\pi} = 2.$$

Which, again, implies deg(G) = 1.

Lemma 4.5 Let G be as in Theorem 4.2 with n = 4 and det $L_i > 0 \forall i = 1, 2, 3, 4$. Then deg(G) = 1.

Proof. For i = 1, ..., 4, let Γ_i , be the image $G(S^1 \cap C_i)$. By Lemma 4.3 we have that $\int_{\Gamma_i} \omega < \pi$ for i = 1, ..., 4 since all the cones are convex. Hence, by Lemma 3.1 and formula (3.2),

$$0 < \deg(\mathsf{G}) < \frac{\pi + \pi + \pi + \pi}{2\pi} = 2.$$

Which, the degree being an integer, implies deg(G) = 1.

Remark 4.4 If det $L_i < 0 \forall i = 1, ..., n$, composing G with the linear maps whose matrix in the standard basis of \mathbb{R}^2 is $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get that $JG : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one so that G is invertible as well.

We are now in a position to prove our main theorem.

Proof. [Proof of Theorem 4.2]

(Case n = 1.) In this case G is linear with nonzero determinant. Thus, there is nothing to prove. (Case n = 2.) See Proposition 3.1 and Remark 4.3.

(Cases n = 3 and n = 4.) In both cases, it follows from Lemmas 4.4 and 4.5 that deg(G) = 1, then the assertion follows from Theorem 2.1.

Example 4.3 Let us consider a decomposition of the space \mathbb{R}^3 into four convex wedges C_1, \ldots, C_4 with a common edge along the straight line r. Let $G: \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous strongly piecewise linear map with respect to this decomposition and with

$$G(x) = L_i x, \qquad x \in C_i, i = 1, \dots, 4$$

and assume that det L_i share the same sign for i = 1, ..., 4. Then, as a consequence of Theorem 4.2, we have that G is invertible with continuous strongly piecewise linear inverse. To see that, consider a plane π orthogonal to r. Clearly, the restriction $G|_{\pi}$ is invertible by Theorem 4.2. Similarly, since $G(x_r) = L_1 x_r = ... = L_4 x_r$, for any point $x_r \in r$, and the L_i 's are isomorphisms, G is invertible on r. Given any vector $y \in \mathbb{R}^3$, we can obtain $G^{-1}(y)$ by the following argument. Write $y = y_{\pi} + y_r$ where y_{π} and y_r denote the orthogonal projections of y onto π and r, respectively. Then one has

$$G^{-1}(y) = L_1^{-1}(y_r) + (G|_{\pi})^{-1}(y_{\pi}).$$

We conclude this section with an example showing that our main result (at least when n = 4) cannot be deduced from the well-known Clarke's Theorem [4, 5]. This important and widely used result on inverse functions states that if $f: \mathbb{R}^k \to \mathbb{R}^k$ satisfies a Lipschitz condition in a neighborhood of $x_0 \in \mathbb{R}^k$ and all the matrices in the Clarke generalized Jacobian $\partial f(x_0)$ (see, e.g., [4, 5] for a definition) are invertible, then f is locally invertible about x_0 . In the case of a continuous strongly piecewise linear function G at 0, the Clarke generalized jacobian $\partial G(0)$ is the closed convex hull of Jac(G, 0).

In Example 4.4 we exhibit a continuous piecewise linear map which is invertible by our main result although it does not satisfy the assumptions of Clarke's Theorem.



Figure 4: The image of S^1 under G in Example 4.4. For clarity's sake, the radial distance of $G(S^1)$ from (0, 0) and the proportion between axes have been both altered.

Example 4.4 Consider a continuous piecewise linear map $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ given as in Definition 2.5 by

$$L_{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_{2} = \begin{pmatrix} \frac{1}{10} & 0 \\ -10 & 1 \end{pmatrix}, \quad L_{3} = \begin{pmatrix} \frac{3}{100} & \frac{5}{100} \\ -\frac{455}{100} & -\frac{445}{100} \end{pmatrix}, \quad L_{4} = \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{10} \end{pmatrix},$$

where the corresponding cones are given, in polar coordinates, by the pairs (ρ, θ) with arbitrary $\rho \ge 0$ and θ chosen as in the following table

$$C_1$$
 C_2 C_3 C_4 $0 \le \theta \le \frac{\pi}{2}$ $\frac{\pi}{2} \le \theta \le \frac{5}{4}\pi$ $\frac{5}{4}\pi \le \theta \le \frac{7}{4}\pi$ $\frac{7}{4}\pi \le \theta \le 2\pi$

This map is illustrated in Figure 4. Trivial computation shows that $det(L_i) > 0$ for any i = 1, 2, 3, 4 and, as the picture suggests, G has degree 1. (The map G has been found with the help of a short FORTRAN program that assisted us in sifting many potential examples.)

