# ON THE FIXED POINT INDEX OF THE FLOW AND APPLICATIONS TO PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS ON MANIFOLDS

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***Sunto*** — *Si dà una formula per il calcolo dell'indice di punto fisso del flusso generato da un campo vettoriale tangente ad una varietà differenziabile. Tale formula, applicata allo studio delle perturbazioni periodiche di equazioni autonome, permette di provare l'esistenza di connessi non limitati di soluzioni armoniche che intersecano i punti di equilibrio dell'equazione non perturbata. Come conseguenza di questo si ottengono alcuni risultati di molteplicità.*

## 0. Introduction

Let $M$ be an $m$-dimensional boundaryless differentiable manifold embedded in some $\mathbb{R}^k$, and $g : M \to \mathbb{R}^k$ a continuous tangent vector field, such that the Cauchy problem

$$\begin{cases} \dot{x} = g(x), \\ x(0) = p, \end{cases} \tag{0.1}$$

admits a unique solution for each $p \in M$. Under these conditions it is defined a local flow $\{\Phi_t\}_{t\in\mathbb{R}}$ over $M$. Assume that for some $T > 0$ the fixed point index of $\Phi_T$, in a relatively compact open subset $\Omega$ of $M$, is well defined (i.e. $\Phi_T$ is defined on $\overline{\Omega}$ and fixed point free on $\partial\Omega$). In this paper we prove that

$$\operatorname{ind}(\Phi_T, \Omega) = \chi(-g, \Omega), \tag{0.2}$$

where $\chi(-g, \Omega)$ is the Euler characteristic (or index) of the vector field $-g$ in $\Omega$. Consequently $\operatorname{ind}(\Phi_T, \Omega)$, when defined, is independent of $T$. We point out that this fact is not a trivial consequence of the homotopy property of the fixed point index, since, in general, the map $\Phi : (x, t) \mapsto \Phi_t(x)$ is not an admissible homotopy in $\Omega$ (unless $t$ is sufficiently small). The above formula has been first proved by Krasnosel'skii in [K] for the case $M = \mathbb{R}^n$ under the assumption that $\Phi$ is admissible in $(0, T] \times \overline{\Omega}$ (i.e. $\Phi$ is defined on $[0, T] \times \overline{\Omega}$ and $\Phi_t(x) \neq x$ for all $(t, x) \in (0, T] \times \partial\Omega$). It has been recently extended in [CMZ], still in the flat case, to any $T > 0$ such that $\operatorname{ind}(\Phi_T, \mathbb{R}^n)$ is well defined. The method used in [CMZ] is based on coincidence degree theory and on a known result due to Kupka and Smale. An alternative proof, which does not require the Kupka-Smale theorem, has been deduced in [M] from a formula for computing the coincidence degree of $S^1$-equivariant maps given in [BM]. Our approach, which is still based on the Kupka-Smale result, is purely finite dimensional and turns out to be very natural in the setting of differentiable manifolds.

For the sake of simplicity, through this paper we consider only smooth vector fields. This fact is not a serious restriction and could be replaced with the uniqueness assumption of the Cauchy problem.

We apply our formula to the study of perturbed problems of the type

$$\dot{x} = g(x) + \lambda f(t, x) \tag{0.3}$$

where $f : \mathbb{R} \times M \to \mathbb{R}^k$ is a $T$-periodic (with respect to the first variable) tangent vector field on $M$. We analyze, in particular, the structure of the set of starting points of (0.3); that is, of those pairs $(\lambda, p) \in [0, +\infty) \times M$ with the property that the solution of (0.3) starting from $p$ (at $t = 0$) is $T$-periodic.

Under reasonable assumptions we prove the existence of an unbounded connected branch of starting points of (0.3) which emanates from the subset $g^{-1}(0)$ (the rest points) of the set of $T$-periodic orbits of the unperturbed equation $\dot{x} = g(x)$. We provide a simple example of an equation where, in spite of the fact that the sufficient conditions ensuring the existence of the emanating branch are satisfied, there are no starting points for $\lambda \neq 0$. The meaning of this phenomenon, in this case, is that the existing branch must be "purely vertical", i.e. entirely contained in the slice $\lambda = 0$. We point out that the weaker fact that the branch emanates merely from the set of $T$-periodic orbits could also be deduced (still from (0.2)) by a method given in [C, FP1] or [FP3], and under the additional assumption that the $T$-periodic orbits lie in a compact set.

