Periodic solutions of retarded functional perturbations of autonomous differential equations on manifolds

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Abstract

Inspired by [1] and [10] we apply the topological tools of fixed point index and of degree of a tangent vector field to the study of the set of harmonic solutions to periodic perturbations of autonomous ODEs on (smooth) boundaryless differentiable manifolds, allowing the perturbation to contain a distributed, possibly infinite, delay. In order to do so, we construct a Poincaré-type T-translation operator on an appropriate function space and, in the unperturbed case, prove a formula for its fixed point index in terms of the degree of the autonomous vector field.

1 Introduction and preliminaries

In this paper we study the set of harmonic solutions to periodic perturbations of autonomous ODEs on (smooth) boundaryless differentiable manifolds, allowing the perturbation to contain a distributed, possibly infinite, delay. Namely, given $T > 0$ and a boundaryless manifold $M \subseteq \mathbb{R}^k$, we consider the T-periodic solutions to the following parametrized Retarded Functional Differential Equation (RFDE) on M:

$$x'(t) = g\big(x(t)\big) + \lambda f(t, x_t), \quad \lambda \geq 0, \tag{1.1}$$

where $g \colon M \to \mathbb{R}^k$ is a tangent vector field, in the sense that $g(p)$ belongs to the tangent space $T_p M$ of M at p for any $p \in M$, and $f \colon \mathbb{R} \times BU\big((-\infty, 0], M\big) \to \mathbb{R}^k$ is continuous, T-periodic in the first variable and such that $f(t, \varphi) \in T_{\varphi(0)} M$ for all $(t, \varphi) \in \mathbb{R} \times BU\big((-\infty, 0], M\big)$. Here $BU\big((-\infty, 0], M\big)$ denotes the metric subspace, consisting of the M-valued functions, of the space of uniformly continuous bounded functions from $(-\infty, 0]$ into $\mathbb{R}^k$, with the supremum norm. As usual in the RFDE context, given $t \in \mathbb{R}$, $x_t \in BU\big((-\infty, 0], M\big)$ is the function $\theta \mapsto x(t + \theta)$. A function f as above is called a *(retarded) functional (tangent vector) field.*

Roughly speaking, we will give conditions ensuring the existence of a connected component of pairs (λ, x), $\lambda \geq 0$ and x a T-periodic solution to the above equation, that emanates from the set of zeros of g and is not compact. In this paper we will always suppose that f and g satisfy some regularity assumptions (see below). So, our results generalize only partially those of [9] and [10], in which f and g are assumed to be merely continuous.

In order to carry out this program, as in [10] we will use the topological tools of fixed point index and degree (also called the rotation or characteristic) of a tangent vector field, that shall be briefly recalled. In fact, in the case when the perturbation is independent of the delay, as in [9], the existence of such a connected component of T-periodic solutions is based on the computation of the fixed point index of the T-translation operator associated to the unperturbed equation (see e.g. [8]). As in [9], since in our case the perturbing term f is a functional field, the T-translation operator P ought to be replaced by an appropriate infinite dimensional adaptation. However, due to the possibly infinite delay, the construction of [9] would not work here directly. In fact, the "natural" translation operator defined on the space $BU\big((-\infty, 0], M\big)$ of initial functions is not locally compact. What we actually do is to define a Poincaré-type T-translation operator on $C([-T, 0], M)$ whose fixed points are the restrictions to $[-T, 0]$ of the T-periodic solutions of (1.1).

Throughout this paper, we shall suppose that g is locally Lipschitz and that f satisfies the following assumption:

(H) Given any compact subset Q of $\mathbb{R} \times BU\big((-\infty, 0], M\big)$, there exists $L \geq 0$ such that
$$|f(t, \varphi) - f(t, \psi)| \leq L \sup_{t \leq 0} |\varphi(t) - \psi(t)|$$
for all $(t, \varphi), (t, \psi) \in Q$.

We will say that condition (H) holds locally in $\mathbb{R} \times BU\big((-\infty, 0], M\big)$ if for any $(\tau, \eta) \in \mathbb{R} \times BU\big((-\infty, 0], M\big)$ there exists a neighborhood of (τ, η) in which (H) holds. Actually, one could show that if (H) is satisfied locally, then it is also satisfied globally. However, the local condition is easier to check. It holds, for instance, when f is C^1 or, more generally, locally Lipschitz in the second variable. Observe also that if $g \colon M \to \mathbb{R}^k$ is a locally Lipschitz tangent vector field, and f is a functional field satisfying (H), then for any $\lambda \in [0, +\infty)$ the map of $\mathbb{R} \times BU\big((-\infty, 0], M\big)$ in $\mathbb{R}^k$, given by

$$(t, \varphi) \mapsto g\big(\varphi(0)\big) + \lambda f(t, \varphi),$$

is a functional vector field that verifies (H) as well.

It can be proved (see e.g. [3]) that if a tangent functional field Φ satisfies (H) locally, then two maximal solutions of equation $x'(t) = \Phi(t, x_t)$ coinciding in the past must coincide also in the future. Thus, if g is locally Lipschitz and f satisfies (H) locally, then the initial value problem associated to (1.1) admits a unique solution.

It will be convenient to assume throughout the paper that the ambient manifold M is a closed subset of $\mathbb{R}^k$. In this way the space $C\big([-T, 0], M\big)$ turns out to be complete. This assumption is not restrictive. In fact, as is well known, any differentiable manifold can be diffeomorphically embedded as a closed submanifold of some Euclidean space.

