HARMONIC SOLUTIONS OF PERIODIC CARATHÉODORY PERTURBATIONS OF AUTONOMOUS ODE'S ON MANIFOLDS

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Abstract. We investigate the set of harmonic solutions of Carathéodory T-periodic perturbations of autonomous ODE's on a differentiable manifold M embedded in some $\mathbf{R}^k$. More precisely, we consider the parametrized differential equation $\dot{x} = g(x) + \lambda f(t, x)$, $\lambda \geq 0$, where g and f are two vector fields tangent to $M \subset \mathbf{R}^k$, with g continuous and f Carathéodory, T-periodic in t. We prove the existence of a non-compact connected set of pairs (λ, x), where $\lambda \geq 0$ and $x : \mathbf{R} \to M$ is a non-constant Carathéodory solution of the equation, emanating from the set $g^{-1}(0)$. This leads to a continuation principle and a multiplicity result for a pendulum like equation.

1. Introduction

This paper, which is inspired by [6], is devoted to extending the results of [8] to the case of Carathéodory perturbations of autonomous vector fields.

Let M be a (not necessarily closed) boundaryless differentiable manifold embedded in $\mathbf{R}^k$. Consider the following parametrized first order ordinary differential equation:

$$\dot{x} = g(x) + \lambda f(t, x)\,,\ \lambda \geq 0\,, \tag{1}$$

where $g : M \to \mathbf{R}^k$ and $f : \mathbf{R} \times M \to \mathbf{R}^k$ are tangent vector fields, g is continuous, and f is Carathéodory and T-periodic in the first variable. We investigate the set X of the T-pairs of (1); i.e. of the pairs (λ, x), where $\lambda \geq 0$ and $x : \mathbf{R} \to M$ is an absolutely continuous T-periodic function such that $\dot{x}(t) = g\left(x(t)\right) + \lambda f\left(t, x(t)\right)$ a.e. in $\mathbf{R}$. Endowing X with a natural topology, we will obtain (Theorem 3.1 below) some conditions ensuring the existence of a non-compact connected component of the set of nontrivial T-pairs (i.e. not of the type $(0, x)$ with x constant) of (1), which emanates from the set of the rest points of the unperturbed equation

$$\dot{x} = g(x)\,. \tag{2}$$

In a particular case, when M is closed in $\mathbf{R}^k$, and $\deg(g, M)$ is well defined and non-zero, this set will turn out to be unbounded.

As applications we get a continuation principle and a generalization of a result from [8] on the multiplicity of forced oscillations of pendulum-type equations.

Some earlier results in this direction may be found in [1] where, under the weaker assertion that the set Y of T-periodic solutions of the unperturbed equation (2) is compact, it is proved (for the case of continuous perturbations) the existence of a

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non-compact connected component of X, emanating from Y. For the results related to the continuation principles see also [2] and [9].

2. The set of T-pairs

Let M be a (not necessarily closed) boundaryless differentiable manifold embedded in $\mathbf{R}^k$ endowed with the standard Euclidean norm $|\cdot|$. Given $p \in M$, T_pM denotes the tangent space to M at p (which is a subspace of $\mathbf{R}^k$). We will say that $f : \mathbf{R} \times M \to \mathbf{R}^k$ is a T-periodic Carathéodory tangent vector field on M if it satisfies the following requirements:

(C1) for each $p \in M$, the map $t \mapsto f(t,p)$ is Lebesgue measurable on $\mathbf{R}$;
(C2) for a.a. $t \in \mathbf{R}$, the map $p \mapsto f(t,p)$ is continuous on M;
(C3) for any compact set $K \subset M$, there exists a function $\gamma_K \in L^1_T(\mathbf{R})$ such that $|f(t,p)| \leq \gamma_K(t)$ for a.a. $t \in \mathbf{R}$ and all $p \in K$, here $L^1_T(\mathbf{R})$ denotes the space of L^1_{loc}, T-periodic maps $x : \mathbf{R} \to \mathbf{R}$;
(C4) for any $p \in M$, $f(t+T,p) = f(t,p) \in T_pM$ a.e. in $\mathbf{R}$.

