BRANCHES OF HARMONIC SOLUTIONS TO PERIODICALLY PERTURBED COUPLED DIFFERENTIAL EQUATIONS ON MANIFOLDS

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Abstract. Fixed point index methods are used to explore the structure of the set of harmonic solutions to periodically perturbed coupled differential equations on differentiable manifolds. The results obtained generalize existing theorems for single differential equation by gathering them in an unique framework.

1. Introduction

This paper explores some properties of the set of periodic solutions of periodically perturbed coupled differential equations on manifolds.

More precisely, let $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^s$ be boundaryless smooth manifolds, let $f : \mathbb{R} \times M \times N \to \mathbb{R}^k$ be tangent to M, and let $g : M \times N \to \mathbb{R}^s$ and $h : \mathbb{R}\times M\times N \to \mathbb{R}^s$ be tangent to N. (This means that, for any $(t, p, q) \in \mathbb{R}\times M\times N$, $g(p, q)$ and $h(t, p, q)$ belong to the tangent space T_qN, and $f(t, p, q)$ is in T_pM.) Given $T > 0$, we assume also that f and h are T-periodic in the first variable. Consider the following system of differential equations for $\lambda \geq 0$:

$$\begin{cases} \dot{x} = \lambda f(t, x, y), \\ \dot{y} = g(x, y) + \lambda h(t, x, y). \end{cases} \tag{1.1}$$

This system is equivalent to a single parameter-dependent differential equation on the product manifold $M \times N \subseteq \mathbb{R}^{k+s}$.

Denote by $C_T(M)$ and $C_T(N)$ the spaces of T-periodic continuous functions from $\mathbb{R}$ to M and N, respectively, with the topology of uniform convergence. We investigate the properties of the set of the T-triples of (1.1), i.e. of those triples $(\lambda, x, y) \in [0,\infty)\times C_T(M)\times C_T(N)$, where (x, y) is a solution to (1.1). In particular, we shall give conditions for the existence of a noncompact connected component of T-triples (i.e. a "*branch*") emanating from the set $\nu^{-1}(0)$, where $\nu : M\times N \to \mathbb{R}^{k+s}$ is the autonomous vector field, tangent to the manifold $M \times N \subseteq \mathbb{R}^{k+s}$, given by

$$\nu(p, q) = \left(\frac{1}{T}\int_0^T f(t, p, q)\,\mathrm{d}t\ ,\ g(p, q)\right).$$

In order to prove our result, we will study the fixed point index of the Poincaré T-translation operator associated with (1.1) for small values of $\lambda > 0$. The techniques used in this paper are similar in spirit to those exposed in the survey [3] and references therein but, in general, the results obtained here cannot be deduced directly from those. For instance, since (1.1) is a periodic perturbation of the nontrivial system $\dot{x} = 0$, $\dot{y} = g(x, y)$, if $g(p, q) \not\equiv 0$, the results in [3] are not applicable when M is noncompact. Thus, in a sense, this paper can be seen as a first step towards the study of the structure of periodic solutions to periodically

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perturbed differential equations on a manifold for which the unperturbed term has a noncompact submanifold of zeros.

This paper offers insights also from a different viewpoint. Consider system (1.1) when no coupling occurs. Namely, consider the following pair of equations on M and N:

$$\dot{x} = \lambda f(t, x), \tag{1.2}$$

$$\dot{y} = g(y) + \lambda h(t, y). \tag{1.3}$$

The results of this paper allow us to relate some known facts about equations (1.2) and (1.3) whose mutual connection was not yet clearly perceived. For instance, the fixed point index of the Poincaré T-translation operator associated with (1.2) and (1.3) can be computed by [3, Thm. 3.15] and [3, Thm. 3.11], respectively. These two result, actually, are consequences of Theorem 3.1 below (see the corollaries 3.4 and 3.5). We will also show that some other known facts on the set of periodic solutions to equations (1.2) and (1.3) are consequences of the more general results discussed in this paper (see Remarks 4.6 and 4.8 below).

The paper is organized as follows: Section 2 gathers some known results about the fixed point index and the degree of tangent vector fields that are needed in the following sections. In Section 3 we compute, for small values of λ, the fixed point index of the Poincaré T-translation operator associated with (1.1). The main result of this section (Theorem 3.1 below) is inspired by [3, Thm. 3.11]. Section 4 contains the main results of this paper concerning the set of periodic solutions to (1.1). Such results are adapted from the analogous ones in [4, 3]. Finally, in Section 5 we describe a simple application to the set of periodic solutions of a family of second order scalar differential equations.

2. Preliminaries and notation

Through this paper, M and N shall denote smooth submanifolds of $\mathbb{R}^k$ and $\mathbb{R}^s$, of dimensions m and n respectively. Thus $M \times N \subseteq \mathbb{R}^{k+s}$ is a smooth manifold of dimension $m+n$. Moreover, if T_pM and T_qN denote the tangent spaces of M at p and of N at q respectively, then $T_{(p,q)}(M \times N) = T_pM \times T_qN$.

A vector field $v : M \to \mathbb{R}^k$ such that $v(p) \in T_pM$ for all $p \in M$ is said to be tangent to M. Similarly, a time-dependent vector field $u : \mathbb{R} \times M \to \mathbb{R}^k$ is tangent to M if $u(t, p) \in T_pM$ for all $(t, p) \in \mathbb{R} \times M$. When $M = \{p\}$ is a singleton, its tangent space is the 0-dimensional vector space $T_pM = \{0\}$. Thus, there exists a unique possible tangent vector field that will be denoted by $\mathbf{0}$.

Given a time-dependent vector field $u : \mathbb{R} \times M \times N \to \mathbb{R}^{k+s}$ tangent to $M \times N \subseteq \mathbb{R}^{k+s}$, we can clearly write $u(t, p, q) = \big(u^1(t, p, q), u^2(t, p, q)\big)$, where $u^1 : \mathbb{R} \times M \times N \to \mathbb{R}^k$ and $u^2 : \mathbb{R} \times M \times N \to \mathbb{R}^s$ are tangent to M and N respectively, in the sense that $u^1(t, p, q) \in T_pM$ and $u^2(t, p, q) \in T_qN$ for $(t, p, q) \in \mathbb{R} \times M \times N$. Conversely, given any pair of tangent vector fields $u^1 : \mathbb{R} \times M \times N \to \mathbb{R}^k$ and $u^2 : \mathbb{R} \times M \times N \to \mathbb{R}^s$ as above, $u(t, p, q) = \big(u^1(t, p, q), u^2(t, p, q)\big)$ is a vector field tangent to $M \times N$.

Let us recall some basic facts about the topological degree of tangent vector fields on manifolds and about the fixed point index of maps on manifolds.