1.1 The fixed point index and the degree of a tangent vector field

In this section we briefly recall the notions of fixed point index of a map and of degree of a tangent vector field.

Let us begin with the fixed point index. We recall that a metrizable space E is an *absolute neighborhood retract* (ANR) if, whenever it is homeomorphically embedded as a closed subset C of a metric space X, there exists an open neighborhood U of C in X and a retraction $r\colon U \to C$ (see e.g. [4, 12]). Polyhedra and differentiable manifolds are examples of ANRs. Let us also recall that a continuous map between topological spaces is called *locally compact* if it has the property that each point in its domain has a neighborhood whose image is contained in a compact set.

Let E be an ANR and let ψ be a locally compact E-valued map defined (at least) on an open subset U of E. If the set $\mathrm{Fix}\,(\psi, U)$ of the fixed points of ψ in U is compact, then it is well defined an integer, $\mathrm{ind}(\psi, U)$, called *the fixed point index of ψ in U* (see, e.g. [11, 12, 16]). Roughly speaking, $\mathrm{ind}(\psi, U)$ counts algebraically the elements of $\mathrm{Fix}\,(\psi, U)$.

The fixed point index enjoys a number of useful properties. We list here a few of them for the purpose of future reference.

Normalization. *Let $\psi\colon E \to E$ be constant. Then $\mathrm{ind}(\psi, E) = 1$.*

Additivity. *Given U open in E and a locally compact map $\psi\colon U \to E$ with $\mathrm{Fix}(\psi, U)$ compact, if U_1 and U_2 are disjoint open subsets of U such that $\mathrm{Fix}(\psi, U) \subseteq U_1 \cup U_2$, then*

$$\mathrm{ind}(\psi, U) = \mathrm{ind}(\psi, U_1) + \mathrm{ind}(\psi, U_2).$$

Homotopy Invariance. *Given U open in E, assume that $H\colon U \times [0,1] \to E$ is an admissible homotopy in U; that is, H is locally compact and the set $\{(x, \lambda) \in U \times [0,1]\colon H(x, \lambda) = x\}$ is compact. Then*

$$\mathrm{ind}\big(H(\cdot, 0), U\big) = \mathrm{ind}\big(H(\cdot, 1), U\big).$$

Commutativity. *Let E_1, E_2 be ANRs and let $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ be open. Suppose $\psi_1\colon U_1 \to E_2$ and $\psi_2\colon U_2 \to E_1$ are locally compact maps. If one of the sets*

$$\{x \in \psi_1^{-1}(U_2) : \psi_2 \circ \psi_1(x) = x\} \quad or \quad \{y \in \psi_2^{-1}(U_1) : \psi_1 \circ \psi_2(y) = y\}$$

is compact, then so is the other and

$$\mathrm{ind}\big(\psi_2 \circ \psi_1, \psi_1^{-1}(U_2)\big) = \mathrm{ind}\big(\psi_1 \circ \psi_2, \psi_2^{-1}(U_1)\big).$$

It is easily shown that the Additivity Property implies the following two important ones:

Solution. Given U open in E, let $\psi\colon U \to E$ be locally compact with empty $\mathrm{Fix}(\psi, U)$. Then $\mathrm{ind}(\psi, U) = 0$.

Excision. *Let U be open in E. Given a locally compact map $\psi\colon U \to E$ with $\mathrm{Fix}(\psi, U)$ compact, and an open subset V of U containing $\mathrm{Fix}(\psi, U)$, one has $\mathrm{ind}(\psi, U) = \mathrm{ind}(\psi, V)$.*

From the Homotopy Invariance and Excision properties one could deduce the following property:

Generalized Homotopy Invariance. *Let W be open in $E \times [0,1]$. Assume that $H \colon W \to E$ is locally compact and such that the set $\{(x, \lambda) \in W : H(x, \lambda) = x\}$ is compact. Let W_λ denote the slice $W_\lambda := \{x \in E : (x, \lambda) \in W\}$. Then, $\operatorname{ind}\big(H(\cdot, \lambda), W_\lambda\big)$ does not depend on $\lambda \in [0,1]$.*

It is worth mentioning that when $E = \mathbb{R}^n$, U is bounded, ψ is defined on $\overline{U}$ and fixed point free on ∂U, then $\operatorname{ind}(\psi, U)$ is just the Brouwer degree $\deg_B(I - \psi, U, 0)$, where I denotes the identity on $\mathbb{R}^n$.

We now briefly discuss the notion of degree of a tangent vector field on a differentiable manifold. Let $M \subseteq \mathbb{R}^k$ be a differentiable manifold and let $w \colon M \to \mathbb{R}^k$ be a (continuous) tangent vector field on M; meaning that, for all $p \in M$, $w(p)$ belongs to the tangent space $T_p M$ of M at p. Let U be an open subset of M in which we assume w admissible for the degree; that is, the set $w^{-1}(0) \cap U$ compact. Then, one can associate to the pair (w, U) an integer, $\deg(w, U)$, called the *degree of the vector field w in U* (see e.g. [6, 14, 15] and references therein).

