ON A CLASS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH INFINITE DELAY

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Abstract. We study the set of T-periodic solutions of a class of T-periodically perturbed Differential-Algebraic Equations, allowing the perturbation to contain a distributed and possibly infinite delay. Under suitable assumptions, the perturbed equations are equivalent to Retarded Functional (Ordinary) Differential Equations on a manifold. Our study is based on known results about the latter class of equations.

1. Introduction

This paper is devoted to the study of some properties of the set of harmonic solutions to retarded functional periodic perturbations of differential-algebraic equations (DAEs) of a particular type. The results we obtain are mainly related, on one hand, with those of [7] concerning the method used to deal with distributed and possibly infinite delay and, on the other hand, with [5, 14] as regards the treatment of DAEs.

Let $g\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^s$ and $f\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^k$ be given. Assume f continuous and $g \in C^\infty(\mathbb{R}^k \times \mathbb{R}^s, \mathbb{R}^s)$ has the property that $\partial_2 g(p,q)$, the partial derivative of g with respect to the second variable, is invertible for any $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s \cong \mathbb{R}^n$. We consider the following DAE in semi-explicit form:

$$\begin{cases} \dot{x} = f(x,y), \\ g(x,y) = 0, \end{cases} \tag{1.1}$$

and perturb it as follows:

$$\begin{cases} \dot{x} = f(x,y) + \lambda h(t, x_t, y_t), \\ g(x,y) = 0, \end{cases} \tag{1.2}$$

where $h\colon \mathbb{R} \times BU((-\infty,0], \mathbb{R}^k \times \mathbb{R}^s) \to \mathbb{R}^k$ is continuous and T-periodic, $T > 0$ given, in the first variable. The resulting equation (1.2) is an example of *Retarded Functional Differential-Algebraic Equation* (RFDAE).

For $\lambda \geq 0$, we are interested in the T-periodic solutions of (1.2) where, given $t \in \mathbb{R}$, we adopt the notation $x_t, y_t\colon (-\infty, 0] \to \mathbb{R}^k \times \mathbb{R}^s$ for the maps $x_t\colon \theta \mapsto x_t(\theta) := x(t+\theta)$ and $\theta \mapsto y_t(\theta) := y(t+\theta)$. Since $\partial_2 g(p,q)$ is invertible for any $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s$, $0 \in \mathbb{R}^s$ is a regular value of g, and so $M := g^{-1}(0)$ is a C^∞ manifold and a closed subset of $\mathbb{R}^k \times \mathbb{R}^s \cong \mathbb{R}^n$. The latter fact is important as we wish to use the results of [7] that depend in an essential manner on M being closed.

The Implicit Function Theorem imply that M can be locally represented as a graph of some map from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^s$. Thus, in principle, equation (1.2) can be locally decoupled. Globally, however, this might be not the case or it could not be convenient to do so (see, e.g. [5, 14]).

As we shall see, proceeding as in [11, §4.5] (compare also [14]) when $\partial_2 g(p,q)$ is invertible for all $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s$, equation (1.2) is equivalent to a retarded functional differential equation (RFDE) on M of the form considered in [7]. Some

Date: April 1, 2011.

related ideas, in the context of constrained mechanical systems, can be found in [13]. In order to obtain information about the set of T-periodic solutions of (1.2), we will use the techniques of [7] combined with a result of [14] about the degree of the tangent vector field on M induced by the unperturbed equation (1.1). Our aim will be to show the existence of a noncompact 'branch' of T-periodic solutions of (1.2) emanating from the set of the constants solutions of (1.1). Namely, denote by $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ the Banach space of the T-periodic, $\mathbb{R}^k \times \mathbb{R}^s$-valued functions, we will prove the existence of a noncompact connected set of triples $(\lambda, \xi) \in [0, \infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s)$, with ξ a T-periodic solution to (1.2), whose closure meets the set of the constant solutions of (1.1).

In the last section of this paper, in order to illustrate our results we provide some applications to a particular class of implicit retarded functional differential equations.

2. Associated vector Fields and RFDEs on M

In this section, following [11, Chapter 4, §5] (compare also [14]), we associate to (1.2) a RFDE on $M = g^{-1}(0)$.

We first discuss the notion of solution to a retarded functional DAE of the form (1.2). Let $f\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^k$ and $g\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^s$ be given maps with f continuous and $g \in C^\infty(\mathbb{R}^k \times \mathbb{R}^s, \mathbb{R}^s)$ has the property that $\partial_2 g(p,q)$ is invertible for any $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s \cong \mathbb{R}^n$. Given $T > 0$, consider also continuous and T-periodic in the first variable map $h\colon \mathbb{R} \times BU((-\infty, 0], \mathbb{R}^k \times \mathbb{R}^s) \to \mathbb{R}^k$. A solution of (1.2), for a given $\lambda \geq 0$ consists of a pair of functions $x \in AC(I, \mathbb{R}^k)$ and $y \in C(I, \mathbb{R}^s)$, $I \subseteq \mathbb{R}$ an interval with $\inf I = -\infty$, such that

$$g\big(x(t), y(t)\big) = 0, \qquad \text{for all } t \in I, \tag{2.1a}$$

and,

$$\dot{x}(t) = f\big(x(t), y(t)\big) + \lambda h(t, x_t, y_t), \tag{2.1b}$$

eventually. In the sense that there exists a subinterval $J \subseteq I$ with $\sup J = \sup I$ on which (2.1b) holds. Observe that, by the Implicit Function Theorem, y has the same regularity as x. Therefore, a solution of (1.2) is an absolutely continuous function $\zeta := (x, y)$ which actually is an eventually C^1 function, i.e., $\zeta \in C^1(J, \mathbb{R}^k \times \mathbb{R}^s)$.

Let us now associate tangent vector fields on $M = g^{-1}(0)$ to f and h. Recall that a continuous map $w\colon M \to \mathbb{R}^n$ with the property that for any $p \in M$, $w(p)$ belongs to the tangent space T_pM to M at p is called a *tangent vector field on M*. Similarly, a time-dependent *functional (tangent vector) field on M* is a map $W\colon \mathbb{R} \times BU\big((-\infty, 0], M\big) \to \mathbb{R}^k \times \mathbb{R}^s$, such that $W(t, \varphi, \psi) \in T_{(\varphi(0), \psi(0))}M$, for all $(t, \varphi, \psi) \in \mathbb{R} \times BU\big((-\infty, 0], M\big)$.