Let $v : M \to \mathbb{R}^k$ be a C^1 tangent vector field and take $p \in v^{-1}(0)$. One can prove (see e.g. [7]) that the image of the Fréchet derivative $v'(p) : T_pM \to \mathbb{R}^k$ of v at p is contained into T_pM. In other words, when p is a zero of v, $v'(p)$ is an endomorphism of T_pM, thus $\det v'(p)$ is well defined. When $\det v'(p) \neq 0$ we say that p is a *nondegenerate zero* of v and its *index* $\mathrm{i}(v, p)$ is defined as $\operatorname{sign}\big(\det v'(p)\big)$.

Let $\omega : M \to \mathbb{R}^k$ be a continuous tangent vector field on M, and let V be an open subset of M in which we assume ω admissible for the degree, that is $\omega^{-1}(0) \cap V$ compact. Then, one can associate to the pair (ω, V) an integer, $\deg(\omega, V)$, called the *degree (or characteristic) of the vector field* ω in V, which, roughly speaking, counts (algebraically) the number of zeros of ω in V (see e.g. [6, 7] and references therein). More precisely, in the case when ω is smooth and $\omega^{-1}(0) \cap V$ consists of a finite number of nondegenerate zeros, we define $\deg(\omega, V)$ as the sum of the indices of these zeros. In the general admissible case the degree is defined by taking a sufficiently close smooth approximation of ω having finitely many nondegenerate zeros in V (see e.g. [3]).

When $M = \mathbb{R}^k$, $\deg(\omega, W)$ is just the classical Brouwer degree, $\deg(w, W, 0)$, of w at 0 in any bounded open neighborhood W of $w^{-1}(0) \cap V$ whose closure is in V. Moreover, when M is a compact manifold, the celebrated Poincaré-Hopf Theorem states that $\deg(\omega, M)$ coincides with the Euler-Poincaré characteristic of M and, therefore, is independent of ω. Observe, in particular, that when $M = \{p\}$ is a singleton, the following convention is justified:

$$\deg(\mathbf{0}, M) = 1. \tag{2.1}$$

We recall an important relation concerning the degree of a vector field: for a given constant $\alpha \neq 0$, one has

$$\deg(v, U) = (\operatorname{sign} \alpha)^m \deg(\alpha v, U), \tag{2.2}$$

where m denotes the dimension of M.

Let V be an open subset of M, and let $\Psi : V \to M$ be continuous. The map Ψ is said to be admissible (for the fixed point index) on V if its set of fixed points is compact. In these conditions it is defined an integer, called the *fixed point index* of Ψ in V and denoted by $\operatorname{ind}(\Psi, V)$, which satisfies all the classical properties of the Brouwer degree: solution, excision, additivity, homotopy invariance, normalization etc. A detailed exposition of this matter can be found, for example, in [8] and references therein. The following fact deserves to be mentioned: if M is an open subset of $\mathbb{R}^m$, then $\operatorname{ind}(\Psi, V)$ is just the Brouwer degree of $I - \Psi$ in V at 0, where $I - \Psi$ is defined by $(I - \Psi)(x) = x - \Psi(x)$.

Clearly, similar statements and properties remain valid also when M is replaced with $M \times N$. A further property will be needed in what follows.

Lemma 2.1. *Consider a tangent vector field $\gamma : M \times N \to \mathbb{R}^{k+s}$ given by $\gamma(p, q) = \big(\gamma^1(p, q), \gamma^2(p, q)\big)$, and let $\alpha \neq 0$ be a given constant. Define $\gamma_\alpha : M \times N \to \mathbb{R}^{k+s}$ by $\gamma_\alpha(p, q) = \big(\alpha\gamma^1(p, q), \gamma^2(p, q)\big)$. If γ is admissible for the degree on an open subset U of $M \times N$, then*

$$\deg(\gamma_\alpha, U) = (\operatorname{sign} \alpha)^m \deg(\gamma, U), \tag{2.3}$$

where m is the dimension of M.

Observe that in the particular case when $\alpha > 0$ the assertion of Lemma 2.1 follows directly from the homotopy property.

Sketch of the proof of Lemma 2.1. Without loss of generality we can assume that U is relatively compact and that γ is C^1. Observe that $\gamma_\alpha^{-1}(0) = \gamma^{-1}(0)$ and that for $(p, q) \in \gamma^{-1}(0)$, the Fréchet derivative $\gamma'_\alpha(p, q) : T_{(p,q)}(M \times N) \to T_{(p,q)}(M \times N)$ can be written in block-matrix form as follows

$$\gamma'_\alpha(p, q) = \begin{pmatrix} \alpha\partial_1\gamma^1(p, q) & \alpha\partial_2\gamma^1(p, q) \\ \partial_1\gamma^2(p, q) & \partial_2\gamma^2(p, q) \end{pmatrix}$$

where ∂_1 and ∂_2 denote the partial derivative with respect to the first and second variable, respectively.

By standard transversality arguments, we can assume that the zeros of γ_α are nondegenerate. That is, for any $(p,q) \in \gamma_\alpha^{-1}(0)$ the determinant of $\gamma'(p,q)$ is nonzero. Thus,

$$\begin{aligned}
\deg(\gamma_\alpha, U) &= \sum_{(p,q)\in\gamma_\alpha^{-1}(0)} \operatorname{sign}\det \gamma'_\alpha(p,q) = \sum_{(p,q)\in\gamma^{-1}(0)} \operatorname{sign}\det \gamma'_\alpha(p,q) \\
&= \sum_{(p,q)\in\gamma^{-1}(0)} \operatorname{sign}\det \begin{pmatrix} \alpha\partial_1\gamma^1(p,q) & \alpha\partial_2\gamma^1(p,q) \\ \partial_1\gamma^2(p,q) & \partial_2\gamma^2(p,q) \end{pmatrix} \\
&= \sum_{(p,q)\in\gamma^{-1}(0)} \operatorname{sign}\left(\alpha^m \det \begin{pmatrix} \partial_1\gamma^1(p,q) & \partial_2\gamma^1(p,q) \\ \partial_1\gamma^2(p,q) & \partial_2\gamma^2(p,q) \end{pmatrix}\right) \\
&= \sum_{(p,q)\in\gamma^{-1}(0)} (\operatorname{sign}\alpha)^m \operatorname{sign}\det \gamma'(p,q) = (\operatorname{sign}\alpha)^m \deg(\gamma, U).
\end{aligned}$$

This proves the assertion. □

Remark 2.2. *Let $\alpha > 0$ be given, and let γ and γ_α be as in Lemma 2.1. Let U be a relatively compact open subset of $M \times N$ and assume that $\gamma^2(p,q) \neq 0$ for all $(p,q) \in \partial U$. Then,*

$$(p,q,\lambda) \mapsto \big(\lambda\alpha\gamma^1(p,q), \gamma^2(p,q)\big), \qquad \lambda \in [0,1]$$

defines an admissible homotopy between γ_α and the tangent vector field γ_0 given by $(p,q) \mapsto \big(0, \gamma^2(p,q)\big)$, thus $\deg(\gamma, U) = \deg(\gamma_0, U)$. In particular, when the dimension of M is odd, from Lemma 2.1 it follows

$$-\deg(\gamma, U) = \deg(\gamma_{-\alpha}, U) = \deg(\gamma_0, U) = \deg(\gamma_\alpha, U) = \deg(\gamma, U).$$

That implies $\deg(\gamma, U) = 0$.