We recall that if w is C^1 and if $p \in M$ is such that $w(p) = 0$, then the Fréchet derivative $w'(p) \colon T_p M \to \mathbb{R}^k$ maps $T_p M$ into itself (see e.g. [15]), so that the determinant $\det w'(p)$ of $w'(p)$ is defined. If, in addition, p is a non-degenerate zero (i.e. $w'(p) \colon T_p M \to \mathbb{R}^k$ is injective) then p is an isolated zero and $\det w'(p) \neq 0$. In this case, the degree of w in U is an algebraic count of its zeros in U, that is

$$\deg(w, U) = \sum_{q \in w^{-1}(0) \cap U} \operatorname{sign} \det w'(q).$$

When $M = \mathbb{R}^k$, $\deg(w, U)$ is just the classical Brouwer degree, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap U$ whose closure is in U. Moreover, when M is a compact manifold, the celebrated Poincaré-Hopf Theorem states that $\deg(w, M)$ coincides with the Euler-Poincaré characteristic $\chi(M)$ of M and, therefore, is independent of w.

As in the case of the fixed point index, we list a few of the properties of the degree of a tangent vector field for the purpose of future reference. In what follows U is an open subset of a manifold $M \subseteq \mathbb{R}^k$ and w is a tangent vector field on M.

Additivity *Let w be admissible in U. If U_1 and U_2 are two disjoint open subsets of U whose union contains $w^{-1}(0) \cap U$, then*

$$\deg(w, U) = \deg(w, U_1) + \deg(w, U_2).$$

Homotopy Invariance *Let $h \colon M \times [0,1] \to \mathbb{R}^k$ be a homotopy of tangent vector fields. Assume that h is admissible in an open subset U of M; that is, $h^{-1}(0) \cap (U \times [0,1])$ is compact. Then $\deg\big(h(\cdot, \lambda), U\big)$ is independent of λ.*

Solution. *If w is admissible in U and $\deg(w, U) \neq 0$, then w has a zero in U.*

Excision *Let w be admissible in U. If $V \subseteq U$ is open and contains $w^{-1}(0) \cap U$, then $\deg(w, U) = \deg(w, V)$.*

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2 Poincaré-type translation operator

Let $M \subseteq \mathbb{R}^k$ be a boundaryless differentiable manifold and assume that is a closed subset of $\mathbb{R}^k$. Let $g \colon M \to \mathbb{R}^k$ be a tangent vector field on M and $f \colon \mathbb{R} \times BU\big((-\infty, 0], M\big) \to \mathbb{R}^k$ a functional field. Given $T > 0$, assume also that f is T-periodic in the first variable. We are interested in the T-periodic solutions of equation (1.1).

Let us introduce some notation. Given $T > 0$, and any $X \subseteq \mathbb{R}^k$, $\widetilde{X}$ denotes the metric space $C\big([-T, 0], X\big)$ with the distance inherited from the Banach space $\widetilde{\mathbb{R}}^k = C([-T, 0], \mathbb{R}^k)$ with the usual supremum norm. Notice that $\widetilde{X}$ is complete if and only if X is closed in $\mathbb{R}^k$.

We now define a Poincaré-type T-translation operator Q^λ, $\lambda \in [0, +\infty)$, on $\widetilde{M}$ whose fixed points are the restrictions to the interval $[-T, 0]$ of the T-periodic solutions of (1.1). For this purpose, given $\psi \in \widetilde{M}$, we construct a suitable backward continuous extension $\widehat{\psi}$ of ψ (see Lemma 2.1 below).

Denote by $C_=$ the set $\big\{ \psi \in \widetilde{M} : \psi(-T) = \psi(0) \big\}$.

Lemma 2.1. *There exist an open neighborhood W of $C_=$ in $\widetilde{M}$ and a continuous map from W to $BU\big((-\infty, 0], M\big)$, $\psi \mapsto \widehat{\psi}$, with the following properties:*

1) $\widehat{\psi}$ is an extension of ψ;

2) $\widehat{\psi}$ is T-periodic on the interval $(-\infty, -T]$;

3) $\widehat{\psi}$ is T-periodic on the interval $(-\infty, 0]$, provided that $\psi \in C_=$.

Proof. Take any $\psi \in \widetilde{M}$ and let $\psi^\sim \colon [-2T, -T] \to \mathbb{R}^k$ be defined as $\psi^\sim(s) = \psi(s + T) + a + bs$, where $a = 2(\psi(-T) - \psi(0))$ and $b = (\psi(-T) - \psi(0))/T$ are the (unique) constants such that $\psi^\sim(-2T) = \psi^\sim(-T) = \psi(-T)$. Observe that, given $\psi \in C_=$, the associated map $\psi^\sim$ may not be a curve in M. However, given a tubular neighborhood U of M, there exists an open neighborhood W of $C_=$ in $\widetilde{M}$ such that if $\psi \in W$, then $\psi^\sim$ takes values in U. Now, given $\psi \in W$, define $\widehat{\psi} \colon (-\infty, 0] \to M$ to be the unique extension of ψ which is T-periodic on $(-\infty, -T]$ and coincides with $r \circ \psi^\sim$ in $[-2T, -T]$, where r denotes the canonical retraction associated with the tubular neighborhood U. The defined map $\widehat{\psi}$ clearly belongs to $BU\big((-\infty, 0], M\big)$ and satisfies 1)–3). Moreover, it is easy to verify that the correspondence $\psi \mapsto \widehat{\psi}$ is continuous. □