Consider the maps $\Psi\colon M \to \mathbb{R}^k \times \mathbb{R}^s$ and $\Upsilon\colon \mathbb{R} \times BU\big((-\infty, 0], M\big) \to \mathbb{R}^k \times \mathbb{R}^s$ defined as follows:

$$\Psi(p,q) = \big(f(p,q), [\partial_2 g(p,q)]^{-1} \partial_1 g(p,q) f(p,q)\big), \text{ and} \tag{2.2a}$$

$$\Upsilon(t, \varphi, \psi) = \big(h(t, \varphi, \psi), -[\partial_2 g(\varphi(0), \psi(0))]^{-1} \partial_1 g(\varphi(0), \psi(0)) h(t, \varphi, \psi)\big). \tag{2.2b}$$

Using the fact that, given a point $(p,q) \in M$, $T_{(p,q)}M$ is the kernel $\operatorname{Ker} d_{(p,q)}g$ of the differential $d_{(p,q)}g$ of g at (p,q), it can be easily proved that Ψ is tangent to M in the sense that $\Psi(p,q)$ belongs to $T_{(p,q)}M$ for all $(p,q) \in M$ (compare, e.g. [14]). Similarly, we have that Υ is tangent to M, in the sense that $\Upsilon(t, \varphi, \psi) \in T_{(\varphi(0), \psi(0))}M$, for all $(t, \varphi, \psi) \in \mathbb{R} \times BU\big((-\infty, 0], M\big)$. In other words, we see that Ψ is a tangent vector field, whereas Υ is a time-dependent functional field on M. Since h is assumed T-periodic in the first variable, so is Υ. Notice that, for any

$\lambda \geq 0$, the map of $\mathbb{R} \times BU\big((-\infty, 0], M\big)$ in $\mathbb{R}^k \times \mathbb{R}^s$, defined by

$$(t, \varphi, \psi) \mapsto \Psi\big(\varphi(0), \psi(0)\big) + \lambda \Upsilon(t, \varphi, \psi),$$

is a functional tangent vector field as well.

We claim that (1.2) is equivalent to the following RFDAE on M, which keep implicitly account of the algebraic condition $g(x, y) = 0$:

$$\dot{\zeta} = \Psi(\zeta) + \lambda \Upsilon(t, \zeta_t), \tag{2.3}$$

where we have used the compact notation $\zeta_t = (x_t, y_t)$, in the sense that $\zeta = (x, y)$ is a solution of (2.3) in an interval $I \subseteq \mathbb{R}$ if and only if so is (x, y) for (1.2). To verify the claim, let $\zeta = (x, y)$ be a solution of (1.2), defined on $I \subseteq \mathbb{R}$. Let $J \subseteq I$ be a subinterval where (2.1) holds. By differentiation of the algebraic equation $g\big(x(t), y(t)\big) = 0$, one gets

$$0 = \partial_1 g(x(t), y(t))\dot{x} + \partial_2 g(x(t), y(t))\dot{y},$$

whence

$$\dot{y}(t) = [\partial_2 g(x(t), y(t))]^{-1} \partial_1 g(x(t), y(t)) \left[f(x(t), y(t)) + \lambda h(t, x_t, y_t)\right], \tag{2.4}$$

when $t \in J$. Hence, the solutions of (1.2) correspond to those of (2.3). The converse correspondence is more straightforward and follows from the fact that a solution $\zeta = (x, y)$ of (2.3) defined on an interval I with $\inf I = -\infty$ satisfies identically $\big(x(t), y(t)\big) \in M$, which implies (2.1), and eventually fulfills

$$\dot{\zeta}(t) = \Psi\big(\zeta(t)\big) + \lambda \Upsilon(t, \zeta_t)$$

whose first component is (2.1b).

We now introduce the important technical assumption (**K**) below on the function h. This hypothesis implies a similar property, called condition (**H**) (discussed e.g. in [2]), for the induced vector Υ on M defined in (2.2b) that plays a central role in [7]. This fact allows us to apply the methods of [7] to our situation.

Throughout this paper, we shall suppose that f is locally Lipschitz and that h satisfies the following assumption (**K**):

Definition 2.1. *We say that $\mathcal{K}\colon \mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^n\big) \to \mathbb{R}^p$ satisfies (**K**) if, given any compact subset C of $\mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^n\big)$, there exists $\ell \geq 0$ such that*

$$|\mathcal{K}(t, \varphi) - \mathcal{K}(t, \psi)|_p \leq \ell \sup_{t \leq 0} |\varphi(t) - \psi(t)|_n$$

*for all $(t, \varphi), (t, \psi) \in C$. Here $|\cdot|_n$ and $|\cdot|_p$ represent the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Furthermore, we say that condition (**K**) holds locally in $\mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^n\big)$ if for any $(\tau, \eta) \in \mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^n\big)$ there exists a neighborhood of (τ, η) in which (**K**) holds.*

One could show that if (**K**) is satisfied locally, then it is also satisfied globally. However, the local condition is easier to check. It holds, for instance, when $\mathcal{K}$ is C^1 or, more generally, locally Lipschitz in the second variable.

The assumption that h satisfies (**K**) means that for any compact subset C of $\mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^k \times \mathbb{R}^s\big)$, there exists a constant $\ell \geq 0$ such that

$$|h(t, \varphi_1, \psi_1) - h(t, \varphi_2, \psi_2)|_k \leq \ell \sup_{t \leq 0} \Big(|\varphi_1(t) - \varphi_2(t)|_k + |\psi_1(t) - \psi_2(t)|_s\Big)$$

for all $(t, \varphi_1, \psi_1), (t, \varphi_2, \psi_2) \in C$. Here $|\cdot|_k$ and $|\cdot|_s$ represent the Euclidean norm in $\mathbb{R}^k$ and $\mathbb{R}^s$, respectively. Observe that if $f\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^k$ is a locally Lipschitz tangent vector field, and h is a functional field satisfying (**K**), then for any $\lambda \in [0, +\infty)$ the map of $\mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^k \times \mathbb{R}^s\big)$ in $\mathbb{R}^k$, given by

$$(t, \varphi, \psi) \mapsto f\big(\varphi(0), \psi(0)\big) + \lambda h(t, \varphi, \psi),$$

verifies (**K**) as well.

If Ψ and Υ are the functional fields on M defined in (2.2), it is easy to see that for any $\lambda \in [0,+\infty)$ the map of $\mathbb{R}\times BU\big((-\infty,0],M\big)$ in $\mathbb{R}^k\times\mathbb{R}^s$, given by

$$(t,\varphi,\psi)\mapsto \Psi\big(\varphi(0),\psi(0)\big)+\lambda\Upsilon(t,\varphi,\psi),$$

verifies the condition (**H**) discussed in [2, 7].

It can be proved (see e.g. [2]) that if a functional field on M satisfies (**H**) locally, then any associated initial value problem admits a unique solution. This shows, given the equivalence of (1.2) and (2.3), that if f and h satisfy (**K**) then any initial value problem associated to (1.2) has unique initial solution.