Our sharp assertion about the emanating set of the unbounded branch allows us to deduce multiplicity results regarding the existence of $T$-periodic solutions of the perturbed equation (0.3).

## 1. Preliminaries

Assume that $M$ is an $m$-dimensional differentiable manifold embedded in some $\mathbb{R}^k$, $\Omega$ a relatively compact open subset of $M$, and $\Psi : \overline{\Omega} \to M$ a continuous map. The map $\Psi$ is said to be admissible on $\Omega$ if $\Psi(x) \neq x$ for all $x \in \partial\Omega$. In these conditions it is defined an integer, called the fixed point index and denoted by $\operatorname{ind}(\Psi, \Omega)$, which satisfies all the classical properties of the Brouwer degree: solution, excision, additivity, homotopy invariance, normalization etc. A detailed exposition of this matter can be found, for example in [B, G, N] and references therein. The following fact deserves to be mentioned: if $M$ is an open subset of $\mathbb{R}^m$, then $\operatorname{ind}(\Psi, \Omega)$ is just the Brouwer degree of $I - \Psi$, where $I - \Psi$ is defined by $(I - \Psi)(x) = x - \Psi(x)$.

Finally recall that an *admissible homotopy* is a continuous map $H : \overline{\Omega} \times [0, 1] \to M$ such that $H(x, \lambda) \neq x$ for all $(x, \lambda) \in \partial\Omega \times [0, 1]$.

Let $g : M \to \mathbb{R}^k$ be a continuous tangent vector field on $M$. If $g^{-1}(0) \cap \Omega$ is compact, it is defined (see e.g. [H, Mi] and [T]) an integer $\chi(g, \Omega)$, called the Euler characteristic of $g$, which, in some sense, counts algebraically the number of zeros of $g$. If $g^{-1}(0) \cap \Omega$ is a finite set, then $\chi(g, \Omega)$ is simply the sum of the indices of the (obviously isolated) zeros of $g$. In the general case, $\chi$ is defined taking a sufficiently close approximation of $g$ with only isolated zeros (such an approximation exists by the Sard's Lemma). It is known that if $M$ is a compact boundaryless manifold, then $\chi(g, M)$ is the same for all continuous tangent vector fields and coincides with $\chi(M)$, the Euler-Poincaré characteristic of $M$. Moreover in the flat case, that is if $M$ is an open subset of $\mathbb{R}^m$, $\chi(g, \Omega)$ coincides with the Brouwer degree (with respect to zero) of $g$. Using the equivalent definition of characteristic of vector fields given in [FP2], one can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc. are still valid in the context of differentiable manifolds.

## 2. The fixed point index of the flow

Let $M$ be an $m$-dimensional differentiable boundaryless manifold embedded in some $\mathbb{R}^k$, and $g : M \to \mathbb{R}^k$ a tangent vector field on it. Consider the following differential equation

$$\dot{x} = g(x). \tag{2.1}$$

We will denote by $\{\Phi_t\}_{t\in\mathbb{R}}$ the local flow associated to the equation (2.1), that is the map $\Phi : W \to M$ defined on an open set $W$ of $\mathbb{R} \times M$ containing $\{0\} \times M$, with the property that, for any $p \in M$, the curve $t \mapsto \Phi_t(p)$ is the maximal solution of $\dot{x} = g(x)$ such that $\Phi_0(p) = p$. Therefore, given $\tau \in \mathbb{R}$, the domain of $\Phi_\tau$ is the open set consisting of those points $p \in M$ for which the maximal solution of (2.1), starting from $p$ at $t = 0$ is defined up to $\tau$.

In what follows, by an orbit we mean the image of a periodic solution of (2.1). Given $T > 0$, with $A_T$ we denote the union of all $\tau$-periodic orbits with $0 < \tau \leq T$. Note that $g^{-1}(0) \subset A_T$ for all $T > 0$.