As above, assume that g is locally Lipschitz and that f satisfies (H), so that uniqueness holds for the maximal solutions of initial value problems associated to (1.1). Let $W \subseteq \widetilde{M}$ be as in Lemma 2.1 and consider the map Q^λ, $\lambda \geq 0$, with domain $\mathcal{D}(Q^\lambda) \subseteq W$, taking values in $\widetilde{M}$ defined by

$$Q^\lambda(\psi)(\theta) = x\big(\lambda, \widehat{\psi}, T + \theta\big), \qquad \theta \in [-T, 0],$$

where $\widehat{\psi}$ denotes the backward extension of ψ as in Lemma 2.1, and $x(\lambda, \eta, \cdot)$ is the unique maximal solution of the initial value problem

$$\begin{cases} x'(t) = g\big(x(t)\big) + \lambda f(t, x_t), & t > 0, \\ x(t) = \eta(t), & t \leq 0. \end{cases} \qquad (2.1)$$

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Notice that a function $\psi \in W$ lies in $\mathcal{D}(Q^\lambda)$ if and only if $t \mapsto x(\lambda, \widehat{\psi}, t)$ is defined up to T. Known properties of functional differential equations (see e.g. [3]) ensure that Q^λ is continuous and that $\mathcal{D}(Q^\lambda)$ is an open subset of W, hence of $C([-T, 0], \mathbb{R}^k)$. Also, it is not difficult to show that Q^λ is a locally compact map due to Ascoli's Theorem.

Proposition 2.2 below asserts that the T-periodic solutions of (1.1) are in a one-to-one correspondence with the fixed points of Q^λ.

Proposition 2.2. *A function $\psi \in \widetilde{M}$ is a fixed point of Q^λ if and only if it is the restriction to $[-T, 0]$ of a T-periodic solution of* (1.1).

Proof. (if) Assume that $t \mapsto y(t)$ is a T-periodic solution of (1.1) and denote by ψ its restriction to the interval $[-T, 0]$. Lemma 2.1 yields $y(t) = \widehat{\psi}(t)$ for $t \le 0$. Thus, one gets $y(t) = x(\lambda, \widehat{\psi}, t)$, for all $t \in \mathbb{R}$. In particular, for $-T \le t \le 0$, we obtain $\psi(t) = y(t) = y(t + T) = x(\lambda, \widehat{\psi}, t + T) = Q^\lambda(\psi)(t)$. That is, $\psi = Q^\lambda(\psi)$.

(only if) Let ψ be a fixed point of Q^λ. This implies that the maximal solution $x(\lambda, \widehat{\psi}, \cdot)$ of (1.1) is defined (at least) up to T and $\psi(t) = x(\lambda, \widehat{\psi}, t + T)$ for all $t \in [-T, 0]$. Thus, $\psi(-T) = x(\lambda, \widehat{\psi}, 0) = \widehat{\psi}(0)$ and, consequently, Lemma 2.1 yields $\psi(-T) = \psi(0)$. It remains to prove that the T-periodic extension $y \colon \mathbb{R} \to M$ of ψ is a solution of (1.1). For $t \in [0, T]$ one has

$$y(t) = y(t - T) = \psi(t - T) = Q^\lambda(\psi)(t - T) = x(\lambda, \widehat{\psi}, t),$$

and Lemma 2.1 implies that $y(t) = \widehat{\psi}(t)$ for $t \le 0$. Hence

$$y'(t) = g\big(y(t)\big) + \lambda f(t, y_t)$$

for all $t \in (0, T]$. The assertion now follows from the T-periodicity of y and the T-periodicity in the first variable of the functional field f. $\qquad\square$

In what follows, the Poincaré-type T-translation operator $Q^0 \colon \mathcal{D}(Q^0) \subseteq \widetilde{M} \to \widetilde{M}$, regarding the unperturbed equation (1.1), will be simply denoted by Q.

We will prove a formula (Theorem 2.4 below) for the computation of the fixed point index of Q in an open subset U of $\mathcal{D}(Q)$, when defined. Clearly, Q is strictly related to the M-valued classical Poincaré map P, given by $P(p) = x(0, \overline{p}, T)$, where $\overline{p} \in C\big((-\infty, 0], M\big)$ is constantly equal to p, and whose domain is the open subset $\mathcal{D}(P)$ of M consisting of those points p such that $x(0, \overline{p}, \cdot)$ is defined up to T.

We shall need the following result of [8] about the fixed point index of P.

Theorem 2.3. *Let g be as above and let $V \subseteq M$ be open and such that $\operatorname{ind}(P, V)$ is defined. Then, $\deg(-g, V)$ is defined as well and*

$$\operatorname{ind}(P, V) = \deg(-g, V). \tag{2.2}$$

Given any $p \in M$, denote by $p^\# \in \widetilde{M}$ the constant function $p^\#(t) \equiv p$ and, for any $U \subseteq M$, define $U^\# = \big\{ p^\# \in \widetilde{M} : p \in U \big\}$. Also, given $V \subseteq \widetilde{M}$, we put $V_\# = \big\{ p \in M : p^\# \in V \big\}$. Observe that, for any given $U \subseteq M$, one has $U^\# \subseteq \widetilde{U}$ and $(\widetilde{U})_\# = U$. There is a simple relation between the domain $\mathcal{D}(Q)$ of Q and

the domain $\mathcal{D}(P)$ of P. In fact $\mathcal{D}(Q) = \{\varphi \in \widetilde{M} : \varphi(0) \in \mathcal{D}(P)\}$. In particular, $\widetilde{\mathcal{D}(P)} \subseteq \mathcal{D}(Q)$. Observe also that $P(p) = Q(p^{\#})(0)$ for all $p \in \mathcal{D}(P)$.

The following result is crucial for the next section. It can be regarded as an infinite dimensional analogue of Theorem 2.3.