3. The degree of the tangent vector field Ψ

In this section we introduce some basic notions about the degree of tangent vector fields on manifolds. Recall that if $w : M \to \mathbb{R}^n$ is a tangent vector field on the differentiable manifold $M \subseteq \mathbb{R}^n$ which is (Fréchet) differentiable at $p \in M$ and $w(p) = 0$, then the differential $d_pw\colon T_pM \to \mathbb{R}^n$ maps T_pM into itself (see e.g. [12]), so that, the determinant $\det d_pw$ of d_pw is defined. In the case when p is a nondegenerate zero (i.e. $d_pw\colon T_pM \to \mathbb{R}^n$ is injective), p is an isolated zero and $\det d_pw \neq 0$.

Let $M \subseteq \mathbb{R}^n$ be a boundaryless differentiable manifold, and let w be a tangent vector field on M. Let W be an open subset of M in which we assume w admissible for the degree, that is we suppose the set $w^{-1}(0)\cap W$ is compact. Then, it is possible to associate to the pair (w,W) an integer, $\deg(w,W)$, called the degree (or characteristic) of the vector field w in W (see e.g. [6, 10]), which, roughly speaking, counts (algebraically) the zeros of w in W in the sense that when the zeros of w are all non-degenerate, then the set $w^{-1}(0)\cap W$ is finite and

$$\deg(w,W)=\sum_{q\in w^{-1}(0)\cap W}\operatorname{sign}\det d_qw.$$

The concept of degree of a tangent vector field is related to the classical one of Brouwer degree (whence its name), but the former notion diverge from the latter when dealing with manifolds. In particular, the former does not need the orientation of the underlying manifolds. However, when $M = \mathbb{R}^n$, the degree of a vector field $\deg(w,W)$ is essentially the well known Brouwer degree $\deg_B(w,W,0)$ of w on W with respect to 0 (recall that in Euclidean spaces vector fields can be regarded as maps). For the main properties of the degree we refer e.g. to [6, 10, 12].

Let now $g\colon \mathbb{R}^k\times\mathbb{R}^s \to \mathbb{R}^s$ and $f\colon \mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^s \to \mathbb{R}^k$ be given maps such that f is continuous and g is C^∞ with the property that $\partial_2 g(p,q)$ is invertible for all $(p,q)\in\mathbb{R}^k\times\mathbb{R}^s$. Define $M = g^{-1}(0)$ and let Ψ be the tangent vector field on M given by (2.2a). We need the following consequence of [14, Th. 4.1] concerning the degree of Ψ.

Proposition 3.1. *Let $F\colon \mathbb{R}^k\times\mathbb{R}^s \to \mathbb{R}^k\times\mathbb{R}^s$ be given by*

$$(p,q)\mapsto \big(f(p,q),g(p,q)\big)$$

and let $V \subseteq \mathbb{R}^k\times\mathbb{R}^s$ be an open set. Then, if either $\deg(\Psi,M)$ *or $\deg(F,V)$ is well defined, so is the other, and*

$$|\deg(\Psi, M\cap V)| = |\deg(F,V)|. \tag{3.1}$$

Proof. Follows immediately from Theorem 4.1 in [14] and the excision property. $\square$

4. Connected sets of T-periodic solutions

This section is concerned with the set of periodic solutions to (1.2). As in Section 2 we are given maps $f\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^k$, $g\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^s$ and $h\colon \mathbb{R} \times BU((-\infty,0],\mathbb{R}^k \times \mathbb{R}^s) \to \mathbb{R}^k$, and we assume that

(1) f is locally Lipschitz;
(2) g is C^∞ and such that $\det \partial_2 g(p,q) \neq 0$ for all $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s$;
(3) h satisfies (**K**) and, given $T > 0$, is T-periodic with respect to its first variable.

Denote by $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ the Banach space of all the continuous T-periodic functions assuming values in $\mathbb{R}^k \times \mathbb{R}^s$ with the usual supremum norm. We say that $(\mu;\xi) \in [0,\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s)$ is a *T-periodic pair* for (1.2) if $\xi = (x,y)$ satisfies (1.2) for $\lambda = \mu$. Here, as well as in what follows, the elements of $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ will be written as pairs whenever convenient. In this way, T-periodic pairs actually will be often written as triples. Moreover, given $(p,q) \in \mathbb{R}^k \times \mathbb{R}^s$, denote by $(\overline{p}(t),\overline{q}(t)) \equiv (p,q)$, for all $t \in \mathbb{R}$, the map in $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ that is constantly equal to (p,q). A T-periodic pair of the form $(0;\overline{p},\overline{q})$ is called *trivial*.

Let $F\colon \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^k \times \mathbb{R}^s$ be the vector field given by

$$F(p,q) = \big(f(p,q), g(p,q)\big). \tag{4.1}$$

It can be easily verified that $(\overline{p},\overline{q})$ is a constant solution of (1.2) for $\lambda = 0$ if and only if $F(p,q) = (0,0)$. Thus, with the above notation, the set of trivial T-periodic pairs can be written as

$$\big\{(0;\overline{p},\overline{q}) \in [0,\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s) : F(p,q) = (0,0)\big\}.$$

The following convention is very handy. Given subsets Ω and X of $[0,\infty)\times C_T(\mathbb{R}^k \times \mathbb{R}^s)$ and of $\mathbb{R}^k \times \mathbb{R}^s$, respectively, with $X \cap \Omega$ we denote the set of points of X that, regarded as constant functions, lie in Ω. Namely,

$$X \cap \Omega = \{(p,q) \in X : (0;\overline{p},\overline{q}) \in \Omega\}.$$

The next result provides an insight into the topological structure of the set of T-periodic solutions of (1.2).

