BRANCHES OF FORCED OSCILLATIONS FOR PERIODICALLY PERTURBED SECOND ORDER AUTONOMOUS ODE'S ON MANIFOLDS

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1. INTRODUCTION

This paper, which is the natural extension of [4] to the case of second order differential equations, is devoted to investigate the structure of the set of forced oscillations of periodic perturbations of second order autonomous ODE's on differentiable manifolds. Namely, we deal with the following parametrized differential equation:

$$(1) \qquad \ddot{x}_\pi = h(x, \dot{x}) + \lambda f(t; x, \dot{x}), \ \lambda \geq 0,$$

where $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ are continuous vector fields, tangent to a differentiable manifold M embedded in some Euclidean space $\mathbf{R}^k$ and f is T-periodic in t. Here $\ddot{x}_\pi$ represents the component of the acceleration parallel to M.

This equation governs the motion of a constrained mechanical system with configuration space M, acted on by the sum of two forces: an autonomous one h and a periodic perturbation λf. This motion problem reduces to that of [3] and [2] when $h = 0$. Nevertheless, our result do not include those of [3] unless the manifold M is compact, as in [2].

We investigate the properties of the set X of T-pairs of (1); i.e. of those pairs $(\lambda, x) \in [0, \infty) \times C_T^1(M)$ with x a T-periodic solution of (1). In particular we give conditions ensuring the existence of a non-compact connected component of T-pairs emanating from the set of equilibria for $\lambda = 0$, i.e. the zeros of $h(\cdot, 0)$. In the case when M is complete, this component turns out to be unbounded. Our result, beyond its intrinsic interest, turns out to be useful in establishing some "topological" multiplicity results for forced oscillations that will be investigated in a forthcoming paper.

2. PRELIMINARIES

In what follows, if M is a differentiable manifold embedded in some $\mathbf{R}^k$, we will denote by $C_T^n(M)$, $n \in \{0, 1\}$, the metric subspace of the Banach space $\left(C_T^n(\mathbf{R}^k), \|\cdot\|_n \right)$ of all the T-periodic C^n maps $x : \mathbf{R} \to M$ with the usual C^n norm (when $n = 0$ we will simply write $C_T(M)$). Observe that $C_T^n(M)$ is not complete unless M is complete (i.e. closed in $\mathbf{R}^k$). Nevertheless, since M is locally compact, $C_T^n(M)$ is always locally complete.

Let us recall some basic facts about second order differential equations on manifolds.

Let M be a smooth submanifold of $\mathbf{R}^k$. By

$$TM = \left\{ (p, v) \in \mathbf{R}^k \times \mathbf{R}^k : p \in M , \ v \in T_p M \right\}$$

we mean the tangent bundle of M (given $p \in M$, $T_p M \subset \mathbf{R}^k$ is the tangent space to M at p).

Given an active force on M, that is a continuous map $\varphi : \mathbf{R} \times TM \to \mathbf{R}^k$ such that $\varphi(t, p, v) \in T_p M$ for all $(t; p, v) \in \mathbf{R} \times TM$, the motion equation associated with φ can be written in the form

$$(2) \qquad\qquad\qquad\qquad \ddot{x}_\pi = \varphi(t; x, \dot{x}).$$

A solution of (2) is a C^2 map $x : J \to M$, defined on an interval J, such that $\ddot{x}_\pi(t) = \varphi(t; x(t), \dot{x}(t))$ for all $t \in J$, where $\ddot{x}_\pi(t)$ denotes the orthogonal projection on $T_{x(t)} M$ of $\ddot{x}(t)$.