**Lemma 2.1.** *Given $T > 0$, let $O \subset M$ be a nontrivial isolated orbit of (2.1) in $A_T$. There exist an open neighborhood $S$ of $O$ such that, for all $0 < \tau \leq T$, $\Phi_\tau$ is defined on $\overline{S}$, admissible on $S$, and* $\mathrm{ind}(\Phi_\tau, S) = 0$.

Proof. Since $O$ is a periodic orbit, $\Phi_t$ is defined on $O$ for all $t \in \mathbb{R}$; thus $\Phi_t$ is defined on some neighborhood $S$ of $O$. Observe that, since $O$ is isolated in $A_T$, one can choose $S$ such that $\Phi_\tau$ is fixed point free on $\partial S$ for all $\tau \in (0, T]$. This implies, by the homotopy property, that $\mathrm{ind}(\Phi_\tau, S)$ is independent of $\tau \in (0, T]$. Moreover, by the non triviality of $O$, there exists a positive minimal period $\sigma$ of $O$. Thus $\mathrm{ind}(\Phi_{\sigma/2}, S) = 0$ since $\Phi_{\sigma/2}$ is fixed point free on $\overline{S}$. ∎

**Lemma 2.2.** *Assume $\Phi_T$ is defined on a relatively compact open subset $\Omega$ of $M$. Suppose that all the orbits with period in $(0, T]$ which meet $\overline{\Omega}$ are isolated in $A_T$. Given $\tau, \sigma \in (0, T]$ such that $\Phi_\tau$ and $\Phi_\sigma$ are fixed point free on $\partial\Omega$, we have*

$$ind(\Phi_\tau, \Omega) = \mathrm{ind}(\Phi_\sigma, \Omega).$$

Proof. As a consequence of our assumption, since $\overline{\Omega}$ is compact, there are only finitely many orbits of period in $(0, T]$ which meet $\overline{\Omega}$. Let $O_1 \dots O_n$ be all the nontrivial ones. Applying Lemma 2.1, there exist open subsets of $M$, $S_1 \dots S_n$, such that $O_i \subset S_i$ and

$$\mathrm{ind}(\Phi_\tau, S_i) = \mathrm{ind}(\Phi_\sigma, S_i) = 0,$$

for all $i = 1 \dots n$. We can clearly assume $\overline{S}_i \cap \overline{S}_j = \emptyset$ when $i \neq j$. Define

$$\Omega_1 = \Omega \setminus \bigcup_{i=1}^{n} \overline{S}_i \ .$$

By the additivity and the excision properties,

$$\mathrm{ind}(\Phi_\tau, \Omega) = \mathrm{ind}(\Phi_\tau, \Omega_1),$$

and

$$\mathrm{ind}(\Phi_\sigma, \Omega) = \mathrm{ind}(\Phi_\sigma, \Omega_1).$$

Using the homotopy invariance, we can write

$$\operatorname{ind}(\Phi_\tau, \Omega_1) = \operatorname{ind}(\Phi_\sigma, \Omega_1),$$

and the claim follows. ■

Let $\Omega$ be an open relatively compact subset of $M$, and let $T > 0$ be given. Consider a continuous tangent vector field $g : M \to \mathbb{R}^k$, such that the solution of the Cauchy problem (0.1) is defined in $[0, T]$, for all $p \in \overline{\Omega}$. Suppose that $\operatorname{ind}(\Phi_T, \Omega)$ is well defined. This clearly implies that there are no zeros of $g$ on $\partial\Omega$, so $\chi(g, \Omega)$ is defined as well.