One can also prove the following fact that is often useful: Let $U_1 \subseteq M$ and $U_2 \subseteq N$ be open sets. Take $U = U_1 \times U_2$, and consider a tangent vector field on $M \times N$ of the form $w(p,q) = \big(w^1(p), w^2(q)\big)$, with $(p,q) \in M \times N$. If (w^1, U_1) and (w^2, U_2) are admissible for the degree of tangent vector fields, then so is (w, U) and we have

$$\deg(w, U) = \deg(w^1, U_1)\, \deg(w^2, U_2). \tag{2.4}$$

Let $v = (v^1, v^2)$ be C^1 and tangent to $M \times N$, for $(p,q) \in M \times N$, and take $t \in \mathbb{R}$. Let $\Phi_t(p,q)$ be the value at t (if defined) of the maximal solution of

$$\begin{cases} \dot{x} = v^1(x,y), \\ \dot{y} = v^2(x,y), \end{cases}$$

from (p,q) at time $t = 0$. We shall also use the (more cumbersome) notation $\Phi_t^v(p,q)$ whenever it will be necessary to emphasize the dependence on v. The map $(p,q) \mapsto \Phi_t(p,q)$, when (and where) defined, is called *flow operator at time t (associated with v)*. Obviously, if M and N are compact, $\Phi_t(p,q)$ is defined for all $(t,p,q) \in \mathbb{R} \times M \times N$. Moreover, given a relatively compact subset U of $M \times N$, $\Phi_t(p,q)$ is defined for any $(p,q) \in \overline{U}$ and $|t|$ small enough. In fact, continuity with respect to initial data implies that the domain of the map $(t,p,q) \mapsto \Phi_t(p,q)$ is an open subset of $\mathbb{R} \times M \times N$ containing the section $\{0\} \times M \times N$.

We shall need the following result (see [4] or [3, Thm. 3.8]).

Theorem 2.3. *Let $v = (v^1, v^2)\colon M \times N \to \mathbb{R}^{k+s}$ be a C^1 tangent vector field on $M \times N \subseteq \mathbb{R}^{k+s}$ and let U be a relatively compact open subset of $M \times N$. Let $T > 0$*

and assume that, for any $(p,q) \in \overline{U}$*, the (maximal) solution of the Cauchy problem*

$$\begin{cases} \dot{x} = v^1(x,y) \\ \dot{y} = v^2(x,y) \\ x(0) = p \\ y(0) = q \end{cases}$$

is defined on $[0,T]$*. If* Φ_T^v *is fixed point free on* ∂U*, then* $\operatorname{ind}(\Phi_T^v, U) = \deg(-v, U)$*.*

3. Computation of the fixed point index

In order to avoid excessive repetition, from now on, T will be a given positive number. Also, unless differently stated, $f : \mathbb{R} \times M \times N \to \mathbb{R}^k$ and $h : \mathbb{R} \times M \times N \to \mathbb{R}^s$ will be continuous T-periodic vector fields tangent to M and N respectively; and $g : M \times N \to \mathbb{R}^s$ will be continuous and tangent to N. We will always set

$$\begin{aligned} w(p,q) =& \frac{1}{T}\int_0^T f(t,p,q)\,\mathrm{d}t, \\ \nu(p,q) =& \big(w(p,q), g(p,q)\big), \end{aligned}$$

and, for $\lambda \geq 0$,

$$\nu_\lambda(p,q) = \big(\lambda w(p,q), g(p,q)\big).$$

Assume that f, g and h as above are such that the initial value problem

$$\begin{cases} \dot{x} = \lambda f(t,x,y) \\ \dot{y} = g(x,y) + \lambda h(t,x,y) \\ x(0) = p \\ y(0) = q \end{cases} \tag{3.1}$$

admits unique a solution for $\lambda \geq 0$. In this case, we shall denote by $P_t^\lambda(p,q)$ the value of the solution of (3.1) at time t (when defined). That is, P_t^λ is the so-called Poincaré t-translation operator associated to (3.1).

As for the flow operator Φ_t discussed in the previous section, known properties of differential equations imply that, for any given $\lambda \geq 0$, the domain of the map $(\lambda, t, p, q) \mapsto P_t^\lambda(p,q)$ is an open subset of $[0,\infty) \times \mathbb{R} \times M \times N$ containing $\{0\} \times \{0\} \times M \times N$.

Theorem 3.1. *Assume that the vector fields* f*,* g *and* h *as above, are of class* C^1*. Let* U *be a relatively compact open subset of* $M \times N$ *and assume that for any* $(p,q) \in \overline{U}$ *the (maximal) solution of the Cauchy problem (in* $M \times N$*)*

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = g(x,y) \\ x(0) = p \\ y(0) = q \end{cases} \tag{3.2}$$

is defined on $[0,T]$*. Assume also that the vector field* ν *is admissible for the degree in* U *and that (3.2) has no nontrivial* T*-periodic solutions for* $(p,q) \in \partial U$*. Then, there exists* $\lambda_0 > 0$ *such that, for* $0 < \lambda \leq \lambda_0$*,* P_T^λ *is defined on* $\overline{U}$*, is fixed point free on* ∂U*, and*

$$\operatorname{ind}(P_T^\lambda, U) = \deg(-\nu, U). \tag{3.3}$$

Proof. Consider the following equation in $M \times N$

$$\begin{cases} \dot{x} = \lambda\big(\mu f(t,x,y) + (1-\mu) w(x,y)\big), \\ \dot{y} = g(x,y) + \lambda\mu h(t,x,y), \end{cases} \qquad \lambda \geq 0,\ \mu \in [0,1]. \tag{3.4}$$

Denote by $H_T = (H_T^1, H_T^2)$ the translation operator that associates to any (λ, p, q, μ) the value at time T (if defined) of the solution of (3.4) starting from (p,q) at time

0. Due to the compactness of $[0,1]\times\overline{U}$, for $\lambda\geq 0$ small enough, $H_T(\lambda,p,q,\mu)$ is defined for $(p,q)\in\overline{U}$ and $\mu\in[0,1]$.