Theorem 2.4. *Let g, T and Q be as above, and let $U \subseteq \widetilde{M}$ be open. If the fixed point index $\mathrm{ind}(Q, U)$ is defined, then so is the degree $\deg(-g, U_{\#})$ and*

$$\mathrm{ind}(Q, U) = \deg(-g, U_{\#}). \tag{2.3}$$

Proof. Let us assume that $\mathrm{ind}(Q, U)$ is defined. This means that $U \subseteq \mathcal{D}(Q)$ and that $\mathrm{Fix}(Q, U)$ is compact. Let us show that $\deg(-g, U_{\#})$ is defined too. We need to prove that $g^{-1}(0) \cap U_{\#}$ is compact. If $p \in g^{-1}(0) \cap U_{\#}$, then the constant function $p^{\#}$ is a fixed point of Q. Thus, given a sequence $\{p_n\}_{n \in \mathbb{N}}$ in $g^{-1}(0) \cap U_{\#}$, consider $\{p_n^{\#}\}_{n \in \mathbb{N}}$ in the compact set $\mathrm{Fix}(Q, U)$. By passing to a subsequence, if necessary, we can assume that $p_n^{\#}$ converges to some function ψ in $\mathrm{Fix}(Q, U)$ that, being $p_n^{\#}$ constant, must be constant as well. Hence, for some $p \in M$, $\psi = p^{\#}$. In particular, since $p^{\#} \in U$, we have $p \in U_{\#}$. Clearly, $p_n \to p$, therefore $p \in g^{-1}(0) \cap U_{\#}$. This shows the compactness of $g^{-1}(0) \cap U_{\#}$.

We now use the Commutativity Property of the fixed point index in order to deduce (2.3) for the fixed point index of Q from the analogous formula (2.2) for P. In order to do so, we define the maps $h \colon \mathcal{D}(P) \to \widetilde{M}$ and $k \colon \widetilde{M} \to M$ by $h(p)(\theta) = x(0, \bar{p}, \theta + T)$ and $k(\varphi) = \varphi(0)$, respectively. (Recall that, given $q \in M$, we denote by $\bar{q}$ the function of $C((-\infty, 0], M)$ constantly equal to q.) Since Q stands for Q^0, one has $Q(\varphi) = Q(\overline{\varphi(0)})$. Thus, we have

$$(h \circ k)(\varphi)(\theta) = x\big(0, \overline{\varphi(0)}, \theta + T\big) = Q(\varphi)(\theta), \quad \varphi \in \mathcal{D}(Q), \quad \theta \in [-T, 0]. \tag{2.4}$$

Moreover

$$(k \circ h)(p) = x(0, \bar{p}, T) = P(p), \qquad\qquad p \in \mathcal{D}(P). \tag{2.5}$$

Define $\gamma = k|_U$. By the Commutativity Property of the fixed point index, one has that $\mathrm{ind}\big(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P))\big)$ is defined if and only if so is $\mathrm{ind}\big(\gamma \circ h, h^{-1}(U)\big)$, and

$$\mathrm{ind}\big(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P))\big) = \mathrm{ind}\big(\gamma \circ h, h^{-1}(U)\big). \tag{2.6}$$

Since $U \subseteq \mathcal{D}(Q)$, then $\gamma^{-1}(\mathcal{D}(P))$ is the whole domain U of γ. Hence, from formulas (2.4)–(2.5), it follows that

$$\mathrm{ind}(Q, U) = \mathrm{ind}\big(h \circ \gamma, \gamma^{-1}(\mathcal{D}(P))\big),$$
$$\mathrm{ind}\big(P, h^{-1}(U)\big) = \mathrm{ind}\big(\gamma \circ h, h^{-1}(U)\big).$$

Thus, by (2.6), we get

$$\mathrm{ind}(Q, U) = \mathrm{ind}\big(P, h^{-1}(U)\big). \tag{2.7}$$

From (2.2) we obtain

$$\mathrm{ind}\big(P, h^{-1}(U)\big) = \deg\big(-g, h^{-1}(U)\big). \tag{2.8}$$

From the definition of h it follows immediately that

$$g^{-1}(0) \cap U_\# = g^{-1}(0) \cap h^{-1}(U),$$

even though the sets $U_\#$ and $h^{-1}(U)$ can be different. Therefore, from the Excision Property of the degree of a vector field, one has

$$\deg\big(-g, h^{-1}(U)\big) = \deg(-g, U_\#), \tag{2.9}$$

and the assertion follows from equations (2.7), (2.8) and (2.9). $\qquad\square$

Let $U \subseteq \mathcal{D}(Q)$ be open in $\widetilde{M}$. We point out that Theorems 2.3 and 2.4 imply that the fixed point index of Q in U actually reduces to the fixed point index of the finite dimensional operator P in $U_\#$. Namely,

$$\mathrm{ind}(Q, U) = \mathrm{ind}(P, U_\#). \tag{2.10}$$

In fact, P is defined on $U_\#$ and $\mathrm{Fix}(P, U_\#)$ can be regarded as a closed subset of $\mathrm{Fix}(Q, U)$. Therefore, if $\mathrm{Fix}(Q, U)$ is compact, then so is $\mathrm{Fix}(P, U_\#)$.

Observe that the above formula (2.10) is more convenient than the reduction formula (2.7) obtained in the proof of Theorem 2.4. In fact, unlike the set $h^{-1}(U)$ that appears in (2.7), $U_\#$ does not depend on the equation (1.1).