Using a corollary of the Kupka-Smale theorem (see [CM]), one can show that there exists a sequence $\{g_k\}$ of $C^1$ tangent vector fields on $M$, uniformly converging to $g$ on compact sets, and such that, for every $k \in \mathbb{N}$, the equation $\dot{x} = g_k(x)$, in any given compact set, admits finitely many periodic orbits with period in $(0, T]$. We will denote by $\{\Psi_t^k\}_{t\in\mathbb{R}}$ the local flow associated to the equation $\dot{x} = g_k(x)$. Since the flow is a continuous map of the twofold variable $(t, x) \in \mathbb{R} \times M$, the "attainable set" $\widehat{\Omega}_T = \Phi_{[0,T]}(\overline{\Omega})$ is a compact subset of $M$. Let $B$ be a relatively compact open set containing $\widehat{\Omega}_T$. Let $c$ be the distance (in $\mathbb{R}^k$) between $\widehat{\Omega}_T$ and $\partial B$. One can choose a sufficiently large $\overline{k}$ such that $\|\Phi_t(x) - \Psi_t^k(x)\| \le c/2$ for all $x \in \overline{\Omega}$, $t \in [0, T]$ and $k > \overline{k}$. This implies that, if $k > \overline{k}$, any solution of $\dot{x} = g_k(x)$, which meets $\overline{\Omega}$, is contained in $\overline{B}$. By the choice of the sequence $\{g_k\}$, $\overline{B}$ contains only finitely many periodic orbits of $\dot{x} = g_k(x)$ with period in $(0, T]$.

It is well known that there exist $\varepsilon > 0$ such that $\operatorname{ind}(\Psi_t, \Omega)$ is well defined and constant for $0 < t \le \varepsilon$. More precisely (see [FP2]) we know that, for $0 < t \le \varepsilon$,

$$\operatorname{ind}(\Phi_t, \Omega) = (-1)^m \chi(g, \Omega) = \chi(-g, \Omega). \tag{2.2}$$

Using the continuous dependence on data and the compactness of $\partial\Omega$ we can assume $\Psi_T^k(x) \neq x$ and $\Psi_\varepsilon^k(x) \neq x$ for all $x \in \partial\Omega$. Moreover, using the homotopy invariance property of the index, we get

$$\operatorname{ind}(\Psi_T^k, \Omega) = \operatorname{ind}(\Phi_T, \Omega), \tag{2.3}$$

$$\operatorname{ind}(\Psi_\varepsilon^k, \Omega) = \operatorname{ind}(\Phi_\varepsilon, \Omega), , \tag{2.4}$$

provided that $k$ is large enough. Applying Lemma 2.2,

$$\operatorname{ind}(\Psi_T^k, \Omega) = \operatorname{ind}(\Psi_\varepsilon^k, \Omega),$$

and, using (2.2), (2.3) and (2.4) we obtain

$$\operatorname{ind}(\Phi_T, \Omega) = \operatorname{ind}(\Phi_\varepsilon, \Omega) = \chi(-g, \Omega).$$

We have proved the following result

**Theorem 2.1.** *Let $g : M \to \mathbb{R}^k$ be a tangent vector field on a boundaryless differentiable manifold $M \subset \mathbb{R}^k$ and $\Omega$ a relatively compact open subset of $M$. Let $T > 0$ and assume that, for any $p \in \overline{\Omega}$, the solution of the Cauchy problem* (0.1) *is defined on $[0, T]$. If $\Phi_T$ is fixed point free on $\partial\Omega$, then*

$$\operatorname{ind}(\Phi_T, \Omega) = \chi(-g, \Omega).$$

In spite oif the fact that the restriction to $\Omega$ of the Poicaré operator $\Phi_T$ may be strongly influenced by the behaviour of $g$ outside $\Omega$, this is not so for its fixed point index.

**Corollary 2.1.** *Let $M$, $g$, $\Omega$ and $T$ be as in the above theorem. Let $h : M \to \mathbb{R}^k$ be a tangent vector field and denote by $\{\Psi_t\}_{t\in\mathbb{R}}$ its local flow. If $g\,|_\Omega = h\,|_\Omega$, then*

$$\mathrm{ind}(\Phi_T, \Omega) = \mathrm{ind}(\Psi_T, \Omega),$$

*provided that they are both well defined.*

Note also that Theorem 2.1 is not a trivial consequence of the homotopy property because, in general, the map $(p,t) \mapsto \Phi_t(p)$ is not an admissible homotopy on $\Omega$. Consider, for example, the following differential equation in $M = \mathbb{R}^2$

$$(\dot{x}, \dot{y}) = (y, -x)$$

and let $\Omega = B(0,1)$ be the unit open disk in $\mathbb{R}^2$. A direct computation shows that $\mathrm{ind}(\Phi_t, \Omega)$ is well defined and equal to 1 for any $t \neq 2k\pi$, and it is not defined for $t = 2k\pi$) $(k \in \mathbb{Z}$. Therefore if $t$ is considered in an interval containing one of these values, the flow does not give an admissible homotopy.