We want to show that when λ is sufficiently small there are no fixed points of H_T on ∂U. Actually we claim a slightly stronger statement: that there exists $\lambda_0>0$ such that $H_T^1(\lambda,p,q,\mu)\neq p$ for $0<\lambda\leq\lambda_0$, $(p,q)\in\partial U$ and $\mu\in[0,1]$. Assume this is not the case. Thus, there exist sequences $\lambda_i\searrow 0$, $\mu_i\in[0,1]$, $(p_i,q_i)\in\partial U$ such that $H_T^1(\lambda_i,p_i,q_i,\mu_i)=p_i$. Since H_T^1 takes values in $M\subseteq\mathbb{R}^k$, this relation can be written as

$$
\begin{aligned}
0 &= H_T^1(\lambda_i,p_i,q_i,\mu_i)-p_i\\
&= \lambda_i\int_0^T\Big(\mu_i f\big(t,x_i(t),y_i(t)\big)+(1-\mu_i)w\big(x_i(t),y_i(t)\big)\Big)\,\mathrm{d}t,
\end{aligned}
\tag{3.5}
$$

where (x_i,y_i) denotes the solution of

$$
\begin{cases}
\dot{x}=\lambda_i\big(\mu_i f(t,x,y)+(1-\mu_i)w(x,y)\big),\\
\dot{y}=g(x,y)+\lambda_i\mu_i h(t,x,y),\\
x(0)=p_i,\\
y(0)=q_i.
\end{cases}
$$

Since $\lambda_i>0$, (3.5) implies

$$
0=\int_0^T\Big(\mu_i f\big(t,x_i(t),y_i(t)\big)+(1-\mu_i)w\big(x_i(t),y_i(t)\big)\Big)\,\mathrm{d}t. \tag{3.6}
$$

Without loss of generality we can assume $\mu_i\to\mu_0\in[0,1]$ and $(p_i,q_i)\to(p_0,q_0)\in\partial U$. Thus, by continuous dependence on initial data, $\big(x_i(t),y_i(t)\big)$ converges uniformly on $[0,T]$ to a solution $\big(x_0(t),y_0(t)\big)$ of (3.2). Since there are no nontrivial T-periodic solutions of (3.2) starting from the boundary of U, one has $\big(x_0(t),y_0(t)\big)\equiv(p_0,q_0)$ and, necessarily,

$$
g(p_0,q_0)=0. \tag{3.7}
$$

Passing to the limit in (3.6), we get

$$
0=\int_0^T\big(\mu_0 f(t,p_0,q_0)+(1-\mu_0)w(p_0,q_0)\big)\,\mathrm{d}t=T\,w(p_0,q_0).
$$

This, along with (3.7), contradicts the assumption, hence proving the claim.

Thus, there exists $\lambda_0>0$ such that, when $0<\lambda\leq\lambda_0$, the map

$$
H_T(\lambda,\cdot,\cdot,\cdot)\colon\overline{U}\times[0,1]\to M\times N
$$

given by $(p,q,\mu)\mapsto H_T(\lambda,p,q,\mu)$ is an admissible homotopy. The homotopy invariance property of the fixed point index shows that for such λ's one has

$$
\operatorname{ind}(\Phi_T^{\nu_\lambda},U)=\operatorname{ind}(P_T^\lambda,U), \tag{3.8}
$$

where $\Phi_T^{\nu_\lambda}$ is the flow operator associated, when $\mu=0$, with (3.4). By Theorem 2.3 and Lemma 2.1, one has

$$
\operatorname{ind}\big(\Phi_T^{\nu_\lambda},U\big)=\deg(-\nu_\lambda,U)=\deg(-\nu,U). \tag{3.9}
$$

Thus (3.3) follows from equations (3.8) and (3.9). □

Remark 3.2. *Let f, g, h, ν and U be as above. If the assumption in Theorem 3.1 that* (3.2) *admits no nontrivial T-periodic solutions starting from ∂U is strenghtened by requiring that no T-periodic orbits at all of* (3.2) *meets ∂U, then* (3.3) *remains valid also for $\lambda=0$. In fact, For $\lambda=0$, P_T^0 coincides with $\Phi_T^{\nu_0}$, the flow operator at time T induced by $\nu_0(p,q)=\big(0,g(p,q)\big)$. If* (3.2) *has no T-periodic solution for $(p,q)\in\partial U$, then $\Phi_T^{\nu_0}$ is admissible (for the fixed point index) in U. Moreover, since* *necessarily* $g(p,q) \neq 0$ *for all* $(p,q) \in \partial U$*,* ν_0 *is admissible for the degree. From Theorem 2.3 and Remark 2.2 it follows that*

$$\operatorname{ind}(P_T^0, U) = \deg(-\nu_0, U) = \deg(-\nu, U).$$

In particular, by Remark 2.2 we get that if the dimension of M *is odd, then*

$$\operatorname{ind}(P_T^\lambda, U) = 0$$

for $0 \leq \lambda \leq \lambda_0$*,* λ_0 *being as in Theorem 3.1.*

Remark 3.3. *Let* f*,* g*, and* h *be as above, and let* $V \subseteq N$ *be open. Consider the tangent vector field to* $M \times N$*, given by* $\nu_0(p,q) = \big(0, g(p,q)\big)$*. Assume that* M *is compact and that* ν_0 *is admissible for the degree of tangent vector fields on* $U = M \times V$*. Using the homotopy property one can easily see that* $\deg(\nu, U) = \deg(\nu_0, U)$*. Thus, for* M *compact and for this particular choice of* U*, the conclusion of Theorem 3.1 can be deduced also from* [3, Thm. 3.15]*. When* M *is not compact, however, this argument does not apply.*

Let $M = \{p\}$. Taking into account the relations (2.1) and (2.4), and arguing as in Remark 3.2, one can easily deduce the following consequence of Theorem 3.1.

Corollary 3.4 ([4, 3])**.** *Let* $h\colon \mathbb{R} \times N \to \mathbb{R}^s$ *and* $g\colon N \to \mathbb{R}^s$ *be* C^1 *tangent vector fields on* N*, and let* h *be* T*-periodic in the first variable. Let* U *be a relatively compact open subset of* N *such that, for any* $q \in \overline{U}$*, the solution* $x(\cdot, q)$ *of* $\dot{x} = g(x)$*,* $x(0) = q$ *is defined up to* T*. Assume that* $x(T,q) \neq q$ *for all* $q \in \partial U$*. Then, there exists* $\lambda_0 > 0$ *such that the fixed point index of the Poincaré* T*-translation operator associated with equation*

$$\dot{x} = g(x) + \lambda h(t,x)$$

is well defined for $0 \leq \lambda \leq \lambda_0$*, and is given by* $\deg(-g, U)$*.*

Similarly, letting $N = \{q\}$, one gets

Corollary 3.5 ([3])**.** *Let* $f\colon \mathbb{R} \times M \to \mathbb{R}^k$ *be a* C^1 *tangent vector field that is* T*-periodic in the first variable. Consider a relatively compact open subset* U *of* M*. Assume that the tangent vector field* $w : M \to \mathbb{R}^k$*, given by* $w(p) = \frac{1}{T}\int_0^T f(t,p)\,\mathrm{d}t$ *is admissible for the degree in* U*. Then, there exists* $\lambda_0 > 0$ *such that, for* $0 < \lambda \leq \lambda_0$*, the fixed point index of the Poincaré* T*-translation operator associated with equation*

$$\dot{x} = \lambda f(t,x)$$

is defined on $\overline{U}$*, and coincides with* $\deg(-w, U)$*.*

4. Branches of periodic solutions

We shall use the result of the previous section to investigate the structure of the set of T-periodic solutions of system (1.1).