Conditions (C1) – (C3) are the so-called Carathéodory type assumptions while (C4) says that f is a time-dependent T-periodic tangent vector field on M.

We will consider the following parametrized differential equation

$$\dot{x} = g(x) + \lambda f(t,x) \,, \tag{3}$$

where λ is a non-negative real parameter, $g : M \to \mathbf{R}^k$ is a continuous tangent vector field and f is as above. We investigate the structure of the set X of T-pairs, i.e. of the pairs (λ, x) with $\lambda \geq 0$ and x a T-periodic solution of (3) corresponding to λ.

Since any solution of (3) is continuous, the set X will be considered a metric subspace of $[0,\infty) \times C_T(M)$, where $C_T(M)$ is the metric subspace of the Banach space $C_T(\mathbf{R}^k)$ of all the T-periodic continuous maps $x : \mathbf{R} \to M$, with norm $\|\cdot\|$ given by $\|x\| = \max_{t\in\mathbf{R}} |x(t)|$. The reason for this choice of the environment space is that X turns out to be closed in $[0,\infty) \times C_T(M)$ (see below). As shown by an interesting example in [6], this would not be true if X was thought as a subspace of $[0,\infty) \times L^1_T(M)$ (which might appear as the most natural choice for the Carathéodory setting); here $L^1_T(M)$ denotes the subspace of the Banach space $L^1_T(\mathbf{R}^k) \cong L^1\big((0,T), \mathbf{R}^k\big)$ of the L^1_{loc}, T-periodic maps $x : \mathbf{R} \to M$. However, by the same method of [6], it is possible to show that both $[0,\infty) \times C_T(M)$ and $[0,\infty) \times L^1_T(M)$ induce the same topology on X.

To see that the space X of T-pairs is closed in $[0,\infty) \times C_T(M)$, consider a sequence $\{(\lambda_n, x_n)\}_{n\in\mathbf{N}}$ in X converging to $(\lambda_0, x_0) \in [0,\infty) \times C_T(M)$. One has, for any $t \in \mathbf{R}$,

$$x_n(t) = x_n(0) + \int_0^t \big[g\left(x_n(s)\right) + \lambda_n f\left(s, x_n(s)\right) \big]\, ds.$$

Since x_n converges uniformly to x_0, there exists a compact subset K of M which contains the image of x_n for any $n \in \mathbf{N}$. The assumption (C3) and the dominated convergence theorem yield $(\lambda_0, x_0) \in X$.

Observe that $C_T(M)$ is not complete unless M is complete (i.e. closed in $\mathbf{R}^k$). Nevertheless, since M is locally compact, $C_T(M)$ is always locally complete. As a consequence, X, being a closed subset of a locally complete space, is locally complete as well.

We will make use of the following version of Ascoli's theorem.

Theorem 2.1. *Let Y be a subset of $\mathbf{R}^k$ and B a bounded equicontinuous subset of $C([a,b]\,,Y)$. Then B is totally bounded in $C([a,b]\,,Y)$. In particular, if Y is closed, B is relatively compact.*

As a consequence, we have the following important property of the set X of all the T-pairs of (3).

Lemma 2.2. *Let M be a closed manifold in $\mathbf{R}^k$. Then any bounded subset of X is actually totally bounded. As a consequence, closed and bounded sets of T-pairs are compact.*

Proof. Since M is complete, given $A \subset X$ bounded, the set

$$\{(\lambda, x(t)) \in [0,\infty) \times M : (\lambda, x) \in A,\ t \in [0,T]\}$$

is contained in a compact set K. Hence, there exists a L^1_T function γ such that $|\dot{x}(t)| = |g(y) + \lambda f(t,y)| \leq \gamma(t)$ for a.a. $t \in [0,T]$ and all $(\lambda, y) \in K$. Thus A can be regarded as an equibounded set of equicontinuous functions from $[0,T]$ into $[0,\infty) \times M$. Ascoli theorem implies that A is totally bounded.