Theorem 4.1. *Let f, h and g be as above. Let also $F : \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^k \times \mathbb{R}^s$ be defined as in* (4.1). *Let Ω be an open subset of $[0,\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s)$ and Assume that $\deg\big(F, \Omega \cap (\mathbb{R}^k \times \mathbb{R}^s)\big)$ is well-defined and nonzero. Then, the set of nontrivial T-periodic pairs of* (1.2), *admits a connected subset whose closure in Ω is noncompact and meets the set of trivial T-periodic pairs in Ω i.e. the set $\big\{(0;\overline{p},\overline{q}) \in \Omega : F(p,q) = (0,0)\big\}$. In particular, the set of T-periodic pairs for* (1.2) *contains a connected component that meets $\big\{(0;\overline{p},\overline{q}) \in \Omega : F(p,q) = (0,0)\big\}$ and whose intersection with Ω is not compact.*

Proof. Let Ψ and Υ be as in (2.2). Then (1.2) is equivalent to (2.3) on M. Denote by $C_T(M)$ the metric subspace of the Banach space $C_T(\mathbb{R}^k \times \mathbb{R}^s)$, of all the continuous T-periodic functions taking values in $M = g^{-1}(0)$. Let also $\mathcal{O}$ be the open subset of $[0,\infty) \times C_T(M)$ given by

$$\mathcal{O} = \Omega \cap \Big([0,\infty) \times C_T(M)\Big).$$

Given $Y \subseteq M$, by $\mathcal{O} \cap Y$ we mean the set of all those points of Y that, regarded as constant functions, lie in $\mathcal{O}$. With this convention one clearly has $Y \cap \Omega = Y \cap \mathcal{O}$ and, in particular, $\Omega \cap M = \mathcal{O} \cap M$. This identity and Proposition 3.1 imply that

$$\deg(\Psi, \mathcal{O} \cap M) = \deg(\Psi, \Omega \cap M) = \pm \deg\big(F, \Omega \cap (\mathbb{R}^k \times \mathbb{R}^s)\big) \neq 0.$$

Thus, Theorem 4.1 in [7] yields the existence of a connected subset Λ of

$$\big\{(\lambda; x, y) \in \mathcal{O} : (x, y) \text{ is a nonconstant solution of (2.3) }\big\},$$

whose closure $\overline{\Lambda}$ in $\mathcal{O}$ is not compact and meets the set

$$\big\{(0; \overline{p}, \overline{q}) \in \mathcal{O} : \Psi(p, q) = (0, 0)\big\},$$

that coincides with $\big\{(0, \overline{p}, \overline{q}) \in \Omega : F(p, q) = (0, 0)\big\}$.

Clearly, each $(\lambda; x, y) \in \Lambda$ is a nontrivial T-periodic pair of (1.2). Since $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$, it is not difficult to prove that any set that is closed in $\mathcal{O}$ is so in Ω and vice versa. Thus, $\overline{\Lambda}$ coincides with the closure of Λ in Ω. The first part of the assertion follows.

Let us prove the last part of the assertion. Consider the connected component Γ of X that contains the connected set Λ. We shall now show that Γ has the required properties. Clearly, Γ meets the set $\big\{(0, \overline{p}) \in \Omega : g(p) = 0\big\}$ because the closure of Λ in Ω does. Moreover, $\Gamma \cap \Omega$ cannot be compact, since it contains the (noncompact) closure of Λ in Ω. □

Remark 4.2. *Let Ω be as in Theorem 3.1, and assume that Γ is a connected component of T-periodic pairs of (1.2) that meets $\{(0, \overline{p}, \overline{q}) \in \Omega : F(p, q) = (0, 0)\}$ and whose intersection with Ω is not compact. Ascoli's Theorem implies that any bounded set of T-periodic pairs is relatively compact. Then, the closed set Γ cannot be both bounded and contained in Ω. In particular, if Ω is bounded, then Γ necessarily meets the boundary of Ω.*

The following corollary ensures the existence of a Rabinowitz-type branch of T-periodic pairs.

Corollary 4.3. *Let f, h and g be as in Theorem 3.1. Let $V \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and assume that $\deg(F, V)$ is well defined and nonzero. Then, there exists a connected component Γ of T-periodic pairs of (1.2) that meets the set*

$$\big\{(0, \overline{p}, \overline{q}) \in [0, +\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s) : (p, q) \in V \cap F^{-1}(0, 0)\big\}$$

and is either unbounded or meets

$$\big\{(0, \overline{p}, \overline{q}) \in [0, +\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s) : (p, q) \in F^{-1}(0, 0) \setminus V\big\}.$$

Proof. Consider the open subset Ω of $[0, +\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s)$ given by

$$\Omega = \big([0, +\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s)\big) \setminus \\ \setminus \big\{(0, \overline{p}, \overline{q}) \in [0, +\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s) : (p, q) \in F^{-1}(0, 0) \setminus V\big\}.$$

Clearly, we have $\Omega \cap \mathbb{R}^k \times \mathbb{R}^s = U$. Hence $\deg(F, \Omega \cap \mathbb{R}^k \times \mathbb{R}^s) \neq 0$. Theorem 3.1 implies the existence of a connected component Γ of T-periodic pairs of (1.2) that meets $\{(0, \overline{p}, \overline{q}) \in \Omega : F(p, q) = 0\}$ and whose intersection with Ω is not compact. Because of Remark 4.2, if Γ is bounded, then it meets the boundary of Ω which is given by

$$\big\{(0, \overline{p}, \overline{q}) \in [0, +\infty) \times C_T(\mathbb{R}^k \times \mathbb{R}^s) : (p, q) \in F^{-1}(0, 0) \setminus V\big\}.$$

And the assertion is proved. □

Example 4.4. *The well-known logistic equation (see, e.g. [3])*

$$\dot{x}(t) = \alpha x - \beta x^2 \tag{4.2}$$

is sometimes used as a model for a population x with birth and mortality rate αx and βx^2, respectively. Consider a generalization of (4.2) where the mortality rate

y is related to the population by the implicit relation $g(x,y)=0$. This generalized model is expressed by the following DAE:

$$\begin{cases} \dot{x}(t) = \alpha x - y, \\ g(x,y) = 0. \end{cases}$$

If we allow the population's fertility to undergo periodic oscillations —say $\lambda h(t,x_t)$ with $\lambda \geq 0$— depending possibly on the history of the population, the above model can be modified into the following RFDAE:

$$\begin{cases} \dot{x}(t) = \alpha x - y + \lambda h(t,x_t), \\ g(x,y) = 0. \end{cases} \tag{4.3}$$

As to the perturbation $h(t,x_t)$ one could look into examples inspired to models describing the dynamics of animals populations (see, e.g. [3, 4]*) in which the delay is distributed on time. For instance, one could take $h(t,x_t) = \sin(t)\int_0^\infty x(t-s)p(s)ds$, where $p(t)$ is continuous with $\int_0^\infty p(s)ds = 1$. Clearly,* (4.3) *makes sense as a population model only as long as $x \geq 0$; here, however, we care only of the 'abstract' properties of this equation. Let us show how Theorem 4.1 can be applied to* (4.3)*. As an example, take $\alpha > 0$ and $g(x,y) = y^3 + y - x^5$, and let $\Omega \subseteq [0,\infty) \times BU((-\infty,0],\mathbb{R}^k \times \mathbb{R}^s)$ be given by $\Omega = \{(x,y) \in BU((-\infty,0],\mathbb{R}^k \times \mathbb{R}^s) : x(t) > 0 \text{ for all } t \in \mathbb{R}\}$. The map F defined in* (4.1) *is given by $F(x,y) = (\alpha x - y, y^3 + y - x^5)$, and a simple direct computation shows that $F^{-1}(0) \cap \Omega$ consists of the singleton $\{p\}$ and that $\deg(F,\mathbb{R}^2 \cap \Omega) = -1$. Hence Theorem 4.1 yields an unbounded connected component of periodic 2π-periodic pairs emanating from the trivial 2π-periodic pair $(0,\overline{p})$.*