Consider the differential equation

$$\dot{x} = \lambda g(x) \quad \lambda \in [0,1], \tag{2.5}$$

where $g$ is as in Theorem 2.1, and denote by $\Phi_t(\lambda, \cdot)$ the flow associated to this equation. Observe that

$$\Phi_T(\lambda, \cdot) = \Phi_{\lambda T}(1, \cdot). \tag{2.6}$$

In particular, any $T$-periodic solution of $\dot{x} = g(x)$ corresponds to a $(T/\lambda)$-periodic one of (2.5). Assume that $\Phi_T(\lambda_1, \cdot)$ and $\Phi_T(\lambda_2, \cdot)$ are fixed point free on $\partial\Omega$ $(\lambda_1, \lambda_2 \in (0,1])$; then, by Theorem 2.1 we have

$$\mathrm{ind}(\Phi_{\lambda_1 T}(1, \cdot), \Omega) = \mathrm{ind}(\Phi_{\lambda_2 T}(1, \cdot), \Omega).$$

Therefore, by (2.6), we get

$$\mathrm{ind}(\Phi_T(\lambda_1, \cdot), \Omega) = \mathrm{ind}(\Phi_T(\lambda_2, \cdot), \Omega). \tag{2.7}$$

Let now $f : \mathbb{R} \times M$ be a $T$-periodic tangent vector field on $M$; assume that the solutions of $\dot{x} = \lambda f(t,x)$ are continuable on $[0,T]$ for any $\lambda \in [0,1]$. One could ask if a formula like (2.7), is still true for the for the parametrized differential equation

$$\dot{x} = \lambda f(t, x(t)) \quad \lambda \in [0,1]. \tag{2.8}$$

More precisely, denote by $P_T(\lambda, \cdot) : \overline{\Omega} \to M$ the translation operator, which associates to any point $p$ the value at time $T$ of the solution of (2.8), satisfying $x(0) = p$. Assuming that $P_T(\lambda_1, \cdot)$ and $P_T(\lambda_2, \cdot)$ are fixed point free on $\partial\Omega$ $(\lambda_1, \lambda_2 \in (0,1])$, the question is if one could write

$$\mathrm{ind}\left(P_T(\lambda_1, \cdot), \Omega\right) = \mathrm{ind}\left(P_T(\lambda_2, \cdot), \Omega\right). \tag{2.9}$$

The answer is affirmative in the case when $\Omega = M$ is a compact, boundaryless manifold (this is an easy consequence of the homotopy property of the fixed point index), but it is false in general. To see this put $M = \mathbb{R}^2$ and let $\Omega$ be the unit open disk in $\mathbb{R}^2$. Consider the following equation in $\mathbb{R}^2$

$$\begin{cases} \dot{x}_1 = \lambda x_2 \\ \dot{x}_2 = -\lambda x_1 + \lambda \sin t \end{cases} \tag{2.10}$$

For $\lambda = 1$, (2.10) does not admit $2\pi$-periodic solution. Thus $\mathrm{ind}(P_{2\pi}(1,\cdot),\Omega) = 0$. On the other hand, for sufficiently small $\lambda$, it is known that (see [FP2]) that $\mathrm{ind}(P_{2\pi}(\lambda,\cdot),\Omega) = \chi(-w,\Omega) = 1$, where $w : \overline{\Omega} \to \mathbb{R}^2$ is defined by

$$w(x_1, x_2) = \frac{1}{2\pi}\int_0^{2\pi} (x_2, -x_1 + \sin t)\, \mathrm{d}t = (x_2, -x_1),$$

contradicting (2.9).