3 Branches of starting pairs

Any pair $(\lambda, \varphi) \in [0, +\infty) \times \widetilde{M}$ is said to be a *starting pair* (for (1.1)) if the following initial value problem has a T-periodic solution:

$$\begin{cases} x'(t) = g\big(x(t)\big) + \lambda f(t, x_t), & t > 0, \\ x(t) = \widehat{\varphi}(t), & t \le 0. \end{cases} \tag{3.1}$$

A pair of the type $(0, p^\#)$ with $g(p) = 0$ is clearly a starting pair and will be called a *trivial starting pair*. Also, we will denote by $\mathcal{V}$ the open set of all pairs $(\lambda, \varphi) \in [0, +\infty) \times \widetilde{M}$ such that the corresponding solution of (3.1) is defined up to time T.

In the sequel, given $A \subseteq \mathbb{R} \times \widetilde{M}$ and $\lambda \in \mathbb{R}$, the symbol A_λ will denote the slice $\{x \in \widetilde{M} : (\lambda, x) \in A\}$. Observe that $(\mathcal{V}_0)_\# = \mathcal{D}(P)$, where P is the classical Poincaré operator defined in the previous section.

We will need the following global connectivity result of [5].

Lemma 3.1. *Let Y be a locally compact metric space and let Z be a compact subset of Y. Assume that any compact subset of Y containing Z has nonempty boundary. Then $Y \setminus Z$ contains a connected set whose closure (in Y) intersects Z and is not compact.*

Proposition 3.2. *Assume that the vector field g is locally Lipschitz and that the functional field f satisfies assumption (H). Let S be the set of all starting pairs for (1.1). Given $W \subseteq [0, +\infty) \times \widetilde{M}$ open, if $\deg\big(g, (W_0)_\#\big)$ is (defined and) nonzero, then the set*

$$(S \cap W) \setminus \big\{(0, p^\#) \in W : g(p) = 0\big\}$$

of nontrivial starting pairs in W admits a connected subset whose closure in $S \cap W$ meets $\big\{(0, p^\#) \in W : g(p) = 0\big\}$ and is not compact.

Proof. Let us define the open set $U = W \cap \mathcal{V}$. Since

$$g^{-1}(0) \cap (U_0)_\# = g^{-1}(0) \cap (W_0)_\#, \qquad (3.2)$$

and $S \cap U = S \cap W$, we need to prove that the set of nontrivial starting pairs in U admits a connected subset whose closure in $S \cap U$ meets $\{(0, p^\#) \in U : g(p) = 0\}$ and is not compact.

Notice that because of Ascoli's Theorem, S is locally totally bounded. Hence, since $\widetilde{M}$ is complete, S is locally compact. Thus, U being open, $S \cap U$ is locally compact as well. Moreover the assumption that $\deg\big(g, (W_0)_\#\big)$ is defined means that the set $\{p \in (W_0)_\# : g(p) = 0\}$ is compact. Thus, by (3.2), so is $\{(0, p^\#) \in U : g(p) = 0\}$ being homeomorphic to $g^{-1}(0) \cap (U_0)_\#$.

The assertion follows applying Lemma 3.1 to the pair

$$(Y, Z) = \big(S \cap U, \{(0, p^\#) \in U : p \in g^{-1}(0)\}\big).$$

Assume, by contradiction, that there exists a compact subset C of the set $S \cap U$ containing $\{(0, p^\#) \in U : p \in g^{-1}(0)\}$ and with empty boundary in $S \cap U$. Thus C is a relatively open subset of $S \cap U$. As a consequence, $(S \cap U) \setminus C$ is closed in $S \cap U$. So the distance, $\delta = \mathrm{dist}\big(C, (S \cap U) \setminus C\big)$, between C and $(S \cap U) \setminus C$ is nonzero (recall that C is compact). Consider the open set

$$A = \big\{(\lambda, \varphi) \in U : \mathrm{dist}\big((\lambda, \varphi), C\big) < \delta/2\big\},$$

which, clearly, does not meet $(S \cap U) \setminus C$.

Since A is bounded, there exists $\overline{\lambda} > 0$ such that $A_{\overline{\lambda}} = \emptyset$. Moreover, because of Proposition 2.2, the set $\{(\lambda, \varphi) \in A : Q^\lambda(\varphi) = \varphi\}$ coincides with $C = S \cap A$ which is compact. Then, from the Generalized Homotopy Invariance Property of the fixed point index,

$$0 = \mathrm{ind}\big(Q^{\overline{\lambda}}, A_{\overline{\lambda}}\big) = \mathrm{ind}\big(Q^\lambda, A_\lambda\big),$$

for all $\lambda \in [0, \overline{\lambda}]$. But, by (2.3) and by the Excision Property of the degree, we get

$$\mathrm{ind}(Q, A_0) = \deg(-g, (A_0)_\#) = \deg(-g, (W_0)_\#) \neq 0,$$

which contradicts the previous formula. $\qquad\square$

4 Branches of T-periodic pairs

Let us introduce the function space where most of the work of this section is done. We will denote by $C_T(M)$ the set of the T-periodic continuous maps from $\mathbb{R}$ into M. This will be regarded as a metric subspace of the Banach space $\big(C_T(\mathbb{R}^k), \|\cdot\|\big)$ of the T-periodic continuous maps from $\mathbb{R}$ into $\mathbb{R}^k$ with the usual supremum norm. Since in this paper M is supposed closed in $\mathbb{R}^k$, $C_T(M)$ is complete.