In what follows we deal with a parametrized second order equation of the following form:

$$(3) \qquad\qquad\qquad\qquad \ddot{x}_\pi = h(x, \dot{x}) + \lambda f(t; x, \dot{x}),$$

where $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ are assumed to be continuous maps such that $h(p, v)$ and $f(t; p, v)$ belong to $T_p M$ for any $(t; p, v) \in \mathbf{R} \times TM$ and f is T-periodic with respect to the first variable. A pair $(\lambda, x) \in [0, \infty) \times C_T^1(M)$ is a T-pair for the second order equation (3), if x is a solution of (3) corresponding to λ. In particular we will say that (λ, x) is *trivial* if $\lambda = 0$ and x is constant.

It is known that (3) can be written, in an equivalent way, as a first order equation on the tangent bundle TM in the form

$$(4) \qquad\qquad\qquad\qquad \dot{\xi} = \widehat{h}(\xi) + \lambda \bar{f}(t, \xi),$$

where $\xi = (x, y)$,

$$\begin{aligned} \widehat{h}(x, y) &= (y, r(x, y) + h(x, y)) , \\ \bar{f}(t; x, y) &= (0, f(t; x, y)) , \end{aligned}$$

and $r : TM \to \mathbf{R}^k$ is a smooth map, quadratic in the second variable $v \in T_p M$ for any $p \in M$, with values in $(T_p M)^\perp$. Such a map is strictly related to the second fundamental form on M and may be interpreted as the reactive force due to the constraint M. Actually $r(p, v)$ is the unique vector in $\mathbf{R}^k$ which makes $(v, r(p, v))$ tangent to TM at (p, v). It is well known that $\widehat{h}$, called the *second order vector field associated to* h, is a tangent vector field on TM. It is also readily verified that $\bar{f}$ is tangent to $TM \subset \mathbf{R}^{2k}$ (even if not a second order one); hence (4) is actually a first order equation on TM.

In order to simplify the statements regarding second order differential equations, it is convenient to identify any space with its image in the following commutative diagram of closed embeddings:

(5)

$$
\begin{array}{ccc}
M & \xrightarrow{\ i_1\ } & C_T^1(M) \\
\end{array}
$$

where:

$s_1 : p \mapsto (0,p)$, that is M is identified with the slice $\{0\} \times M$,

$s_2 : (p,v) \mapsto (0;p,v)$,

$s_3 : x \mapsto (0,x)$,

$s_4 : (x,y) \mapsto (0;x,y)$,

$j_1 : p \mapsto (p,0)$, i.e. M is identified with the null section of TM,

$j_2 : (\lambda,p) \mapsto (\lambda;p,0)$,

$j_3 : x \mapsto (x,\dot{x})$,

$j_4 : (\lambda,x) \mapsto (\lambda;x,\dot{x})$,

$i_1 : p \mapsto \bar{p}$, where $\bar{p}(t) \equiv p$,

$i_2 : (\lambda,p) \mapsto (\lambda,\bar{p})$,

$i_3 : (p,v) \mapsto \bar{q}$, where $\bar{q}(t) \equiv (p,v)$,

$i_4 : (\lambda;p,v) \mapsto (\lambda,\bar{q})$.

Since in the above diagram any space can be identified with its image, it makes sense to consider expressions that otherwise would be meaningless. For instance if Ω is open in $[0,\infty) \times C_T^1(M)$, $\Omega \cap M$ consists of the open subset of M made up of all the $p \in M$ such that the pair $(0,\bar{p})$ belongs to Ω. Moreover $\Omega \cap C_T^1(M)$ consists of those functions $x \in C_T^1(M)$ such that $(0,x) \in \Omega$. We observe, in particular, that it makes sense to consider the intersections of a given subset A of the "big" space $[0,\infty) \times C_T(TM)$ with any other space in the diagram. Clearly, if A is open, all these intersections turn out to be open as well.

Let us define the notion of T-pair for a general first order differential equation on a manifold M. This is important because, as shown below, there is a very strict correlation between the set of T-pairs of (3) and the corresponding set for the associated first order equation (4).