By $C_T(M)$ and $C_T(N)$ we mean the spaces of T-periodic continuous functions from $\mathbb{R}$ to M and N respectively, endowed with the topology induced by the Banach spaces $C_T(\mathbb{R}^k)$ and $C_T(\mathbb{R}^s)$ respectively. There is a natural homeomorphism between $C_T(M) \times C_T(N)$ and $C_T(M \times N)$; therefore, without further comments, we shall use $C_T(M \times N)$ and $C_T(M) \times C_T(N)$ interchangeably, and denote the elements of $C_T(M \times N)$ as pairs (x,y) with $x \in C_T(M)$ and $y \in C_T(N)$.

A triple $(\lambda, p, q) \in [0, \infty) \times M \times N$ is a *starting point* (of T-periodic solutions) if the Cauchy problem (3.1) has a T-periodic solution. A starting point (λ, p, q) is *trivial* if $\lambda = 0$ and $g(p,q) = 0$.

Although the concept of starting point is essentially finite-dimensional, there is an infinite-dimensional notion strictly related to it: that of T*-triple*. We say that $(\lambda, x, y) \in [0, \infty) \times C_T(M \times N)$ is a T-triple if (x,y) satisfies (3.1). If $\lambda = 0$ and (x,y)

is constant, then (λ, x, y) is said to be *trivial*. Notice that if $(0, y)$ is a nonconstant T-periodic solution of (3.2) then $(0, 0, y)$ is a nontrivial T-triple.

Remark 4.1. *Observe that our notion of T-triple actually coincides with that of T-pair in* [3, 5] *when* (1.1) *is considered as a single differential equation on the manifold $M \times N$. In this paper, however, we wish to look at* (1.1) *from the viewpoint of coupled equations.*

Denote by $X \subseteq [0, \infty) \times C_T(M \times N)$ the set of the T-triples of (1.1) and by $S \subseteq [0, \infty) \times M \times N$ the set of the starting points. Note that, as a closed subset of a locally complete space, X is locally complete. One can show that, no matter whether or not $M \times N$ is closed in $\mathbb{R}^{k+s}$, X is always locally compact. Moreover, by the Ascoli-Arzelà Theorem, when $M \times N$ is closed in $\mathbb{R}^{k+s}$, any bounded closed set of T-triples is compact.

Remark 4.2. The map $\sigma : X \to S$ given by $(\lambda, x, y) \mapsto \big(\lambda, x(0), y(0)\big)$ is continuous and onto. Notice that, if (λ, x, y) is trivial, then $\big(\lambda, x(0), y(0)\big)$ is a trivial starting point.

In case f, g and h are C^1, σ is also one to one. Furthermore, by the continuous dependence on initial data, we get the continuity of $\sigma^{-1} : S \to X$. Clearly, trivial T-triples correspond to trivial starting points under this homeomorphism.

As in [5], we tacitly assume some natural identifications. That is, we will regard every space as its image in the following diagram of closed embeddings:

$$\begin{array}{ccc} [0,\infty) \times M \times N & \longrightarrow & [0,\infty) \times C_T(M \times N) \\ \Big\uparrow & & \Big\uparrow \\ M \times N & \longrightarrow & C_T(M \times N) \end{array} \tag{4.1}$$

where the horizontal arrows are defined by regarding any (p, q) in $M \times N$ as the pair of constant maps $\big(\hat{p}(t), \hat{q}(t)\big) \equiv (p, q)$ in $C_T(M \times N)$, and the two vertical arrows are the natural identifications $(p, q) \mapsto (0, p, q)$ and $(x, y) \mapsto (0, x, y)$.

According to these embeddings, if Ω is an open subset of $[0, \infty) \times C_T(M \times N)$, by $\Omega \cap (M \times N)$ we mean the open subset of $M \times N$ given by all $(p, q) \in M \times N$ such that the triple $(0, p, q)$ belongs to Ω. If U is an open subset of $[0, \infty) \times M \times N$, then $U \cap (M \times N)$ represents the open set $\{(p, q) \in M \times N : (0, p, q) \in U\}$.

Let ν be the vector field defined in the previous section. Observe that any $(p, q) \in \nu^{-1}(0)$ can be seen –in the sense specified above– as a T-periodic solution of the unperturbed equation (1.1).

We will need the following global connectivity result.

Lemma 4.3 ([1])**.** *Let Y be a locally compact metric space and let $K \subseteq Y$ be nonempty and compact. Assume that any compact subset of Y containing K has nonempty boundary. Then $Y \setminus K$ contains a connected set whose closure intersects K and is not compact.*

By continuous dependence, the set $D \subseteq [0, \infty) \times M \times N$ given by

$$D = \left\{ (\lambda, p, q) : \begin{array}{l} \text{the solution } \big(x(\cdot), y(\cdot)\big) \text{ of} \\ (3.1) \text{ is defined on } [0, T] \end{array} \right\}, \tag{4.2}$$

is open. Thus it is locally compact. Clearly D contains the set S of all starting points of (1.1). Observe that S is closed in D, though not necessarily closed in $[0, \infty) \times M \times N$. Therefore it is locally compact.

As in the previous section, let ν and ν_0 be the tangent vector fields on $M \times N$ given by $\nu(p, q) = \big(w(p, q), g(p, q)\big)$ and $\nu_0(p, q) = \big(0, g(p, q)\big)$, respectively. Let

$$\mathfrak{S} = S \setminus \big(\{0\} \times \nu_0^{-1}(0)\big)$$

be the set of all the nontrivial starting points, and define

$$\mathfrak{S}_\nu = \mathfrak{S} \cup \big(\{0\} \times \nu^{-1}(0)\big).$$

We will need the following fact:

Lemma 4.4. *Assume the vector fields f, g and h are C^1. Then, the set $\mathfrak{S}_\nu$ is closed in S and, therefore, it is locally compact.*

Proof. It is enough to show that given a sequence $\{(\lambda_i, p_i, q_i)\}_{i\in\mathbb{N}} \subseteq \mathfrak{S}$ with $\lambda_i \searrow 0$ and $(p_i, q_i) \to (p_0, q_0) \in g^{-1}(0)$ one necessarily has $(p_0, q_0) \in \nu^{-1}(0)$.