5. An application

This technical section is primarily intended as an illustration of our main result Theorem 4.1; for this reason we will not pursue maximal generality but restrict ourselves to simple situations. Below, we consider retarded periodic perturbations of a particular class of implicit functional differential equations. Namely, we study equations of the following form:

$$E\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) + \lambda\mathcal{H}(t,\mathbf{x}_t), \qquad \lambda \geq 0, \tag{5.1}$$

where $E\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear endomorphism of $\mathbb{R}^n$, $\mathcal{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{H}\colon \mathbb{R} \times BU\big((-\infty,0],\mathbb{R}^n\big) \to \mathbb{R}^n$ are continuous maps with $\mathcal{F}$ locally Lipschitz and $\mathcal{H}$ verifies condition $(\mathbf{K})$.

We will show how, in some circumstances, (5.1) can be transformed into RFDAEs of type (1.2) by the means of relatively simple linear transformations. We will apply the results of the previous section to the resulting RFDAEs. A first example of the above mentioned transformation is considered in the following remark:

Remark 5.1. *Consider equation* (5.1) *and assume that there exists a decomposition of $\mathbb{R}^n$ as $\mathbb{R}^r \times \mathbb{R}^{n-r}$ such that E and $\mathcal{H}$ can be represented as follows:*

$$E \simeq \begin{pmatrix} E_{11} & E_{12} \\ 0 & 0 \end{pmatrix}, \ \textit{with } E_{11} \in \mathbb{R}^{r\times r} \textit{ invertible and } E_{12} \in \mathbb{R}^{r\times(n-r)}, \tag{5.2a}$$

$$\mathcal{H}(t,\varphi) = \begin{pmatrix} \mathcal{H}_1(t,\varphi) \\ 0 \end{pmatrix}, \quad \textit{with } \mathcal{H}_1\colon \mathbb{R} \times BU\big((-\infty,0],\mathbb{R}^n\big) \to \mathbb{R}^r. \tag{5.2b}$$

In $\mathbb{R}^n \simeq \mathbb{R}^r \times \mathbb{R}^{n-r}$ put $\mathbf{x} = (\xi,\eta)$ and let $J_E\colon \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^r \times \mathbb{R}^{n-r}$ be the linear transformation represented by the following block matrix:

$$\begin{pmatrix} E_{11}^{-1} & -E_{11}^{-1}E_{12} \\ 0 & I \end{pmatrix}.$$

Let $(x,y) = J_E^{-1}(\xi,\eta)$, and let $\mathcal{F}_1(\xi,\eta)$ and $\mathcal{F}_2(\xi,\eta)$ denote the projection of $\mathcal{F}(\xi,\eta)$ onto the first and second factor, respectively, of $\mathbb{R}^r \times \mathbb{R}^{n-r}$. Then, in the new variables x and y equation (5.1) *becomes, with a slight abuse of notation,*

$$EJ_E \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1\big(J_E(x,y)\big) \\ \mathcal{F}_2\big(J_E(x,y)\big) \end{pmatrix} + \lambda \begin{pmatrix} \mathcal{H}_1\big(t, J_E(x_t,y_t)\big) \\ 0 \end{pmatrix}$$

or, equivalently,

$$\begin{cases} \dot{x} = \widetilde{\mathcal{F}}_1(x,y) + \lambda \widetilde{\mathcal{H}}_1(t,x_t,y_t), \\ \widetilde{\mathcal{F}}_2(x,y) = 0, \end{cases} \tag{5.3}$$

*where $\widetilde{\mathcal{F}}_i(x,y) = \mathcal{F}_i\big(J_E(x,y)\big)$, $i=1,2$, and $\widetilde{\mathcal{H}}_1(t,\varphi) = \mathcal{H}_1\big(t, J_E \circ \varphi\big)$, for any $(t,\varphi) \in \mathbb{R} \times BU\big((-\infty,0],\mathbb{R}^n\big)$. Furthermore, since $\mathcal{H}$ satisfies (**K**), it is not difficult to prove that $\widetilde{\mathcal{H}}_1$ satisfies (**K**) as well.*

Example 5.2. *Consider the following DAE in $\mathbb{R}^3$:*

$$\begin{cases} \dot{\xi}_1 + \dot{\xi}_2 + \dot{\xi}_3 = \xi_2, \\ \dot{\xi}_1 = -\xi_1 + \xi_2^2 + \xi_3, \\ 0 = \xi_3^3 + \xi_3 + \xi_1, \end{cases} \tag{5.4}$$

which can be written as the implicit ODE below

$$E\dot{\xi} = \mathcal{F}(\xi) \tag{5.5}$$

where E is the endomorphism of $\mathbb{R}^3 \simeq \mathbb{R}^2 \times \mathbb{R}$ represented by the block matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

and $\mathcal{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\mathcal{F}(\xi_1,\xi_2,\xi_3) = (\xi_2, -\xi_1 + \xi_2^2 + \xi_3, \xi_3^3 + \xi_3 + \xi_1)$. Let J_E be the linear transformation of $\mathbb{R}^3 \simeq \mathbb{R}^2 \times \mathbb{R}$ represented by the block matrix

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & -1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right),$$

and put $(x_1,x_2,y) = J_E^{-1}(\xi_1,\xi_2,\xi_3)$. One has that

$$\mathcal{F}\big(J_E(x_1,x_2,y)\big) = \big(x_1 - x_2 + y, -x_1 + (x_1 - x_2 + y)^2 + y, y^3 + y + x_1\big),$$

As in Remark 5.1, for $\xi = (\xi_1,\xi_2,\xi_3)$, let $\mathcal{F}_1(\xi)$ and $\mathcal{F}_2(\xi)$ denote the projection of $\mathcal{F}(\xi)$ onto the first and second factor, respectively, of $\mathbb{R}^2 \times \mathbb{R}$. Put also $\widetilde{\mathcal{F}}_i(x,y) = \mathcal{F}_i\big(J_E(x,y)\big)$, $i=1,2$, where $x = (x_1,x_2)$. Proceeding as in Remark 5.1, we transform Equation (5.4) *into*

$$\begin{cases} \dot{x} = \widetilde{\mathcal{F}}_1(x,y), \\ \widetilde{\mathcal{F}}_2(x,y) = 0, \end{cases}$$

that can be written more explicitly as follows:

$$\begin{cases} \dot{x}_1 = x_1 - x_2 + y, \\ \dot{x}_2 = -x_1 + (x_1 - x_2 + y)^2 + y, \\ y^3 + y + x_1 = 0. \end{cases}$$

Theorem 4.1 combined with the above Remark 5.1, yields Proposition 5.3 below concerning the set of T-periodic solutions of (5.1). We use here the convention on the subsets of $[0,\infty) \times C_T(\mathbb{R}^r \times \mathbb{R}^{n-r})$ introduced in section 4. We also need to introduce some further notation.