In this case $\partial G(0)$ is just the closed convex hull of the matrices L_i, i = 1, 2, 3, 4, i.e.

$$\partial G(0) = \left\{ \sum_{i=1}^{4} a_i L_i, \ a_i \ge 0 \ i = 1, 2, 3, 4, \ \sum_{i=1}^{4} a_i = 1 \right\}.$$

Choosing

$$a_1 = \frac{637}{7208}, \quad a_2 = \frac{1165}{7208}, \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{4}$$

we get det $\left(\sum_{i=1}^{4} a_i L_i\right) = 0$, i.e. $\partial G(0)$ contains at least one singular matrix. Moreover, choosing

$$a_1 = \frac{9}{136}, \quad a_2 = \frac{25}{136}, \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{4},$$

we get det $\left(\sum_{i=1}^{4} \alpha_i L_i\right) = \frac{-1}{10}$, i.e. $\partial G(0)$ contains at least one matrix with negative determinant.

5 Application: Piecewise differentiable functions

We now provide some applications of the results of the previous section to the local invertibility of PC^1 functions. In this section, for a Frechét differentiable map φ at a point x_0 , the (Frechét) differential at x_0 is denoted by $d\varphi(x_0)$. The basis for our considerations is the following consequence of Theorem 2.1.

Theorem 5.1 Let f be an \mathbb{R}^k -valued PC^1 function in a neighborhood of $x_0 \in \mathbb{R}^k$. Assume that

- 1. All the determinants of all the elements of $Jac(f, x_0)$ have the same sign;
- 2. The Bouligand differential of f at x_0 is an invertible piecewise linear map.

Then f *is locally invertible at* x_0 *.*

Proof. It is not difficult to show that since $f'(x_0, \cdot)$ is invertible,

$$\deg(f'(x_0,\cdot),V,0) = \mathfrak{s}_{\mathfrak{s}}$$

where \mathfrak{s} denotes the common sign of the determinants of the elements of Jac(f, \mathfrak{x}_0).

Consider the map F with the same domain as f given by $F(x) = f(x) - f(x_0)$. Clearly, f is locally invertible if and only if so is F. We claim that in a sufficiently small neighborhood V of x_0 the map F is admissibly homotopic to $f'(x_0, \cdot)$ so that, by homotopy invariance, deg(F, V, 0) = \mathfrak{s} . The assertion follows from Theorem 2.1.

We now prove the claim. Since f is Bouligand differentiable in x_0 there exists a continuous function ε such that $\varepsilon(0) = 0$ and

$$f(x) = f(x_0) + f'(x_0, x - x_0) + |x - x_0| \varepsilon(x - x_0).$$

Thus, $F(x) = f'(x_0, x - x_0) + |x - x_0| \varepsilon(x - x_0)$. Consider the homotopy

$$H(x,\lambda) = f'(x_0, x - x_0) + \lambda |x - x_0| \varepsilon(x - x_0), \quad \lambda \in [0, 1],$$

and let $\{1, \ldots, n\}$ be the active index set of f (hence of F) at x_0 . Observe that

$$m := \inf \left\{ |f'(x_0, \nu)| : |\nu| = 1 \right\} = \min_{i=1,...,n} || df_i(x_0) || > 0.$$

Thus,

$$|\mathsf{H}(\mathbf{x},\lambda)| \ge \left(\mathsf{m} - |\varepsilon(\mathbf{x} - \mathbf{x}_0)|\right) |\mathbf{x} - \mathbf{x}_0|.$$

This shows that in a conveniently small ball centered at x_0 , the homotopy H is admissible. The claim follows from the homotopy invariance property of the degree.