Since $x_i(t) \in M \subseteq \mathbb{R}^k$ for any $i \in \mathbb{N}$ and $t \in [0, T]$, one has

$$0 = x_i(T) - x_i(0) = \lambda_i \int_0^T f\big(t, x_i(t), y_i(t)\big)\,\mathrm{d}t, \tag{4.3}$$

where (x_i, y_i) is the (T-periodic) solution of

$$\begin{cases} \dot{x} = \lambda_i f(t, x, y), \\ \dot{y} = g(x, y) + \lambda_i h(t, x, y), \\ x(0) = p_i, \\ y(0) = q_i. \end{cases}$$

By continuos dependence (x_i, y_i) converge uniformly on $[0, T]$ to the constant solution

$$\big(x_0(t), y_0(t)\big) \equiv (p_0, q_0).$$

Since $\lambda_i \neq 0$, by (4.3) one has that

$$0 = \int_0^T f\big(t, x_i(t), y_i(t)\big)\,\mathrm{d}t.$$

Passing to the limit, one gets $\int_0^T f(t, p_0, q_0)\,\mathrm{d}t = 0$ and, being $g(p_0, q_0) = 0$, we have $(p_0, q_0) \in \nu^{-1}(0)$. $\square$

In the sequel, given any subset A of $[0, \infty) \times M \times N$ and $\lambda \geq 0$, the symbol A_λ denotes the slice $\{(x, y) \in M \times N : (\lambda, x, y) \in A\}$ of A.

Theorem 4.5. *Assume that the vector fields f, g and h are C^1. Let W be an open subset of D such that $\nu^{-1}(0) \cap W_0$ is compact. If $\deg\big(\nu, W_0\big) \neq 0$, then the set $\mathfrak{S} \cap W$ of the nontrivial starting points in W admits a connected subset whose closure in W meets $\{0\} \times \big(\nu^{-1}(0) \cap W_0\big)$ and is not compact.*

Proof. Since $\nu^{-1}(0) \cap W_0$ is compact, one can prove that if $\mathfrak{S} \cap W$ does not contain a connected subset as in the assertion, then the same is true for $\mathfrak{S}_\nu \cap W$. Thus, it is sufficient to prove that there is a connected subset of starting points in $\mathfrak{S}_\nu \cap W$ whose closure in W intersects $\{0\} \times \big(\nu^{-1}(0) \cap W_0\big)$ and is not compact. Therefore, to prove the assertion it is enough to show that the pair

$$(Y, K) = \Big(\mathfrak{S}_\nu \cap W, \{0\} \times \big(\nu^{-1}(0) \cap W_0\big)\Big)$$

satisfies the assumptions of Lemma 4.3.

The set $\mathfrak{S}_\nu \cap W$ is open in $\mathfrak{S}_\nu$ that, by Lemma 4.4, is locally compact. Thus $\mathfrak{S}_\nu \cap W$ is locally compact as well. Moreover, as $\deg(\nu, W_0)$ is nonzero, the compact set K is nonempty. Assume, by contradiction, that there exists a compact subset C of $\mathfrak{S}_\nu \cap W$ containing K and with empty boundary in the space $\mathfrak{S}_\nu \cap W$. Thus C is open in $\mathfrak{S}_\nu \cap W$ (in fact it is clopen). As W is open in $[0, \infty) \times M \times N$, C is actually open as a subspace of $\mathfrak{S}_\nu$. Thus there exists an open subset A of $[0, \infty) \times M \times N$ such that $\mathfrak{S}_\nu \cap A = C$ and $A \subseteq W$. Because of the compactness of the slice C_0 of C, we may choose A is such a way that the neighborhood A_0 of C_0 turns out to be relatively compact in $M \times N$. Moreover, without loss of generality, we may assume that the boundary of A_0 in $M\times N$ does not contain neither points of $\nu^{-1}(0)\cap W_0$ (i.e. fixed points of Φ_T^ν) nor elements of $\mathfrak{S}$. Similarly, C being compact, it is not restrictive to assume that there exists $\varepsilon>0$ such that $A_\lambda=A_\varepsilon$, for $0\leq\lambda\leq\varepsilon$.

Applying Theorem 2.3, formula (2.2) and the excision property of the degree, one gets

$$\begin{aligned}\operatorname{ind}(\Phi_T^\nu,A_0) &= \deg(-\nu,A_0)=(-1)^{m+n}\deg(\nu,A_0)\\ &= (-1)^{m+n}\deg(\nu,W_0)\neq 0,\end{aligned}\tag{4.4}$$

where m and n are the dimensions of M and N, respectively. Since no T-periodic solution of (3.2) meets the boundary of A_0, by Theorem 3.1 there exists $\lambda_0>0$ such that for any $0<\lambda\leq\lambda_0$,

$$\operatorname{ind}(P_T^\lambda,A_0)=\operatorname{ind}(\Phi_T^\nu,A_0)\neq 0.\tag{4.5}$$

Since C is compact, there exists $\delta>0$ such that the Poincaré operator P_T^δ is fixed point free on the slice A_δ. Then, from the generalized homotopy property and the solution property of the index, we get

$$\operatorname{ind}\big(P_T^\lambda,A_\lambda\big)=\operatorname{ind}\big(P_T^\delta,A_\delta\big)=0,$$

for $\lambda\in(0,\delta]$. In particular,

$$\operatorname{ind}\big(P_T^\lambda,A_0\big)=\operatorname{ind}\big(P_T^\lambda,A_\lambda\big)=0,$$

when $0<\lambda\leq\min\{\varepsilon,\lambda_0,\delta\}$. This contradicts equation (4.5). □

Similarly to the discussion in Remark 3.3, one can observe that when M is compact, an important particular case of Theorem 4.5 could be deduced also from [4, Thm. 3.1] (or [3, Thm. 4.2]).

Remark 4.6. *By the same argument of the Corollaries 3.4 and 3.5, and taking into account the definition of trivial starting point, one can check that Theorem 4.5 implies both* [1, Thm. 2.1] *and* [4, Thm. 3.1] *(respectively, theorems 4.4 and 4.2 in* [3]*) that apply only to the case of a single differential equation of the form* (1.2) *and* (1.3) *respectively.*

We are now in a position to state and prove our main result concerning the "branches" of T-triples.

Theorem 4.7. *Let Ω be an open subset of $[0,\infty)\times C_T(M\times N)$, and assume that $\deg\big(\nu,\Omega\cap(M\times N)\big)$ is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-triples in Ω whose closure in $[0,\infty)\times C_T(M\times N)$ meets $\nu^{-1}(0)\cap\Omega$ and is not contained in any compact subset of Ω. In particular, if $M\times N$ is closed in $\mathbb{R}^{k+s}$ and $\Omega=[0,\infty)\times C_T(M\times N)$, then Γ is unbounded.*

Proof. Let X denote the set of T-triples of (1.1). Since X is closed, it is enough to show that there exists a connected set Γ of nontrivial T-triples in Ω whose closure in $X\cap\Omega$ meets $\nu^{-1}(0)$ and is not compact.