For the sake of simplicity, we will identify M with its image in $[0, +\infty) \times C_T(M)$ under the embedding which associates to any $p \in M$ the pair $(0, \overline{p})$, $\overline{p} \in C_T(M)$ being the map constantly equal to p. According to this identification, if E is a subset of $[0, +\infty) \times C_T(M)$, by $E \cap M$ we mean the subset of M given by all $p \in M$ such that the pair $(0, \overline{p})$ belongs to E. Observe that if $\Omega \subseteq [0, +\infty) \times C_T(M)$ is open, then so is $\Omega \cap M$.

A pair $(\lambda, x) \in [0, +\infty) \times C_T(M)$, where x is a solution of (1.1), is called a *T-periodic pair* (for (1.1)). Those T-periodic pairs that are of the particular form $(0, \bar{p})$ are said to be *trivial*. Observe that $(0, \bar{p}) \in [0, +\infty) \times C_T(M)$ is a trivial T-periodic pair if and only if $g(p) = 0$. We point out that if x is a nonconstant T-periodic solution of the unperturbed equation $x'(t) = g(x(t))$, then $(0, x)$ is a nontrivial T-periodic pair.

Theorem 4.1. *Let $g \colon M \to \mathbb{R}^k$ be a locally Lipschitz tangent vector field on M and $f \colon \mathbb{R} \times BU((-\infty, 0], M) \to \mathbb{R}^k$ a functional field T-periodic in the first variable satisfying (H). Let Ω be an open subset of $[0, +\infty) \times C_T(M)$, and assume that $\deg(g, \Omega \cap M)$ is defined and nonzero. Then Ω contains a connected set of nontrivial T-periodic pairs whose closure in Ω is not compact and meets the set $\{(0, \bar{p}) \in \Omega : g(p) = 0\}$. In particular, the set of T-periodic pairs for (1.1) contains a connected component that meets $\{(0, \bar{p}) \in \Omega : g(p) = 0\}$ and whose intersection with Ω is not compact.*

Proof. Denote by X the set of T-periodic pairs of (1.1) and by S the set of starting pairs of the same equation.

Define the map $h \colon X \to S$ by $h(\lambda, x) = (\lambda, x|_{[-T, 0]})$ and observe that h is continuous, onto and, since the functional field

$$(t, \varphi) \mapsto g(\varphi(0)) + \lambda f(t, \varphi)$$

satisfies (H), it is also one to one. Furthermore, by the continuous dependence on data, h is actually a homeomorphism. Thus, $h(\Omega \cap X)$ is an open subset of S and, consequently, we can find an open subset W of $[0, +\infty) \times \widetilde{M}$ such that $S \cap W = h(\Omega \cap X)$. This implies

$$\begin{aligned} \{p \in (W_0)_\# : g(p) = 0\} &= \{p \in M : (0, p^\#) \in W,\ g(p) = 0\} \\ &= \{p \in M : (0, \bar{p}) \in \Omega,\ g(p) = 0\} = \{p \in \Omega \cap M : g(p) = 0\}. \end{aligned}$$

Thus, by excision, one has $\deg(g, (W_0)_\#) = \deg(g, \Omega \cap M) \neq 0$. Applying Proposition 3.2, we get the existence of a connected set

$$\Sigma \subseteq (S \cap W) \setminus \{(0, p^\#) \in W : g(p) = 0\}$$

whose closure in $S \cap W$ meets $\{(0, p^\#) \in W : g(p) = 0\}$ and is not compact.

Observe that the trivial T-periodic pairs correspond to the trivial starting pairs under the homeomorphism h. Thus, $\Xi = h^{-1}(\Sigma) \subseteq X \cap \Omega$ is a connected set of nontrivial T-periodic pairs whose closure in $X \cap \Omega$ is not compact and meets $\{(0, \bar{p}) \in \Omega : g(p) = 0\}$. Since X is closed in $[0, +\infty) \times C_T(M)$, the closures of Ξ in $X \cap \Omega$ and in Ω coincide. This proves that Ξ satisfies the requirements of the first part of the assertion.

Let us prove the last part of the assertion. Consider the connected component Γ of X that contains the connected set Ξ. We shall now show that Γ has the required properties. Clearly, Γ meets the set $\{(0, \bar{p}) \in \Omega : g(p) = 0\}$ because the closure of Ξ in Ω does. Moreover, $\Gamma \cap \Omega$ cannot be compact, since it contains the closure of Ξ in Ω which is not compact. $\qquad\square$

Remark 4.2. *Let Ω be as in Theorem 4.1, and assume that Γ is a connected component of T-periodic pairs of (1.1) that meets $\{(0,\overline{p}) \in \Omega : g(p) = 0\}$ and whose intersection with Ω is not compact. Ascoli's Theorem implies that any bounded set of T-periodic pairs is relatively compact. Then, the closed set Γ cannot be both bounded and contained in Ω. In particular, if Ω is bounded, then $\Gamma \cap \partial\Omega \neq \emptyset$.*

To better understand the meaning of Theorem 4.1, consider for example the case when $M = \mathbb{R}^m$. If $g^{-1}(0)$ is compact and $\deg(g, \mathbb{R}^m) \neq 0$, then there exists an unbounded connected set of T-periodic pairs in $[0, +\infty) \times C_T(\mathbb{R}^m)$ which meets the set $\{(0, \overline{p}) \in [0, +\infty) \times C_T(M) : g(p) = 0\}$, that can be identified with $g^{-1}(0)$. The existence of this unbounded connected set cannot be destroyed by a particular choice of f. However it is possibly "completely vertical", i.e. contained in the slice $\{0\} \times C_T(M)$. This peculiarity is exhibited, for instance, by the set of T-periodic pairs of the equation

$$\begin{cases} x' = y, \\ y' = -x + \lambda \sin t, \end{cases}$$

where $M = \mathbb{R}^2$ and $T = 2\pi$.