Consider the following first order differential equation on a manifold $N \subset \mathbf{R}^s$:

(6) $$\dot{x} = \gamma(x) + \lambda\psi(t,x),$$

where $\gamma : N \to \mathbf{R}^s$ and $\psi : \mathbf{R} \times N \to \mathbf{R}^s$ are (continuous) tangent vector fields on N with ψ T-periodic in t. We say that $(\lambda,x) \in [0,\infty) \times C_T(N)$ is a T-pair if x satisfies (6). If $\lambda = 0$ and x is constant, then (λ,x) is said to be *trivial*. Clearly one may have nontrivial T-pairs even with $\lambda = 0$.

Denote by Y the set of all the T-pairs of (6). Known properties of the set of solutions of differential equations imply that Y is closed, hence it is locally complete,

as a closed subset of a locally complete space. We will use the following fact from
[5].

Lemma 2.1. *The set Y is locally compact. Moreover, if N is complete, any bounded subset of Y is actually totally bounded. As a consequence, in this case, closed and bounded sets of T-pairs are compact.*

Denote by $X \subset [0, \infty) \times C_T^1(M)$ and by $\widehat{X} \subset [0, \infty) \times C_T(TM)$ the set of all the T-pairs of equations (3) and (4), respectively. The diagram (5) establishes a correspondence between the sets $\widehat{X}$ and X which "preserves" the notion of triviality for a T-pair. More precisely:

Remark 2.2. *Let $X_* \subset X$ and $\widehat{X}_* \subset \widehat{X}$ denote the sets of the trivial T-pairs of (3) and (4) respectively. The map j_4 of (5), when restricted to X, is a homeomorphism of X onto $\widehat{X}$ under which X_* corresponds to $\widehat{X}_*$. Furthermore, j_4 is a homeomorphism of $[0, \infty) \times C_T^1(M)$ onto its image; thus, as a restriction of a linear map (defined on $\mathbf{R} \times C_T^1(\mathbf{R}^k)$), it is actually a Lipschitz map with Lipschitz inverse. Consequently, under this homeomorphism, bounded sets correspond to bounded sets and totally bounded sets correspond to totally bounded sets.*

By this remark and Lemma 2.1, we get the local compactness of X. Moreover we have the following useful property of the set X.

Remark 2.3. *Assume M to be complete. If $A \subset X$ is bounded, by Remark 2.2, $j_4(A)$ is bounded as well. Since TM is complete, Lemma 2.1 implies that $j_4(A)$ is totally bounded, thus, again from Remark 2.2, it follows that $A = j_4^{-1}(j_4(A))$ is totally bounded. As a consequence, $C_T^1(M)$ being complete, closed and bounded subsets of X are compact.*

3. THE DEGREE OF A SECOND ORDER VECTOR FIELD

Let U be an open subset of the differentiable manifold $M \subset \mathbf{R}^k$, and $v : M \to \mathbf{R}^k$ be a continuous tangent vector field such that the set $v^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (v, U) an integer, often called the Euler characteristic (or Hopf index) of v in U, which, roughly speaking, counts (algebraically) the number of zeros of v in U (see e.g. [6], [7], [8], and references therein), and which, for reasons that will became clear in the sequel, we will call degree of the vector field v and denote by $\deg(v, U)$. If $v^{-1}(0) \cap U$ is a finite set, then $\deg(v, U)$ is simply the sum of the indices at the zeros of v. In the general admissible case, i.e. when $v^{-1}(0) \cap U$ is a compact set, $\deg(v, U)$ is defined by taking a convenient smooth approximation of v having finitely many zeros (provided that these zeroes are sufficiently close to $v^{-1}(0) \cap U$).

The celebrated Poincaré-Hopf theorem says that, if M is a compact manifold (possibly with boundary ∂M), then $\deg(v, M \setminus \partial M) = \chi(M)$ for any tangent vector field v which points outward along ∂M.

