

Corrigendum to on a class of differential-algebraic equations with infinite delay

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Abstract

We present here a corrected version of Lemma 5.5 and Corollary 5.7 of EJTDE 2011, No. 81.

1 Introduction

In Section 5 of [1] we investigated examples of applications of that paper's results to a particular class of implicit differential equations. For so doing we used a technical lemma from linear algebra that, unfortunately, turns out to be flawed. As briefly discussed below this affects only marginally our paper's results (just a corollary in Section 5 of [1]).

The simple example below shows that there is something wrong with Lemma 5.5 in [1]. In the next section we provide an amended version of this result.

Example 1.1. *Consider the matrices*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, $\ker C^T = \ker E^T = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ for all $t \in \mathbb{R}$. The matrices

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

realize a singular value decomposition for E . Nevertheless

$$P^T C Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which is not the form expected from Lemma 5.5 in [1]. The problem, as it turns out, is that $\ker C \neq \ker E$.

2000 Mathematics Subject Classifications: 34A09, 34K13, 34C40

Keywords and phrases: differential-algebraic equations, retarded functional equations, periodic solution, periodic perturbation, infinite delay

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Luckily, the impact of the wrong statement of [1, Lemma 5.5] on [1] is minor: all results and examples (besides Lemma 5.5, of course) remain correct, with the exception of Corollary 5.7 where it is necessary to assume the following further hypothesis:

$$\ker C(t) = \ker E, \quad \forall t \in \mathbb{R}.$$

(A corrected statement of Corollary 5.7 of [1] can be found in the next section, Corollary 2.2.)

2 Corrected Lemma and its consequences

We present here a corrected version of Lemma 5.5 in [1].

Lemma 2.1. *Let $E \in \mathbb{R}^{n \times n}$ and $C \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ be respectively a matrix and a matrix-valued function such that*

$$\ker C^T(t) = \ker E^T, \quad \forall t \in \mathbb{R}, \quad \text{and } \dim \ker E^T > 0, \quad (2.1)$$

Put $r = \text{rank } E$, and let $P, Q \in \mathbb{R}^{n \times n}$ be orthogonal matrices that realize a singular value decomposition for E . Then it follows that

$$P^T C(t) Q = \begin{pmatrix} \tilde{C}_{11}(t) & \tilde{C}_{12}(t) \\ 0 & 0 \end{pmatrix}, \quad \forall t \in \mathbb{R}, \quad (2.2)$$

with $\tilde{C}_{11} \in C(\mathbb{R}, \mathbb{R}^{r \times r})$ and $\tilde{C}_{12} \in C(\mathbb{R}, \mathbb{R}^{r \times n})$.

If, furthermore,

$$\ker C(t) = \ker E, \quad \forall t \in \mathbb{R}, \quad (2.3)$$

then $\tilde{C}_{12}(t) \equiv 0$. Namely, in this case,

$$P^T C(t) Q = \begin{pmatrix} \tilde{C}_{11}(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall t \in \mathbb{R}, \quad (2.4)$$

with $\tilde{C}_{11}(t)$ nonsingular for all $t \in \mathbb{R}$.

Proof. Our proof is essentially a singular value decomposition (see, e.g., [2]) argument, based on a technical result from [3].

Observe that (2.1) imply $\text{rank } E = \text{rank } C(t) = r > 0$ for all $t \in \mathbb{R}$. In fact,

$$\begin{aligned} \text{rank } E &= \text{rank } E^T = n - \dim \ker E^T = \\ &= n - \dim \ker C(t)^T = \text{rank } C(t)^T = \text{rank } C(t). \end{aligned}$$

Since $\text{rank } C(t)$ is constantly equal to $r > 0$, by inspection of the proof of Theorem 3.9 of [3, Chapter 3, §1] we get the existence of orthogonal matrix-valued functions $U, V \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ and $C_r \in C(\mathbb{R}, \mathbb{R}^{r \times r})$ such that, for all $t \in \mathbb{R}$, $\det C_r(t) \neq 0$ and

$$U^T(t) C(t) V(t) = \begin{pmatrix} C_r(t) & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.5)$$

Let $U_r, V_r \in C(\mathbb{R}, \mathbb{R}^{n \times r})$ and $U_0, V_0 \in C(\mathbb{R}, \mathbb{R}^{n \times (n-r)})$ be matrix-valued functions formed, respectively, by the first r and $n - r$ columns of U and V . An

argument involving Equation (2.5) shows that, for all $t \in \mathbb{R}$, the space $\text{im } C(t)$ is spanned by the columns of $U_r(t)$. Also, (2.5) imply that the columns of $V_0(t)$, $t \in \mathbb{R}$, belong to $\ker C(t)$ for all $t \in \mathbb{R}$. A dimensional argument shows that they constitute a basis $\ker C(t)$. Analogously, transposing (2.5), we see that the columns of $V_r(t)$ and $U_0(t)$ are bases of $\text{im } C(t)^T$ and $\ker C(t)^T$ respectively.¹

Let now P_r, Q_r and P_0, Q_0 be the matrices formed taking the first r and $n - r$ columns of P and Q , respectively. Since P and Q realize a singular value decomposition of E , proceeding as above one can check that the columns of P_r, Q_r, P_0 and Q_0 span $\text{im } E, \text{im } E^T, \ker E^T$, and $\ker E$, respectively.