A somewhat opposite behavior is shown by the set X of T-periodic pairs for (1.1) in the "degenerate" situation when $f(t, \varphi) \equiv 0$. In this case, X consists of the pairs (λ, x), where $\lambda \geq 0$ and x is a T-periodic solution to $x' = g(x)$. In particular, the "horizontal" set $(0, +\infty) \times \{\overline{p}\}$ satisfies the requirement of Theorem 4.1.

The following corollary, in the case of a compact boundaryless manifolds, is in the spirit of a result of [1] in which g is identically zero.

Corollary 4.3. *Let f and g be as in Theorem 4.1 and let $M \subseteq \mathbb{R}^k$ be compact with nonzero Euler-Poincaré characteristic $\chi(M)$. Then, there exists an unbounded connected set of nontrivial T-periodic pairs whose closure meets $\{(0, \overline{p}) \in [0, +\infty) \times C_T(M) : g(p) = 0\}$. In particular, equation (1.1) has a T-periodic solution for any $\lambda \geq 0$.*

Proof. The Poincaré-Hopf Theorem yields $\deg(g, M) = \chi(M) \neq 0$. Thus, Theorem 4.1 and Remark 4.2 imply the existence of an unbounded connected set Γ of nontrivial T-periodic pairs whose closure in $[0, +\infty) \times C_T(M)$ meets $\{(0, \overline{p}) \in [0, +\infty) \times C_T(M) : g(p) = 0\}$. The last assertion follows from the fact that $C_T(M)$ is bounded while Γ is unbounded. $\square$

The following corollary ensures the existence of a Rabinowitz-type branch of T-periodic pairs.

Corollary 4.4. *Let f and g be as in Theorem 4.1. Let $U \subseteq M$ be open and assume that $\deg(g, U)$ is well defined and nonzero. Then, there exists a connected component Γ of T-periodic pairs of (1.1) that meets the set*

$$\left\{ (0, \overline{p}) \in [0, +\infty) \times C_T(M) : p \in U, \ g(p) = 0 \right\}$$

and is either unbounded or meets

$$\left\{ (0, \overline{p}) \in [0, +\infty) \times C_T(M) : p \in M \setminus U, \ g(p) = 0 \right\}.$$

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Proof. Consider the open subset Ω of $[0, +\infty) \times C_T(M)$ given by

$$\Omega = \big([0, +\infty) \times C_T(M)\big) \setminus \big\{(0, \overline{p}) \in [0, +\infty) \times C_T(M) : p \in M \setminus U, \ g(p) = 0\big\}.$$

Clearly, we have $\Omega \cap M = U$. Hence $\deg(g, \Omega \cap M) \neq 0$. Theorem 4.1 implies the existence of a connected component Γ of T-periodic pairs of (1.1) that meets $\{(0, \overline{p}) \in \Omega : g(p) = 0\}$ and whose intersection with Ω is not compact. Because of Remark 4.2, if Γ is bounded, then it meets

$$\partial\Omega = \big\{(0, \overline{p}) \in [0, +\infty) \times C_T(M) : p \in M \setminus U, \ g(p) = 0\big\},$$

as claimed. $\qquad\qquad\square$

Example 4.5. *Consider a simple pendulum lying in a vertical plane and acted on by a bounded, continuous, T-periodic force λf, $\lambda \geq 0$, depending possibly on the whole history of the pendulum's motion. The behavior of this pendulum is described by a second order RFDE on the unit circle S^1 or, equivalently, by a first order one on the tangent bundle $TS^1 \simeq S^1 \times \mathbb{R}$.*

The unperturbed pendulum (i.e. corresponding to $\lambda = 0$) has just two equilibria: the upper one, denoted by N, which is unstable, and the lower one, S, which is stable. These points can be identified with the sole two trivial T-periodic pairs $(0; \overline{N}, 0)$ and $(0; \overline{S}, 0)$ in $[0, +\infty) \times C_T(S^1 \times \mathbb{R})$. Considerations analogous to those in the proof of Corollary 4.4 show that there exist connected components C_{N}, C_{S} of T-periodic pairs for the first order RFDE, containing N and S, respectively, that either coincide or are both unbounded. Since the force acting on the pendulum is bounded by assumption, an argument as in Theorem 4.1 in [9], based on the local constance of the winding number, allows us to conclude that any forced oscillation corresponding to a T-periodic pair belonging to C_{N} or C_{S} must have a speed that is bounded by a function of λ growing at most linearly. This shows that, actually, at least one of the following two possibilities must happen:

1. *$C_{\mathsf{N}} = C_{\mathsf{S}}$;*

2. *There are at least two forced oscillations for each $\lambda \geq 0$.*

If for some $\lambda \geq 0$ there are no forced oscillations, then C_{N} coincides with C_{S} and the same argument based on the winding number yields that this branch is bounded. In any case, arguing as in Corollary 4.2 of [9], for sufficiently small $\lambda \geq 0$ there are at least two forced oscillations.

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