Let $f : \mathbb{R} \times M \to \mathbb{R}^k$ be as above and $\Omega$ a relatively compact open subset of $M$, and assume that, for each $p \in \Omega$, the solution of the Cauchy problem

$$\begin{cases} \dot{x} = f(t,x), \\ x(0) = p, \end{cases} \tag{2.11}$$

is defined on $[0,T]$. Denote by $P_T : \overline{\Omega} \to M$ the Poincaré $T$-translation operator which associates to any point $p$ the value at time $T$ of the solution $x(\cdot,p)$ of (2.11). Following Krasnosel'skii (see [K]) a point $p \in M$ is said to be of *$T$-irreversibility* if $x(t,p) \neq p$ $\forall t \in (0,T]$. Using the homotopy property of the degree, Krasnosel'skii proves a formula for computing the fixed point index of the operator of translation along trajectories of a nonautonomous differential equation; his result (reformulated in the framework of differentiable manifolds) is the following.

**Theorem 2.2.** *Suppose that all points of $\partial\Omega$ are of $T$-irreversibility and that $f(0,x) \neq 0$ on $\partial\Omega$. Then*

$$\mathrm{ind}(P_T,\Omega) = \chi(-f(0,\cdot),\Omega).$$

Theorem 2.1 shows that, at least in the case of autonomous differential equations, the hypothesis of $T$-irreversibility can be removed: the essential fact is the absence of fixed points for $P_T$ on $\partial\Omega$ (i.e. the admissibility on $\Omega$ of the Poincaré $T$-translation operator). Now, the question is if one can eliminate the $T$-irreversibility hypothesis also for the nonautonomous case. Equation (2.10), with $\lambda = 1$, shows that this is not possible. In fact, let $\Omega$ be the open unit disk in $\mathbb{R}^2$. A direct computation gives $\chi(-f(0,\cdot),\Omega) = 1$ and $\mathrm{ind}(P_{2\pi},\Omega) = 0$ (since (2.10) has no $2\pi$-periodic orbits for $\lambda = 1$).

## 3. Applications

Let $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ be two tangent vector fields on an $m$-dimensional boundaryless manifold $M \subset \mathbb{R}^k$, with $f$ $T$-periodic. Let us give some notation. In the sequel, given $X \subset \mathbb{R} \times M$ and $\lambda \in \mathbb{R}$, we will denote the slice $\{x \in M : (\lambda, x) \in X\}$ with the symbol $X_\lambda$. Any pair $(\lambda, p)$ is said to be a *starting point* (for (0.3)) if the equation (0.3) has a $T$-periodic solution satisfying $x(0) = p$. We will use the following global connectivity result (see [FP4]).

**Lemma 3.1.** *Let $Y$ be a locally compact metric space and let $Y_0$ be a compact subset of $Y$. Assume that any compact subset of $Y$ containing $Y_0$ has nonempty boundary. Then $Y \setminus Y_0$ contains a not relatively compact component whose closure (in $Y$) intersects $Y_0$.*

By known properties of differential equations the set $V \subset [0,\infty) \times M$ given by

$$\{(\lambda, p) : \text{the solution } x(\cdot) \text{ of (0.3) satisfying } x(0) = p \text{ is defined in } [0,T]\},$$

is open. Thus it is locally compact. Clearly $V$ contains the set $S$ of all starting points of (0.3). Observe that $S$ is closed in $V$, even if it could be not so in $[0,+\infty) \times$

$M$. Therefore $S$ is locally compact. Let $U$ be an open subset of $V$ and consider the set $S_U = S \cap U$. Since $S_U$ is open in $S$, it is locally compact as well. The set $S_U$ will be called *the starting point set relative to* $U$ of (0.3).

**Theorem 3.1.** *Let $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ be two tangent vector fields on a boundaryless manifold $M \subset \mathbb{R}^k$, with $f$ $T$-periodic. Assume $S$ and $U$ are as above. If $g^{-1}(0) \cap U_0$ is compact and $\chi(g, U_0)$ is nonzero, then $S_U$ admits a connected subset which meets $\{0\} \times \big(g^{-1}(0) \cap U_0\big)$ and is not contained in any compact subset of $U$.*