Assume first that f, g and h are smooth. As above, denote by S the set of all starting points of (1.1), and take

$$\tilde S=\big\{(\lambda,p,q)\in S:\text{the solution of (3.1) belongs to }\Omega\big\}.$$

Obviously $\tilde S$ is an open subset of S, thus we can find an open subset U of the set D defined in (4.2) such that $S\cap U=\tilde S$. Since clearly $\{0\}\times\nu^{-1}(0)\subseteq S$, we have that

$$\nu^{-1}(0)\cap\Omega=\nu^{-1}(0)\cap\tilde S=\nu^{-1}(0)\cap U_0$$

(according to the embedding diagram (4.1) the set $\nu^{-1}(0) \subseteq M \times N$ is identified with $\{0\} \times \nu^{-1}(0) \subseteq \mathbb{R} \times M \times N$, and the intersections of $\nu^{-1}(0)$ with Ω and with $\tilde{S}$ are all seen as subsets of $M \times N$). Thus,

$$\deg\big(\nu, U_0\big) = \deg\big(\nu, \Omega \cap (M \times N)\big) \neq 0.$$

Applying Theorem 4.5, we get the existence of a connected set of nontrivial starting points $\Sigma \subseteq \big(S \cap U\big) \setminus \big(\{0\} \times \nu^{-1}(0)\big)$ such that its closure in $S \cap U$ is not compact and meets $\nu^{-1}(0) \cap U_0$. Let $\sigma : X \to S$ be the map which assigns to any T-triple (λ, x, y) the starting point $\big(\lambda, x(0), y(0)\big)$. By Remark 4.2, σ is a homeomorphism and trivial T-triples correspond to trivial starting points under σ. This implies that $\Gamma = \sigma^{-1}(\Sigma)$ satisfies the requirements.

Let us remove the smoothness assumption on f, g and h. Put $K = \nu^{-1}(0) \cap \Omega$ and $Y = X \cap \Omega$. We have only to prove that the pair (Y, K) satisfies the hypothesis of Lemma 4.3. Assume the contrary. We can find a relatively open compact subset C of Y containing K. Thus there exists an open subset W of Ω such that its closure $\overline{W}$ in $[0, \infty) \times C_T(M \times N)$ is contained in Ω, $W \cap Y = C$ and $\partial W \cap Y = \emptyset$. Since C is compact and $[0, \infty) \times M \times N$ is locally compact, we can choose W in such a way that the set

$$\Big\{\big(\lambda, x(t), y(t)\big) \in [0, \infty) \times M \times N : (\lambda, x, y) \in W,\ t \in [0, T]\Big\}$$

is contained in a compact subset Z of $[0, \infty) \times M \times N$. This implies that W is bounded with complete closure in Ω and $W \cap (M \times N)$ is a relatively compact subset of $\Omega \cap (M \times N)$. In particular ν is nonzero on the boundary of $W \cap (M \times N)$ (relative to $M \times N$). By known approximation results, there exist sequences $\{g_i\}$ and $\{f_i\}$ of smooth tangent vector fields uniformly approximating g and f on $M \times N$ and on $\mathbb{R} \times M \times N$ respectively, with f_i of period T in the first variable. Put $w_i(p, q) = \frac{1}{T} \int_0^T f_i(t, p, q)\, \mathrm{d}t$ and $\nu_i(p, q) = \big(w_i(p, q), g_i(p, q)\big)$. For $i \in \mathbb{N}$ large enough, we get

$$\deg\big(\nu_i, W \cap (M \times N)\big) = \deg\big(\nu, W \cap (M \times N)\big).$$

Furthermore, by excision,

$$\deg\big(\nu, W \cap (M \times N)\big) = \deg\big(\nu, \Omega \cap (M \times N)\big) \neq 0.$$

Therefore, given i large enough, the first part of the proof can be applied to the equation

$$\left\{\begin{array}{l} \dot{x} = \lambda f_i(t, x, y) \\ \dot{y} = g_i(x, y) + \lambda h_i(t, x, y), \end{array}\right. \tag{4.6}$$

where $\{h_i\}$ is a sequence of T-periodic vector fields on $M \times N$ tangent to N and uniformly approximating h on compact sets.

Let X_i denote the set of T-triples of (4.6). There exists a connected subset Γ_i of $\Omega \cap X_i$ whose closure in Ω meets $\nu_i^{-1}(0) \cap W$ and is not contained in any compact subset of Ω. Let us prove that, for i large enough, $\Gamma_i \cap \partial W \neq \emptyset$. It is sufficient to show that $X_i \cap \overline{W}$ is compact. In fact, if $(\lambda, x, y) \in X_i \cap \overline{W}$ we have, for any $t \in [0, T]$,

$$\big|\big(\dot{x}(t), \dot{y}(t)\big)\big|_{k+s}^2 \leq \max_{\substack{(\mu, p, q) \in Z \\ \tau \in [0, T]}} \big\{|g_i(p, q) + \mu h_i(\tau, p, q)|_s^2 + |\mu f_i(\tau, p, q)|_k^2\big\},$$

where $|\cdot|_k$, $|\cdot|_s$ and $|\cdot|_{k+s}$ denote the usual norms in $\mathbb{R}^k$, $\mathbb{R}^s$ and $\mathbb{R}^{k+s}$, respectively. Hence, by Ascoli-Arzelà theorem, $X_i \cap \overline{W}$ is totally bounded and, consequently, compact since X_i is closed and $\overline{W}$ is complete. Thus, for i large enough, there exists a T-triple $(\lambda_i, x_i, y_i) \in \Gamma_i \cap \partial W$ of (4.6). Again by Ascoli-Arzelà theorem, we may assume that $(x_i, y_i) \to (x_0, y_0)$ in $C_T(M \times N)$, and $\lambda_i \to \lambda_0$ with $(\lambda_0, x_0, y_0) \in \partial W$. Therefore

$$\begin{cases} \dot{x}_0(t) = \lambda f\big(t, x_0(t), y_0(t)\big) \\ \dot{y}_0(t) = g\big(x_0(t), y_0(t)\big) + \lambda h\big(t, x_0(t), y_0(t)\big), \end{cases}$$

Hence (λ_0, x_0, y_0) is a T-pair in ∂W. This contradicts the assumption $\partial W \cap Y = \emptyset$.

It remains to prove the last assertion. Let $M \times N \subseteq \mathbb{R}^{k+s}$ be closed. There exists a connected set Γ of nontrivial T-triples of (1.1) whose closure is not compact and meets $\{0\} \times \nu^{-1}(0)$. We need to show that Γ is unbounded. Assume the contrary. As we already observed, when $M \times N$ is closed any bounded closed set of T-triples is compact. Thus the closure of Γ in $[0, \infty) \times C_T(M \times N)$ is compact. This yields a contradiction. □

Observe that the connected set Γ of Theorem 4.7 might be completely contained in the slice $\{0\} \times C_T(M \times N)$ as in the following example where $M = \mathbb{R}$, $N = \mathbb{R}^2$ and $T = 2\pi$:

$$\begin{cases} \dot{x} = \lambda(x + \sin t), \\ \dot{y} = z, \\ \dot{z} = -y + \lambda \sin t, \end{cases} \qquad \lambda \geq 0.$$

Similarly to the discussion in Remark 3.3, one can observe that when M is compact, an important particular case of Theorem 4.7 follows from [5, Thm. 3.3].