A pair $(\lambda,\mathbf{x}) \in [0,\infty) \times C_T(\mathbb{R}^n)$ is a *T-periodic pair for* (5.1) if $\mathbf{x}$ is a solution of (5.1) corresponding to λ. A T-periodic pair $(0,\mathbf{x})$ for (5.1) is *trivial* if $\mathbf{x}$ is constant.

Proposition 5.3. *Consider Equation* (5.1) *where* $E\colon \mathbb{R}^n \to \mathbb{R}^n$ *is linear,* $\mathcal{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ *and* $\mathcal{H}\colon \mathbb{R} \times BU\big((-\infty,0],\mathbb{R}^n\big) \to \mathbb{R}^n$ *are continuous maps such that* $\mathcal{F}$ *is locally Lipschitz and* $\mathcal{H}$ *verifies condition* ***(K)*** *and is* T*-periodic in the first variable. Assume, as in Remark 5.1, that there exists a decomposition of* $\mathbb{R}^n \simeq \mathbb{R}^r \times \mathbb{R}^{n-r}$ *such that* E *and* $\mathcal{H}$ *can be represented as in* (5.2)*. Relatively to this decomposition suppose that* $\partial_2\mathcal{F}_2(\xi,\eta)$ *is invertible for all* $(\xi,\eta) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$*.*

Let Ω *be an open subset of* $[0,\infty) \times C_T(\mathbb{R}^n)$ *and suppose that* $\deg(\mathcal{F},\Omega\cap\mathbb{R}^n)$ *is well-defined and nonzero. Then, there exists a connected subset* Γ *of nontrivial* T*-periodic pairs for* (5.1) *whose closure in* Ω *is noncompact and meets the set* $\big\{(0,\overline{\mathrm{p}}) \in \Omega : \mathcal{F}(\mathrm{p}) = 0\big\}$*.*

Proof. Let J_E be the linear transformation introduced in Remark 5.1, and consider the map $\widehat{J_E}\colon [0,\infty)\times C_T(\mathbb{R}^n) \to [0,\infty)\times C_T(\mathbb{R}^n)$ given by

$$\widehat{J_E}(\lambda,\psi) = (\lambda, J_E\circ\psi).$$

Observe that since J_E is invertible, $\widehat{J_E}$ is continuous and invertible, with $\widehat{J_E}^{-1}$ given by $\widehat{J_E}^{-1}(\lambda,\psi) = (\lambda, {J_E}^{-1}\circ\psi)$. With the convention on the subsets of $[0,\infty)\times C_T(\mathbb{R}^n)$ introduced in section 4, we have

$$\widehat{J_E}^{-1}(\Omega)\cap\mathbb{R}^n = {J_E}^{-1}(\Omega\cap\mathbb{R}^n).$$

According to Remark 5.1, under our assumptions Equation (5.1) is equivalent to the RFDAE (5.3). We now show that Theorem 4.1 can be applied to Equation (5.3). Define $\widetilde{\mathcal{F}}\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\widetilde{\mathcal{F}}(\mathrm{p}) = \big(\widetilde{\mathcal{F}}_1(\mathrm{p})\,,\,\widetilde{\mathcal{F}}_2(\mathrm{p})\big) = \mathcal{F}\big(J_E(\mathrm{p})\big)$. The property of invariance under diffeormorphism of the degree (also called topological invariance, see e.g. [6]) yields

$$\begin{aligned}\deg\big(\mathcal{F},\Omega\cap\mathbb{R}^n\big) &= \deg\big({J_E}^{-1}\circ\mathcal{F}\circ J_E, J_E^{-1}(\Omega\cap\mathbb{R}^n)\big)\\ &= \deg\big({J_E}^{-1}\circ\widetilde{\mathcal{F}},\widehat{J_E}^{-1}(\Omega)\cap\mathbb{R}^n\big).\end{aligned} \tag{5.6}$$

Also, it is not difficult to show that

$$\deg\big({J_E}^{-1}\circ\widetilde{\mathcal{F}},\widehat{J_E}^{-1}(\Omega)\cap\mathbb{R}^n\big) = \operatorname{sign}\det(J_E)\deg\big(\widetilde{\mathcal{F}},\widehat{J_E}^{-1}(\Omega)\cap\mathbb{R}^n\big), \tag{5.7}$$

so that, being $\deg\big(\mathcal{F},\Omega\cap\mathbb{R}^n\big)$ nonzero by assumption, (5.6)–(5.7) yield

$$\deg\big(\widetilde{\mathcal{F}},J_E^{-1}(\Omega)\cap\mathbb{R}^n\big)\neq 0.$$

Hence, Theorem 4.1 implies the existence of an unbounded connected set $\Xi \subseteq \widehat{J_E}^{-1}(\Omega)$ of T-periodic pairs of (5.3) whose closure in $\widehat{J_E}^{-1}(\Omega)$ is noncompact and meets the set $\big\{(0,\overline{\mathrm{p}}) \in \widehat{J_E}^{-1}(\Omega) : \widetilde{\mathcal{F}}(\mathrm{p}) = 0\big\}$. It is not difficult to show that $\Gamma = \widehat{J_E}(\Xi)$ has the required properties. □

Example 5.4. *Let* $\mathcal{H}\colon \mathbb{R}\times BU\big((-\infty,0],\mathbb{R}^3\big) \to \mathbb{R}^3$ *be a* T*-periodic continuous map that affects only the first two coordinates, and consider the retarded perturbation* $\lambda\mathcal{H}(t,\xi_t)$ *of Equation* (5.4) *in Example 5.2. Namely, consider*

$$E\dot{\xi} = \mathcal{F}(\xi) + \lambda\mathcal{H}(t,\xi_t),$$

with the same notation of Example 5.2. Take $\Omega = [0,\infty)\times C_T(\mathbb{R}^2\times\mathbb{R})$ *and observe that* $\deg(\mathcal{F},\Omega\cap\mathbb{R}^3)$ *is well-defined and nonzero. Thus, Proposition 5.3 yields the existence of a connected subset of nontrivial* T*-periodic pairs for the above equation whose closure in* Ω *is noncompact and meets the set*

$$\big\{(0,\overline{\mathrm{p}})\in\Omega : \mathcal{F}(\mathrm{p}) = 0\big\} = \{(0,\overline{0})\in\Omega\}.$$