In the flat case, namely if U is an open subset of $\mathbf{R}^k$, $\deg(v, U)$ is just the Brouwer degree (with respect to zero) of v in U. Using the equivalent definition of degree given in [1], one can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds.

Let M be a boundaryless differentiable manifold embedded in $\mathbf{R}^k$, and let $h : TM \to \mathbf{R}^k$ be (continuous and) tangent to M; that is, assume $h(p,v) \in T_pM$ for any $(p,v) \in TM$. As in section 2, we associate to h the second order tangent vector field $\widehat{h}$ on $TM \subset \mathbf{R}^{2k}$.

We want to show that the degree of the vector field $\widehat{h}$ can be expressed in terms of the degree of $h(\cdot, 0)$ (Lemma 3.2 below). The following Lemma of [3] is in order.

Lemma 3.1. *Let $\gamma : M \to \mathbf{R}^k$ be a tangent vector field on M and let $\widehat{\gamma} : TM \to \mathbf{R}^k \times \mathbf{R}^k$, $\widehat{\gamma}(p,v) = (v, r(p,v) + \gamma(p))$, be the second order vector field associated to γ. Then, given an open subset U of TM, $\widehat{\gamma}$ is admissible on U if and only if γ is so on $U \cap M$, and*

$$\deg(\widehat{\gamma}, U) = \deg(-\gamma, U \cap M).$$

In addition to h, let us consider the vector field $h|_M$, tangent to M, given by the restriction of h to the zero section of TM that, according to (5), is identified with M. In other words, let $h|_M : M \to \mathbf{R}^k$ be $h|_M(p) = h(p, 0)$.

The following result is an extension of Lemma 3.1.

Lemma 3.2. *Let $h : TM \to \mathbf{R}^k$ be tangent to a differentiable manifold M and let U be an open subset of TM. Then, $h|_M$ is admissible on $U \cap M$ if and only if the second order vector field $\widehat{h}$ associated to h is admissible on U, and*

$$\deg(\widehat{h}, U) = \deg\left(-h|_M, U \cap M\right).$$

Proof. Since

$$\left[(h|_M)^{-1}(0)\right] \times \{0\} = \widehat{h}^{-1}(0),$$

$h|_M$ is admissible on $U \cap M$ if and only if $\widehat{h}$ is so on U. It remains to show that the claimed relation between the degrees of $\widehat{h}$ and $-h|_M$ holds.

Let $\widehat{h|_M} : TM \to \mathbf{R}^k \times \mathbf{R}^k$ be the second order vector field associated to $h|_M$. That is

$$\widehat{h|_M}(p,v) = (v, r(p,v) + h|_M(p)).$$

Assume that $h|_M$ is admissible on $U \cap M$ (or, equivalently, $\widehat{h}$ is admissible on U). Then, by Lemma 3.1, also $\widehat{h|_M}$ is so on U, and

$$(7) \qquad \deg(\widehat{h|_M}, U) = \deg(-h|_M, U \cap M).$$

Consider the homotopy of vector fields $H : TM \times [0,1] \to \mathbf{R}^k \times \mathbf{R}^k$ defined by

$$\begin{aligned} H(p,v;\lambda) &= \lambda \widehat{h|_M}(p,v) + (1-\lambda)\widehat{h}(p,v) \\ &= (v, r(p,v) + \lambda h(p,0) + (1-\lambda)h(p,v)). \end{aligned}$$

Observe that H is an admissible homotopy, i.e. the set $H^{-1}(0) = \widehat{h}^{-1}(0) \times [0,1]$ is compact. Hence, by the homotopy invariance

$$(8) \qquad \deg(\widehat{h|_M}, U) = \deg(\widehat{h}, U),$$

and, finally, from (7) and (8) we get

$$\deg(\widehat{h}, U) = \deg\left(\widehat{h|_M}, U\right) = \deg\left(-h|_M, U \cap M\right).$$

■

In the case when M is a compact boundaryless manifold, if U is an open subset of TM such that $U \cap M = M$, by the Poincaré-Hopf Theorem we have

$$\deg\left(- \left. h\right|_M, M\right) = \chi(M).$$

Thus, from Lemma 3.2 we immediately get.