Example 5.1 Let $R_1 = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$, $R_2 = \{(x, y) \in \mathbb{R}^2 : y < -x^2\}$, and $R_3 = \mathbb{R}^2 \setminus (R_1 \cup R_2)$. Consider the PC¹ map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \begin{cases} (x,2y-x^2) & \text{for } (x,y) \in R_1, \\ (x,2y+x^2) & \text{for } (x,y) \in R_2, \\ (x,y) & \text{for } (x,y) \in R_3. \end{cases}$$

The map f is Frechét, hence Bouligand, differentiable (but it is not C^1). In fact

$$\mathbf{f}'((0,0),\cdot) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The determinant of $f'((0,0), \cdot)$ is positive. Hence, by Theorem 5.1, f is locally invertible about the origin. Observe in passing that the local invertibility of f does not follow directly from Theorem 2.2 because $Jac(f, x_0) \neq Jac(f'(x_0, \cdot), 0)$.

Example 5.2 Let $R_1 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, $R_2 = \{(x, y) \in \mathbb{R}^2 : y < -x^2\}$, and $R_3 = \mathbb{R}^2 \setminus (R_1 \cup R_2)$. Consider the PC^1 map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \begin{cases} (x,y/2) & \text{for } (x,y) \in R_1, \\ (x,2y+x^2) & \text{for } (x,y) \in R_2, \\ (x,y) & \text{for } (x,y) \in R_3. \end{cases}$$

The map f is Bouligand differentiable. Indeed, the Bouligand differential is as follows:

$$f'((0,0),(x,y)) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } y \ge 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } y \le 0. \end{cases}$$

Both the linear operators in the Bouligand differential have positive determinant. Hence, by Theorem 5.1, f is locally invertible about the origin.

In order to apply Theorem 5.1 above one needs to know when the linearization of a PC^1 map (which is a continuous piecewise linear map) is invertible. This is what all the previous section is about. Criteria for the local invertibility of PC^1 map will be deduced from Theorem 5.1 combined with the results of the previous section.

Let f be an \mathbb{R}^k -valued PC¹ function in a sufficiently small ball $B(x_0, \rho) \subseteq \mathbb{R}^k$, and let $\mathcal{I}_0 = \{1, \ldots, n\}$ be the active index set in $B(x_0, \rho)$. For each $i \in \mathcal{I}_0$ define

$$S_{i} := \{ x \in B(x_{0}, \rho) : f(x) = f_{i}(x) \}.$$
(5.1)

Let C_1, \ldots, C_n be the tangent cones (in the sense of Bouligand) at x_0 to the sectors S_1, \ldots, S_n . Assume that the C_i 's are admissible cones and that

$$df_i(x_0)x = df_j(x_0)x \text{ for any } x \in C_i \cap C_j, i, j \in \{1, \dots, n\}, i \neq j.$$

Define

$$F(x) = df_i(x_0)x$$
 $x \in C_i, i = 1,...,n$ (5.2)

so that F is a continuous piecewise linear map (compare [7]).

This section is concerned with local invertibility of such maps. We first consider arbitrary dimension.

Corollary 5.1 Let f and F be as above, with F nondegenerate at 0. Assume also that there exists $p \in \mathbb{R}^k$ whose preimage under F, $F^{-1}(p)$, is a singleton that belongs to at most two of the cones C_i . Then f is a Lipschitz homeomorphism in a sufficiently small neighborhood of x_0 .

Proof. From Theorem 4.1 it follows that F is invertible. The assertion follows from Theorem 5.1.

The above Corollary 5.1 can be greatly simplified when the number of cones is n = 2, in the sense that the assumption on the existence of the special point p can be dropped altogether. In fact, in dimension k = 2 this is true also for n = 3 and, when n = 4, one can replace it by merely requiring the convexity of the tangent cones to the sectors.

Corollary 5.2 Let f and F be as above, with F nondegenerate at 0 and n = 2. Then f is a Lipschitz homeomorphism in a sufficiently small neighborhood of x_0 .

Proof. From Proposition 3.1 it follows that F is invertible. The assertion follows from Theorem 5.1.

We finally consider dimension k = 2 of the ambient space.

Corollary 5.3 Let f and F be as above, with F nondegenerate at 0. We have that if either

•
$$n \in \{1, 2, 3\},$$

- or
- n = 4 and all the cones C_i 's are convex,

then f is a Lipschitz homeomorphism in a sufficiently small neighborhood of x_0 .

Proof. Since F is nondegenerate then it is invertible by Theorem 4.2. Theorem 5.1, yields the assertion.

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