We claim that $P_0^T U_r(t)$ is constantly the null matrix in $\mathbb{R}^{(n-r) \times r}$. To prove this, it is enough to show that for all $t \in \mathbb{R}$, the columns of P_0 are orthogonal to those of $U_r(t)$. Let v and $u(t)$, $t \in \mathbb{R}$, be any column of P_0 and of $U_r(t)$, respectively. Since for all $t \in \mathbb{R}$ the columns of $U_r(t)$ are in $\text{im } C(t)$, there is a vector $w(t) \in \mathbb{R}^n$ with the property that $u(t) = C(t)w(t)$, and

$$\langle v, u(t) \rangle = \langle v, C(t)w(t) \rangle = \langle C(t)^T v, w(t) \rangle = 0, \quad \forall t \in \mathbb{R},$$

because $v \in \ker E^T = \ker C(t)^T$ for all $t \in \mathbb{R}$. This proves the claim. A similar argument shows that $P_r^T U_0(t)$ is identically zero as well.

Since for all $t \in \mathbb{R}$

$$P^T U(t) = \begin{pmatrix} P_r^T U_r(t) & 0 \\ 0 & P_0^T U_0(t) \end{pmatrix}$$

is nonsingular, we deduce in particular that so is $P_r^T U_r(t)$.

Let us compute the matrix product $P^T C(t)Q$ for all $t \in \mathbb{R}$. We omit here, for the sake of simplicity, the explicit dependence on t .

$$\begin{aligned} P^T C Q &= P^T U U^T C V V^T Q = \begin{pmatrix} P_r^T U_r & 0 \\ 0 & P_0^T U_0 \end{pmatrix} \begin{pmatrix} C_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_r^T Q_r & V_r^T Q_0 \\ V_0^T Q_r & V_0^T Q_0 \end{pmatrix} \\ &= \begin{pmatrix} P_r^T U_r C_r V_r^T Q_r & P_r^T U_r C_r V_r^T Q_0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

which proves (2.2).

Let us now assume that also (2.3) holds. We claim that in this case $V_0^T Q_r$ is identically zero. To see this we proceed as done above for the products $P_0^T U_r$ and $P_r^T U_0$. Let $v(t)$, $t \in \mathbb{R}$, be any column of $V_0(t)$, hence a vector of $\ker C(t)$ for all $t \in \mathbb{R}$, and let q be a column of $Q_r(t)$. Since the columns of Q_r lie in $\text{im } E^T$, there is a vector $\ell \in \mathbb{R}^n$ with the property that $q = E^T \ell$, and

$$\langle v(t), q \rangle = \langle v(t), E^T \ell \rangle = \langle E v(t), \ell \rangle = 0, \quad \forall t \in \mathbb{R},$$

because $v(t) \in \ker C(t) = \ker E$ for all $t \in \mathbb{R}$. This proves the claim. A similar argument shows that $V_r^T Q_0(t)$ is identically zero as well. Hence,

$$V(t)^T Q = \begin{pmatrix} V_r(t)^T Q_r & 0 \\ 0 & V_0(t)^T Q_0 \end{pmatrix}$$

¹In fact, the orthogonality of the matrices $V(t)$ and $U(t)$ for all $t \in \mathbb{R}$, imply that the columns of $U_r(t), V_r(t), U_0(t)$ and $V_0(t)$ are respective orthogonal bases of the spaces $\text{im } C(t), \text{im } C(t)^T, \ker C(t)^T$ and $\ker C(t)$.

thus $V_r^T(t)Q_0$, and $V_0^T(t)Q_r$ are nonsingular. Also, plugging $V_0^T Q_r = 0$ in the above expression for $P^T C Q$ one gets (we omit again the explicit dependence on t)

$$P^T C Q = \begin{pmatrix} P_r^T U_r C_r V_r^T Q_r & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.6)$$

Which proves the assertion because $P_r^T U_r$, C_r , and $V_r^T Q_r$ are nonsingular. \square

In view of the corrected version of the above lemma, the statement of Corollary 5.7 of [1] can be rewritten as follows:

Corollary 2.2. *Consider Equation*

$$E\dot{\mathbf{x}}(t) = \mathcal{F}(\mathbf{x}(t)) + \lambda C(t)S(\mathbf{x}_t), \quad (2.7)$$

where the maps $C: \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ and $S: BU((-\infty, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ are continuous, E is a (constant) $n \times n$ matrix, \mathcal{F} is locally Lipschitz and S verifies condition (\mathbf{K}) in [1]. Suppose also that C and E satisfy (2.1) and (2.3), and that C is T -periodic. Let $r > 0$ be the rank of E and assume that there exists an orthogonal basis of $\mathbb{R}^n \simeq \mathbb{R}^r \times \mathbb{R}^{n-r}$ such that E has the form

$$E \simeq \begin{pmatrix} E_{11} & E_{12} \\ 0 & 0 \end{pmatrix}, \text{ with } E_{11} \in \mathbb{R}^{r \times r} \text{ invertible and } E_{12} \in \mathbb{R}^{r \times (n-r)}.$$

Assume also that, relatively to this decomposition of \mathbb{R}^n , $\partial_2 \mathcal{F}_2(\xi, \eta)$ is invertible for all $x = (\xi, \eta) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$.

Let Ω be an open subset of $[0, +\infty) \times C_T(\mathbb{R}^n)$ and suppose that $\deg(\mathcal{F}, \Omega \cap \mathbb{R}^n)$ is well-defined and nonzero. Then, there exists a connected subset Γ of nontrivial T -periodic pairs for (2.7) whose closure in Ω is noncompact and meets the set $\{(0, \bar{\mathbf{p}}) \in \Omega : \mathcal{F}(\bar{\mathbf{p}}) = 0\}$.

This result follows as in [1] taking into account the modified version of the lemma.

References

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