Corollary 3.3. *Let $M \subset \mathbf{R}^k$ be a compact boundaryless manifold and let $h : TM \to \mathbf{R}^k$ be tangent to M. If U is an open subset of TM such that $U \cap M = M$, then*

$$\deg(\widehat{h}, U) = \chi(M).$$

4. THE MAIN RESULT

Theorem 4.2 below, which is our main result, describes a property of the set of T-pairs for a periodically perturbed second order autonomous differential equation. It is based on an analogous result of [4] (Th. 4.1 below) for first order equations on a differentiable manifold $N \subset \mathbf{R}^s$.

For the sake of simplicity, in analogy with the diagram (5), we will regard every space as its image in the following commutative diagram of natural inclusions:

$$(9) \qquad \begin{array}{ccc} N & \longrightarrow & C_T(N) \\ \downarrow & & \downarrow \\ [0, \infty) \times N & \longrightarrow & [0, \infty) \times C_T(N) \end{array}$$

In particular, we will identify N with its image in $C_T(N)$ under the embedding which associates to any $p \in N$ the map $\bar{p} \in C_T(N)$ constantly equal to p. Moreover we will regard N as the slice $\{0\} \times N \subset [0, \infty) \times N$ and, analogously, $C_T(N)$ as $\{0\} \times C_T(N)$. We point out that the images of the above inclusions are closed.

According to these identifications, if Ω is an open subset of $[0, \infty) \times C_T(N)$, by $\Omega \cap N$ we mean the open subset of N given by all $p \in N$ such that the pair $(0, \bar{p})$ belongs to Ω. If U is an open subset of $[0, \infty) \times N$, then $U \cap N$ represents the open set $\{p \in N : (0, p) \in U\}$.

Theorem 4.1. *Let $\varphi : \mathbf{R} \times N \to \mathbf{R}^s$ and $\gamma : N \to \mathbf{R}^s$ be continuous tangent vector fields defined on a (boundaryless) differentiable manifold $N \subset \mathbf{R}^s$, with φ T-periodic in the first variable. Let Ω be an open subset of $[0, \infty) \times C_T(N)$, and assume that $\deg(\gamma, \Omega \cap N)$ is well defined and nonzero. Then Ω contains a connected set $\mathcal{G}$ of nontrivial T-pairs for the equation*

$$\dot{x} = \gamma(x) + \lambda\varphi(t, x),$$

whose closure in $[0, \infty) \times C_T(N)$ meets $\gamma^{-1}(0) \cap \Omega$ and is not contained in any compact subset of Ω. In particular, if N is closed in $\mathbf{R}^s$ and $\Omega = [0, \infty) \times C_T(N)$, then $\mathcal{G}$ is unbounded.

Applying Lemma 3.2 and Theorem 4.1 we obtain our main result about the set of T-pairs of equation (3).

Theorem 4.2. *Let $M \subset \mathbf{R}^k$ be a boundaryless manifold, and let $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ be tangent to M, with f T-periodic in the first variable. Given an open subset Ω of $[0, \infty) \times C_T^1(M)$ such that $\deg\left(\left. h\right|_M, \Omega \cap M\right)$ is well defined and nonzero, Ω contains a connected set Γ of nontrivial T-pairs for (3)*

whose closure meets $\Omega \cap (h|_M)^{-1}(0)$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbf{R}^k$ and $\Omega = [0, \infty) \times C_T^1(M)$, then Γ is unbounded.

Proof. Since Ω is relatively open in the subspace $[0, \infty) \times C_T^1(M)$ of $[0, \infty) \times C_T(TM)$, there exists an open subset $\widehat{\Omega}$ of $[0, \infty) \times C_T(TM)$ such that

$$\widehat{\Omega} \cap \left([0, \infty) \times C_T^1(M)\right) = \Omega$$

and, consequently, $\widehat{\Omega} \cap M = \Omega \cap M$.