Proof. Observe first that $g^{-1}(0) \cap U_0$ is nonempty since $\chi(g, U_0)$ is nonzero. Thus $S_U$ is nonempty as well. The theorem follows applying Lemma 3.1 to the pair

$$(Y, Y_0) = \big(S_U, \{0\} \times (g^{-1}(0) \cap U_0)\big).$$

In fact, if $\Sigma$ is a component as in the assertion of Lemma 3.1, its closure satisfies the requirement. Assume, by contradiction, that there exist a compact subset $C$ of $S_U$, containing $\{0\} \times (g^{-1}(0) \cap U_0)$ and with empty boundary in $S_U$. Thus $C$ is a relatively open subset of $S_U$. As a consequence, $S_U \setminus C$ is closed in $S_U$, so the distance, $\delta = \mathrm{dist}(C, S_U \setminus C)$, between $C$ and $S_U \setminus C$ is nonzero (recall that $C$ is compact). Consider the set

$$W = \left\{ (\lambda, p) \in U : \mathrm{dist}\big((\lambda, p), C\big) < \frac{\delta}{2} \right\},$$

which, clearly, does not meet $S_U \setminus C$. Because of the compactness of $S_U \cap W = C$, there exists $\overline{\lambda} > 0$ such that $(\{\overline{\lambda}\} \times W_{\{\overline{\lambda}\}}) \cap S_U = \emptyset$. Moreover, the set $S_U \cap W$ coincides with $\{(\lambda, p) \in W : P_T(\lambda, p) = p\}$, where $P_T : V \to M$ denotes the translation operator which associates to any pair $(\lambda, p) \in V$ the value at time $T$ of the solution of (0.3) satisfying $x(0) = p$. Then from the generalized homotopy property of the index (see e.g. [N]),

$$0 = \mathrm{ind}\,\big(P_T(\overline{\lambda}, \cdot), W_{\overline{\lambda}}\big) = \mathrm{ind}\,\big(P_T(\lambda, \cdot), W_\lambda\big),$$

for all $\lambda \in [0, \overline{\lambda}]$. Observe that our contradictory assumption implies that $P_T(0, \cdot)$ is fixed point free on the boundary of $W_0$, therefore $\mathrm{ind}\,\big(P_T(0, \cdot), W_0\big)$ is well defined. Applying the excision property of the Euler-Poincaré characteristic and Theorem 2.1

$$\mathrm{ind}\,\big(P_T(0, \cdot), W_0\big) = (-1)^m \chi(g, W_0) = (-1)^m \chi(g, U_0) \neq 0,$$

contradicting the previous formula. ∎

Below, we give some simple consequences of Theorem 3.1 which illustrate its utility in describing the structure of the starting point set.

**Corollary 3.1.** *Let $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^m$ be two vector fields, with $f$ $T$-periodic. Assume that there exist constants $a, b, c, d \in \mathbb{R}$ such that $\|f(t, x)\| \leq a + b\|x\|$ and $\|g(x)\| \leq c + d\|x\|$ for each $x \in \mathbb{R}^m$ and $t \in \mathbb{R}$. If $g^{-1}(0)$ is compact and $\chi(g, \mathbb{R}^m) \neq 0$, then there exists an unbounded connected set of starting points for $T$-periodic solutions of (0.3) which meets $\{0\} \times g^{-1}(0)$.*

Proof. By the assumptions on $f$ and $g$ any solution of (0.3) is defined on the whole real line. Thus, in this case taking $U = [0, +\infty) \times \mathbb{R}^m$, by Theorem 3.1 there exists a connected set $\Sigma$ of starting points for the equation (0.3) which meets $\{0\} \times g^{-1}(0)$ and is not contained in any compact subset of $[0, +\infty) \times \mathbb{R}^m$. This implies that $\Sigma$ is unbounded. ∎

The following simple two dimensional example shows that, in some cases, the unbounded set of starting points ensured by Corollary 3.1 may be contained in the slice $\{0\} \times \mathbb{R}^2$.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \lambda \sin t \end{cases}$$

The following is an extension of a result of [FP1] to the case $g \neq 0$.