Remark 4.8. *With the same argument of the Corollaries 3.4 and 3.5 and taking into account the definition of trivial T-triple, one can deduce from Theorem 4.7 the results* [2, Thm 2.3] *and* [5, Thm. 3.3] *that apply only to the case of a single differential equation of the form* (1.2) *and* (1.3) *respectively.*

5. An application

In this section we will consider the very simple situation of a single second order scalar differential equation and obtain a result that cannot be directly deduced from those exposed, e.g., in [3] and references therein. Although fairly more general situations could be treated, our sole purpose here is to illustrate Theorem 4.7.

Consider the following second order differential equation in $\mathbb{R}$:

$$\ddot{\eta} = -g(\eta)\dot{\eta} + \lambda f(t, \eta), \quad \lambda \geq 0, \tag{5.1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are C^1 functions and f is T-periodic in t.

A pair $(\lambda, x) \in [0, \infty) \times C^1(\mathbb{R})$ is called a *T-pair* for (5.1) if x is a T-periodic solution of (5.1). A T-pair (λ, x) is *trivial* if x is constant and $\lambda = 0$. Clearly, by identifying any $q \in \mathbb{R}$ with the constant function $\hat{q}(t) \equiv 0$ we can regard $[0, \infty) \times \mathbb{R}$ as a subset of $[0, \infty) \times C^1(\mathbb{R})$. Similarly, we can identify $\mathbb{R}$ with $\{0\} \times \mathbb{R} \subseteq [0, \infty) \times \mathbb{R}$. In this way, given an open subset Ω of $[0, \infty) \times C^1(\mathbb{R})$, by $\Omega \cap \mathbb{R}$ we mean those constants $q \in \mathbb{R}$ such that $(0, q) \in \Omega$. Observe also that $[0, \infty) \times C^1(\mathbb{R})$ can be regarded as a subset of $[0, \infty) \times C_T(\mathbb{R} \times \mathbb{R})$ by means of the injection $j : (\lambda, x) \mapsto (\lambda, x, \dot{x})$.

Introducing a new variable ξ, Equation (5.1) can be equivalently rewritten in $\mathbb{R}^2$ (as in the so-called Liénard plane technique) as follows:

$$\begin{cases} \dot{\xi} = \lambda f(t, \eta), \\ \dot{\eta} = \xi - G(\eta), \end{cases} \qquad \lambda \geq 0, \tag{5.2}$$

where G is a primitive of g. Take $M = N = \mathbb{R}$ and let $\Omega \subseteq [0, \infty) \times C^1_T(\mathbb{R})$ be open. Consider an open subset $\hat{\Omega}$ of $[0, \infty) \times C_T(\mathbb{R} \times \mathbb{R})$ such that $\hat{\Omega} \cap j\big([0, \infty) \times C^1_T(\mathbb{R})\big) = j(\Omega)$. Assume that the vector field ν in $\mathbb{R}^2$, given by

$$(p, q) \mapsto \big(w(q)\ ,\ p - G(q)\big),$$

where $w(q) = \frac{1}{T}\int_0^T f(t,q)\,\mathrm{d}t$, is admissible for the degree in $\hat\Omega \cap \mathbb{R}^2$ (recall the disussion following the inclusion diagram (4.1)), and that $\deg(\nu, \hat\Omega\cap\mathbb{R}^2) \neq 0$. Then, by Theorem 4.7, there exists a connected set Γ of nontrivial T-triples for (5.2) whose closure meets

$$\{0\} \times \big(\hat\Omega \cap \nu^{-1}(0)\big) = \Big\{(0,p,q) \in [0,\infty) \times (\mathbb{R}^2 \cap \hat\Omega) : w(q) = 0,\ p = G(q)\Big\}$$

and is not compact.

Observe that to any $(\lambda, x, y) \in \Gamma$ one can associate the (nontrivial) T-pair (λ, y) for (5.1). In this way, one gets a connected set Ξ of nontrivial T-pairs for (5.1) whose closure meets the set $w^{-1}(0) \cap \Omega$ and is not compact. One can easily prove that for $\lambda = 0$ all the periodic solutions of (5.1) are constant. Consequently, all the T-pairs for (5.1) of the form $(0, x)$, $x \in C^1_T(\mathbb{R})$, are trivial. Thus, being noncompact, the closure of Ξ cannot be confined to $\{0\} \times C^1_T(\mathbb{R})$. In other words, for small values of λ equation (5.1) has necessarily a T-periodic solution.

In summary, we have proved the following fact:

Theorem 5.1. *Let f, Ω, w and ν be as above. Assume that there is $\hat\Omega$ as above, such that $\deg(\nu, \hat\Omega\cap\mathbb{R}^2)$ is well-defined and nonzero. Then, there exists a connected set of nontrivial T-pairs for (5.1) whose closure meets the set $w^{-1}(0)\cap\Omega$ and is not compact. Moreover, there exists $\lambda_0 > 0$ such that equation (5.1) has a T-periodic solution for $0 \leq \lambda \leq \lambda_0$.*

Finally, we illustrate Theorem 5.1 with an elementary example:

Example 5.2. *Take $T = 2\pi$ and consider the equation*

$$\ddot\eta = -\dot\eta + \lambda(\eta + \sin t) \tag{5.3}$$

Since this equation is linear with constant coefficient, we could write explicitly the 2π-periodic solution for any $\lambda \geq 0$. We wish, however, to demonstrate the usage of Theorem 5.1.

Let $\Omega = [0,\infty) \times C^1_T(\mathbb{R})$ and consider $\hat\Omega = [0,\infty) \times C_T(\mathbb{R}^2)$. The vector field $\nu(p,q) = (q, p-q)$ is clearly admissible in $\mathbb{R}^2$ and has degree equal to -1. Thus, by Theorem 4.7, there exists a connected set of nontrivial 2π-pairs for (5.3) whose closure meets the set $w^{-1}(0) = \{0\}$ and is not compact. In particular, for λ sufficiently small, equation (5.3) has a 2π-periodic solution. The situation is represented in Figure 1.

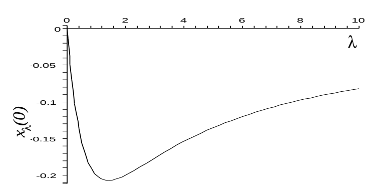

FIGURE 1. To any $\lambda \geq 0$ there corresponds a unique 2π-pair (λ, x_λ) for (5.3). Here, the initial point $x_\lambda(0)$ is plotted against λ.

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