If $\widehat{h}$ is the second order vector field associated to h, by Lemma 3.2

$$\begin{aligned} \deg\left(\widehat{h}, TM \cap \widehat{\Omega}\right) &= \deg\left(-h|_M, \widehat{\Omega} \cap M\right) = (-1)^m \deg\left(h|_M, \widehat{\Omega} \cap M\right) \\ &= (-1)^m \deg\left(h|_M, \Omega \cap M\right) \neq 0, \end{aligned}$$

where m is the dimension of M.

By Theorem 4.1, $\widehat{\Omega}$ contains a connected set $\mathcal{G}$ of nontrivial T-pairs for equation (4) whose closure in $[0, \infty) \times C_T(TM)$ meets $\widehat{h}^{-1}(0) \cap \widehat{\Omega}$ and is not contained in any compact subset of $\widehat{\Omega}$. By Remark 2.2, the set

$$\Gamma = (j_4|_X)^{-1}(\mathcal{G}),$$

of T-pairs for (3), is connected and its closure intersects $\Omega \cap (h|_M)^{-1}(0)$. We claim that Γ is not contained in any compact subset of Ω. Assume by contradiction that there exists a compact set $K \subset \Omega$ containing Γ. Since X is closed, $K \cap X$ is compact. Thus its image $j_4(K \cap X)$ is a compact subset of $\widehat{\Omega}$ containing $\mathcal{G}$, which is a contradiction.

Finally, if M is assumed to be complete, TM is complete as well; thus, the last assertion follows from Remark 2.2 and Theorem 4.1. ∎

A special case of Theorem 4.2 is when $M = \mathbf{R}^k$. In this situation, if the degree $\deg\left(h(\cdot, 0), \mathbf{R}^k\right)$ is well defined and nonzero, then there exists an unbounded connected set of T-pairs of (3) which meets $(h(\cdot, 0))^{-1}(0)$, and the existence of such a connected branch cannot be destroyed by a particular choice of f. However this branch is possibly contained in the slice $\{0\} \times C_T^1(M)$, as in the case of the resonant harmonic oscillator (here $M = \mathbf{R}$ and $T = 2\pi$):

$$\ddot{x} = -x + \lambda \sin t.$$

As a straightforward consequence of Theorem 4.2 we get the following extension of a result of [2] in which h is the zero vector field.

Corollary 4.3. *Let $M \subset \mathbf{R}^k$ be a compact boundaryless manifold with $\chi(M) \neq 0$ and let $h : TM \to \mathbf{R}^k$ and $f : \mathbf{R} \times TM \to \mathbf{R}^k$ be tangent to M, with f T-periodic in the first variable. Then there exists an unbounded connected set of nontrivial T-pairs for (3) whose closure in $[0, \infty) \times C_T^1(M)$ meets $(h|_M)^{-1}(0)$.*

In the case when M is a complete manifold, Theorem 4.2 provides the following "geometric" property of the set T-pairs of (3).

Corollary 4.4. *Let M, h and f be as in Theorem 4.2 and assume in addition M to be closed in $\mathbf{R}^k$. Let U be an open subset of M. If $\deg(h|_M, U)$ is well defined and nonzero, then (3) admits a connected set Γ of nontrivial T-pairs whose closure $\overline{\Gamma}$ meets $U \cap (h|_M)^{-1}(0)$ and satisfies at least one of the following properties:*

1. $\overline{\Gamma}$ *is unbounded;*
2. $\overline{\Gamma}$ *meets* $(h|_M)^{-1}(0)$ *outside* U.