**Corollary 3.2.** *Assume $M$ to be a compact boundaryless manifold with $\chi(M) \neq 0$. Let $f : \mathbb{R} \times M \to \mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ be as in Theorem 3.1. Then there exists a connected set of starting points $\Sigma$ which meets $\{0\} \times g^{-1}(0)$ and such that $\pi_1(\Sigma) = [0, +\infty)$, where $\pi_1$ denotes the projection on the first factor of $[0, +\infty) \times M$.*

Proof. By the compactness of $M$ we have $V = [0, +\infty) \times M$. We apply Theorem 3.1 to the open set $U = V$. By the Poincaré-Hopf theorem (see e.g. [Mi]) and the assumptions, we have

$$\chi(M) = \chi(g, U_0) \neq 0.$$

Therefore there exists a connected set $\Sigma$ of starting points for (0.3) which meets $\{0\} \times g^{-1}(0)$ and is not contained in any compact subset of $U$. In particular, for each $\lambda \geq 0$ fixed, $\Sigma$ intersects $\{\lambda\} \times M$, i.e. $\pi_1(\Sigma) = [0, +\infty)$. ■

The fact that the global branch ensured by Theorem 3.1 emanates from the set of zeros of $g$, and not merely from the set of all $T$-periodic orbits of $\dot{x} = g(x)$, allows us to obtain information about the starting point set of equation (0.3) also in the case of a compact manifold with zero Euler-Poincaré characteristic.

**Corollary 3.3.** *Let $M \subset \mathbb{R}^k$ be a compact boundaryless manifold. Assume that $f$ and $g$ are as in Theorem 3.1 and, in addition, $g$ has exactly two distinct zeros $z_1$ and $z_2$ with nonzero index. Denote by $S_1$ and $S_2$ the connected components of the set of starting points of (0.3) which contain respectively $z_1$ and $z_2$. Then just one of the following two possibilities holds:*

1) *$S_1 = S_2$,*
2) *$S_1$ and $S_2$ are disjoint and both unbounded (in $[0, +\infty) \times M$).*

*In particular, if 2) holds, there exist at least two distinct $T$-periodic solutions of (0.3) for each $\lambda \in [0, +\infty)$.*

Proof. Since $M$ is compact, we have $V = [0, +\infty) \times M$. Take

$$U_1 = [0, +\infty) \times M \setminus (0, z_2),$$
$$U_2 = [0, +\infty) \times M \setminus (0, z_1).$$

Obviously $(0, z_i) \in U_i$, and by the excision property $\chi(g, U_i) \neq 0$ for $i \in 1, 2$. We may assume $S_1 \neq S_2$. In this case $S_1$ and $S_2$, being connected components, are clearly disjoint and, consequently, $S_1 \subset U_1$ and $S_2 \subset U_2$. Because of Theorem 3.1 $S_1$ and $S_2$ are not contained in any compact subset of $U_1$ and $U_2$ respectively and, in particular, they are not compact. Now $S_1$ and $S_2$ are closed in the set $S$ of all the starting points of (0.3). Since $S$ is closed in $V = [0, +\infty) \times M$, which is closed in $\mathbb{R}^{k+1}$, the two components $S_1$ and $S_2$ must be unbounded. ■

**Corollary 3.4.** *Let $g : M \to \mathbb{R}^k$ be a tangent vector field on a compact boundaryless manifold $M \subset \mathbb{R}^k$ and assume $g$ has exactly two zeros $z_1$ and $z_2$ with nonzero index. If $z_1$ and $z_2$ belong to different connected components of the set of all $T$-periodic* *orbits of* $\dot{x} = g(x)$*, then for any* $T$*-periodic tangent vector field* $f : \mathbb{R} \times M \to \mathbb{R}^k$ *there exist* $\lambda_f > 0$ *such that the equation* (0.3) *has at least two* $T$*-periodic solutions for all* $\lambda \in [0, \lambda_f]$*.*

Proof. Let $S_1$ and $S_2$ be as in Corollary 3.3. It is enough to show that both $S_1$ and $S_2$ are not contained in the slice $\{0\} \times M$. Assume, for example, $S_1 \subset \{0\} \times M$. Thus, by the Corollary 3.3, the alternative 1) holds. This implies $z_1$ and $z_2$ belong to the same connected component of $T$-periodic orbits of $\dot{x} = g(x)$, a contradiction. ∎

An infinite dimensional analogous of Theorem 3.1 and some extensions of the results of this section will appear in a forthcoming paper.

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