Here, by $\overline{0}$ *we mean the function constantly equal to* $0\in\mathbb{R}^3$*.*

Observe that Proposition 5.3 seems to impose some rather severe constraint on the form of E and $\mathcal{H}$ in Equation (5.1). In fact, with the help of some linear transformation, one can sometimes lift these restrictions. This is the case when the perturbing term $\mathcal{H}$ has a particular 'separated variables' form that agrees with E in the sense of Equation (5.9) below. Namely, we consider the following equation:

$$E\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) + \lambda C(t) S(\mathbf{x}_t), \tag{5.8}$$

where $C\colon \mathbb{R} \to \mathbb{R}^{n\times n}$, $S\colon BU\big((-\infty,0],\mathbb{R}^n\big) \to \mathbb{R}^n$ are continuous maps, E is a (constant) $n \times n$ matrix, and $\mathcal{F}$ is as in Equation (5.1). We also assume that C and E agree in the following sense:

$$\ker\, C^T(t) = \ker\, E^T,\ \forall\, t \in \mathbb{R},\ \text{and} \dim\ker\, E^T > 0, \tag{5.9}$$

As a consequence of the well-known Rouché-Capelli Theorem we get

$$\begin{aligned} n - \operatorname{rank}\, E = n - \operatorname{rank}\, E^T = \dim\ker\, E^T = \\ = \dim\ker\, C(t)^T = n - \operatorname{rank}\, C(t)^T = n - \operatorname{rank}\, C(t) \end{aligned}$$

Thus, we have that

$$\operatorname{rank}\, E = \operatorname{rank}\, C(t) \text{ is constant and greater than 0 for all } t \in \mathbb{R}, \tag{5.10}$$

Ours is a singular value decomposition (see, e.g., [9]) argument based on the following technical result from linear algebra:

Lemma 5.5. *Let $E \in \mathbb{R}^{n\times n}$ and $C \in C\big(\mathbb{R},\mathbb{R}^{n\times n}\big)$ be as in* (5.9)*. Put $r = \operatorname{rank}\, E$, and let $P, Q \in \mathbb{R}^{n\times n}$ be orthogonal matrices that realize a singular value decomposition for E. Then it follows that*

$$PC(t)Q^T = \begin{pmatrix} \widetilde{C}_{11}(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall t \in \mathbb{R}, \tag{5.11}$$

with $\widetilde{C}_{11} \in C\big(\mathbb{R},\mathbb{R}^{r\times r}\big)$ invertible for any $t \in \mathbb{R}$.

We will provide a proof for Lemma 5.5 for the sake of completeness but, before doing that, we show how it can be used to convert equation (5.8) into (5.1). We begin with an example.

Example 5.6. *Consider Equation* (5.8) *with*

$$E = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C(t) = \begin{pmatrix} c(t) & 0 & 0 & 0 \\ 0 & c(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d(t) \end{pmatrix},$$

where, for any $t \in \mathbb{R}$, $c(t) = \sin(t) + 2$ and $d(t) = cos(t) + 3$. It can be easily verified that, with this choice of E and C, (5.9) *is satisfied. Here, clearly, $r = 3$ and $n = 4$. Consider the following orthogonal matrices*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

that realize a singular value decomposition for E, that is, in block-matrix form in $\mathbb{R}^4 \simeq \mathbb{R}^3 \times \mathbb{R}$,

$$PEQ^T = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \text{ and } PC(t)Q^T = \left(\begin{array}{ccc|c} 0 & c(t) & 0 & 0 \\ c(t) & 0 & 0 & 0 \\ 0 & 0 & d(t) & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

Let us consider the orthogonal change of coordinates $\mathbf{x} = Q^T x$*. Multiplying* (5.8) *by* P *on the left we get the following equivalent equation:*

$$PEQ^T\dot{x} = P\mathcal{F}(Q^T x) + \lambda PC(t)Q^T QS(Q^T x_t). \tag{5.12}$$

Set $\widetilde{E} = PEQ^T$*,* $\widetilde{\mathcal{F}}(x) = P\mathcal{F}(Q^T x)$ *for all* $x \in \mathbb{R}^4$*, and finally, put* $\widetilde{\mathcal{H}}(t,\varphi) = PC(t)Q^T QS(Q^T\varphi)$ *for all* $(t,\varphi) \in \mathbb{R}\times BU\big((-\infty,0],\mathbb{R}^4\big)$*. Thus* (5.12) *can be rewritten as*

$$\widetilde{E}\dot{x} = \widetilde{\mathcal{F}}(x) + \lambda\widetilde{\mathcal{H}}(t,x_t).$$

It is easily verified that the so defined $\widetilde{E}$ *and* $\widetilde{\mathcal{H}}$ *satisfy* (5.2)*, so that* (5.12) *is precisely of the form* (5.1)*. In other words, we have transformed* (5.8)*, for* E *and* C *as above, into an equation of the form considered in Proposition 5.3.*

Let us now consider Equation (5.8) more in general. Let $r > 0$ be the rank of E, and assume that (5.9) is satisfied. Then Lemma 5.5 yields orthogonal matrices P and Q in $\mathbb{R}^{n\times n}$ such that, for every $t \in \mathbb{R}$, $PC(t)Q^T$ is as in (5.11) and realize a singular value decomposition of E. That is

$$PEQ^T = \begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.13}$$

where $\widetilde{E}_{11} \in \mathbb{R}^{r\times r}$ is a diagonal matrix with positive diagonal elements. As in the above example, consider the orthogonal change of coordinates $\mathbf{x} = Q^T x$ in Equation (5.8) and multiply by P on the left. We get the equivalent equation

$$\widetilde{E}\dot{x} = \widetilde{\mathcal{F}}(x) + \lambda\widetilde{\mathcal{H}}(t,x_t). \tag{5.14}$$

where $\widetilde{E}$, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{H}}$ are given by $\widetilde{E} = PEQ^T$, $\widetilde{\mathcal{F}}(x) = P\mathcal{F}(Q^T x)$ for all $x \in \mathbb{R}^n$, and $\widetilde{\mathcal{H}}(t,\varphi) = PC(t)Q^T QS(Q^T\varphi)$ for all $(t,\varphi) \in \mathbb{R}\times BU\big((-\infty,0],\mathbb{R}^n\big)$. A straightforward computation shows that $\widetilde{E}$ and $\widetilde{\mathcal{H}}$ satisfy conditions (5.2). Therefore, (5.14) is of the form considered in Proposition 5.3 from which we deduce the following consequence:

Corollary 5.7. *Consider Equation* (5.8) *where the maps* $C\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ *and* $S\colon BU\big((-\infty,0],\mathbb{R}^n\big) \to \mathbb{R}^n$ *are continuous,* E *is a (constant)* $n\times n$ *matrix, and* $\mathcal{F}$ *is as in Equation* (5.1) *such that* $\mathcal{F}$ *is locally Lipschitz and* S *verifies condition* ($\boldsymbol{K}$)*. Suppose also that* C *and* E *satisfy* (5.9) *and that* C *is* T*-periodic. If* $r > 0$ *is the rank of* E *consider the decomposition* $\mathbb{R}^n \simeq \mathbb{R}^r \times \mathbb{R}^{n-r}$ *and assume that, relatively to this decomposition,* $\partial_2\mathcal{F}_2(\xi,\eta)$ *is invertible for all* $x = (\xi,\eta) \in \mathbb{R}^r\times\mathbb{R}^{n-r}$

Let Ω *be an open subset of* $[0,\infty)\times C_T(\mathbb{R}^n)$ *and suppose that* $\deg(\mathcal{F},\Omega\cap\mathbb{R}^n)$ *is well-defined and nonzero. Then, there exists a connected subset* Γ *of nontrivial* T*-periodic pairs for* (5.8) *whose closure in* Ω *is noncompact and meets the set* $\big\{(0,\overline{\mathrm{p}}) \in \Omega : \mathcal{F}(\mathrm{p}) = 0\big\}$*.*

Proof. Consider the map $\widehat{Q}\colon [0,\infty)\times C_T(\mathbb{R}^n) \to [0,\infty)\times C_T(\mathbb{R}^n)$ given by $\widehat{Q}(\lambda,\psi) = (\lambda, Q\psi)$. Clearly, T-periodic pairs of (5.8) correspond to those of (5.14) under $\widehat{Q}$. The invariance under diffeomorphisms of the degree (or topological invariance, compare e.g. [6]) implies

$$\deg\big(\widetilde{\mathcal{F}}, Q(\Omega)\cap\mathbb{R}^n\big) \neq 0.$$

The assertion follows immediately applying Proposition 5.3 to Equation (5.14). □

We conclude this section with a proof of our technical Lemma.

Proof of Lemma 5.5. Since the dimension of $\ker C(t)$ is constantly equal to $r > 0$, by inspection of the proof of Theorem 3.9 of [11, Chapter 3, §1] we get the existence of orthogonal matrix-valued functions $U, V \in C\big(\mathbb{R}, \mathbb{R}^{n\times n}\big)$ and $C_r \in C(\mathbb{R}, \mathbb{R}^{r\times r})$ such that, for all $t \in \mathbb{R}$, $\det C_r(t) \neq 0$ and

$$U^T(t)C(t)V(t) = \begin{pmatrix} C_r(t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.15}$$

Let $U_r, V_r \in C\big(\mathbb{R}, \mathbb{R}^{n\times r}\big)$ and $U_0, V_0 \in C(\mathbb{R}, \mathbb{R}^{n\times(n-r)})$ be matrix-valued functions formed, respectively, by the first r and $n-r$ columns of U and V. A simple argument involving Equation (5.15) shows that the columns of $V_0(t)$, $t \in \mathbb{R}$, are in $\ker C(t)$ and, since there are $n - r = \dim\ker C(t)$ of them, we have that the columns of $V_0(t)$ actually span $\ker C(t)$. In fact, the orthogonality of the matrix $V(t)$, $t \in \mathbb{R}$, imply that the columns of $V_0(t)$ form an orthogonal basis of $\ker C(t)$. A similar argument proves that the columns of $U_0(t)$ are vectors of $\mathbb{R}^n$ that constitute an orthogonal basis of $\ker C(t)^T$ for all $t \in \mathbb{R}$. Observe also that since $\operatorname{im} C(t)$ is orthogonal to $\ker C(t)^T$ for all $t \in \mathbb{R}$, it follows that the columns of $U_r(t)$ are an orthogonal basis for $\operatorname{im} C(t)$ and that those of $V_r(t)$ so are for $\operatorname{im} C(t)^T$.

Similarly, let P_r, Q_r and P_0, Q_0 be the matrices formed taking the first r and $n-r$ columns of P and Q, respectively. Since P and Q realize a singular value decomposition of E, one can check that the columns of P_r, Q_r, P_0 and Q_0 span $\operatorname{im} E$, $\operatorname{im} E^T$, $\ker E^T$, and $\ker E$, respectively.

We claim that $P_0^T U_r(t)$ is constantly the null matrix. To prove this, it is enough to show that for all $t \in \mathbb{R}$, the columns of P_0 are orthogonal to those of $U_r(t)$. Let v and $u(t)$, $t \in \mathbb{R}$, be any column of P_0 and of $U_r(t)$, respectively. Since for all $t \in \mathbb{R}$ the columns of $U_r(t)$ are in $\operatorname{im} C(t)$, there is a vector $w(t) \in \mathbb{R}^n$ with the property that $u(t) = C(t)w(t)$, and

$$\langle v, u(t)\rangle = \langle v, C(t)w(t)\rangle = \langle C(t)^T v, w(t)\rangle = 0, \quad t \in \mathbb{R},$$

because $v \in \ker E^T = \ker C(t)^T$ for all $t \in \mathbb{R}$. This proves the claim. A similar argument shows that also $P_r^T U_0(t)$, $V_r^T(t)Q_0$, and $V_0^T(t)Q_r$ are also identically zero.

Since for all $t \in \mathbb{R}$

$$P^T Q = \begin{pmatrix} P_r^T U_r(t) & 0 \\ 0 & P_0^T U_0(t) \end{pmatrix} \quad \text{and} \quad V(t)^T Q = \begin{pmatrix} V_r(t)^T Q_r & 0 \\ 0 & V_0(t)^T Q_0 \end{pmatrix}$$

are nonsingular, we deduce in particular that so are $P_r^T U_r(t)$ and $V_r(t)^T Q_r$.

Let us compute the matrix product $P^T C(t) Q$ for all $t \in \mathbb{R}$. We omit here, for the sake of simplicity, the explicit dependence on t.

$$\begin{aligned} P^T C Q = P^T U U^T C V V^T Q &= \begin{pmatrix} P_r^T U_r & 0 \\ 0 & P_0^T U_0 \end{pmatrix} \begin{pmatrix} C_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_r^T Q_r & 0 \\ 0 & V_0^T Q_0 \end{pmatrix} \\ &= \begin{pmatrix} P_r^T U_r C_r V_r^T Q_r & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Which proves the assertion because $P_r^T U_r$, C_r, and $V_r^T Q_r$ are nonsingular. □

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