In particular, if $(h|_M)^{-1}(0) \subset U$, *then (1) holds.*

Proof. Consider the following open subset of $[0, \infty) \times C_T^1(M)$:

$$\Omega = \left([0, \infty) \times C_T^1(M)\right) \setminus \left((h|_M)^{-1}(0) \setminus U\right).$$

Since $\Omega \cap (h|_M)^{-1}(0) = U \cap (h|_M)^{-1}(0)$, by the excision property of the degree,

$$\deg(h|_M, \Omega \cap M) = \deg(h|_M, U) \neq 0.$$

Thus, by Theorem 4.2 we get the existence of a connected set Γ of nontrivial T-pairs whose closure $\overline{\Gamma}$ in $[0, \infty) \times C_T^1(M)$ is not contained in any compact subset of Ω.

Assume that

$$\overline{\Gamma} \cap \left((h|_M)^{-1}(0) \setminus U\right) = \emptyset,$$

in this case $\overline{\Gamma} \subset \Omega$. Since M is complete, by Remark 2.3, $\overline{\Gamma}$ cannot be both bounded and complete. Thus $\overline{\Gamma}$, being a closed subset of the complete metric space $[0, \infty) \times C_T^1(M)$, must be unbounded. ∎

A particular case of this corollary deserves to be mentioned:

Corollary 4.5. *Let* M, h *and* f *be as in Corollary 4.4. If* $p \in h(\cdot, 0)^{-1}(0)$ *is such that* $h(\cdot, 0)'(p): T_p M \to \mathbf{R}^k$ *is injective, then (3) admits a connected set of* T-pairs *which contains* p *and is either unbounded or meets* $h(\cdot, 0)^{-1}(0) \setminus \{p\}$.

Proof. Since $h(p, 0) = 0$, the derivative $h(\cdot, 0)'$ maps $T_p M$ into itself. Thus p is an isolated zero of $h(\cdot, 0)$ with index ± 1. The assertion follows from Corollary 4.4 taking U a sufficiently small neighborhood of p. ∎

In order to give insight into Corollary 4.4, we give an application. Consider for example the following perturbed pendulum equation:

$$(10) \qquad\qquad \ddot{\theta} = -\sin\theta + \lambda f(t, \theta),$$

with $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous, 2π-periodic in θ and T-periodic in t. Clearly the functions $\theta \mapsto -\sin\theta$ and f can be regarded as tangent vector fields on the manifold S^1. Thus (10) may be considered as a second order differential equation on $S^1 \subset \mathbf{R}^2$, with the "south pole" $\mathsf{S} = \{\theta = 0\}$ and the "north pole" $\mathsf{N} = \{\theta = \pi\}$ as the unique zeros of the unperturbed vector field.

Let us show that, if (10) does not have forced oscillations for some $\lambda_0 > 0$, then the two poles are joined by a connected set of T-pairs.

Let C be the connected component of the set of T-pairs containing S. Corollary 4.5 implies that if C does not meet N, it is unbounded. Thus it is enough to show that if (10) has no T-periodic solutions for $\lambda_0 > 0$, then C must be bounded. Consider the (continuous) map $w: [0, \infty) \times C_T^1(S^1) \to \mathbf{Z}$ which associates to any (λ, x) the winding number of the closed curve $t \in [0, T] \to x(t) \in S^1$. Regarding the poles S and N as T-pairs, we have that $w(\mathsf{N}) = w(\mathsf{S}) = 0$. By the continuity of the winding number, w is identically zero on C. This means that, given any T-pair $(\lambda, x) \in C$, the T-periodic map $t \mapsto x(t)$ may be seen as a T-periodic real

function (that is x is actually a solution of (10)). This implies that the derivative $\dot{x}(t)$ vanishes for some $t \in [0, T]$. Consequently one has

$$|\dot{x}(t)| \leq T \left(1 + \lambda_0 \max_{(s,\theta) \in \mathbf{R} \times \mathbf{R}} |f(s,\theta)| \right) ,$$

and this shows that C is bounded.

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