If A is assumed to be closed in X then, $[0,\infty) \times C_T(M)$ being complete, A is complete and therefore compact. ∎

Even when M is not complete, the proof of the above lemma shows that

Remark 2.3. *The space X is always locally totally bounded. Thus, being locally complete, X is actually locally compact.*

3. The main result

Before stating our main result, let us recall some basic facts and definitions. Let U be an open subset of a (boundaryless) differentiable manifold $M \subset \mathbf{R}^k$, and $v : M \to \mathbf{R}^k$ be a continuous tangent vector field such that the set $v^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (v, U) an integer, which we will call degree of the vector field v in U and denote by $\deg(v, U)$ (often called also Euler characteristic or Hopf index or rotation), which, roughly speaking, counts (algebraically) the number of zeros of v in U (see e.g. [10], and references therein).

In the flat case, namely if U is an open subset of $\mathbf{R}^k$, $\deg(v, U)$ is just the Brouwer degree (with respect to zero) of v in U (i.e. in any bounded open set V containing $v^{-1}(0)$ and such that $\bar{V} \subset U$). Using the equivalent definition of degree given in [4], one can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds.

In order to avoid formal complication in our statements, we will identify every space with its image in the following diagram of closed embeddings

$$\begin{array}{ccc} [0,\infty) \times M & \longrightarrow & [0,\infty) \times C_T(M) \\ \uparrow & & \uparrow \\ M & \longrightarrow & C_T(M) \end{array} \quad . \tag{4}$$

Where, the orizzontal arrows are defined by $p \mapsto \hat{p}$ and $(\lambda, p) \mapsto (\lambda, \hat{p})$, with $\hat{p}(t) \equiv p$; the vertical arrows by $p \mapsto (0,p)$ and, analogously, $x \mapsto (0,x)$.

According to these identifications, if Z is a subset of $[0,\infty) \times C_T(M)$, by $Z \cap M$ we mean the subset of M given by all $p \in M$ such that the pair $(0, \hat{p})$ belongs to Z. If Y is a subset of $[0,\infty) \times M$, then $Y \cap M$ represents the set $\{p \in M : (0,p) \in U\}$.

Notice that if Z is open in $[0,\infty)\times C_T(M)$ then $Z\cap M$ turns out to be open in M; analogously, if Y is open then $Y\cap M$ is open as well.

Recall that a pair $(\lambda,x)\in[0,\infty)\times C_T(M)$ is a T-pair if x is a T-periodic solution of (3) corresponding to λ. Within the set X of the T-pairs of (3) we distinguish the set of *trivial T-pairs*, that is of those T-pairs $(0,x)$ with x constant. By the above diagram of embeddings, we regard $g^{-1}(0)\subset M$ as the set of trivial T-pairs and $X\setminus g^{-1}(0)$ stands for the set of nontrivial T-pairs.

Theorem 3.1 below is the main result of this paper; it is a generalization of Theorem 3.3 of [8] to the case of Carathéodory periodic perturbations of an autonomous vector field.

Theorem 3.1. *Let $f:\mathbf{R}\times M\to\mathbf{R}^k$ and $g:M\to\mathbf{R}^k$ be tangent vector fields defined on a (boundaryless) differentiable manifold $M\subset\mathbf{R}^k$. Assume f satisfies (C1) – (C4), and g is continuous. Let Ω be an open subset of $[0,\infty)\times C_T(M)$, and assume that* $\deg(g,\Omega\cap M)$ *is well defined and nonzero. Then, Ω contains a connected set Γ of nontrivial T-pairs for (3) whose closure in $[0,\infty)\times C_T(M)$ meets $g^{-1}(0)\cap\Omega$ and is not contained in any compact subset of Ω. Moreover, if M is closed in $\mathbf{R}^k$, then Γ cannot be contained in a bounded complete subset of Ω. In particular, if $\Omega=[0,\infty)\times C_T(M)$, then Γ is unbounded.*

In order to prove Theorem 3.1, some preliminary results are needed.

A pair $(\lambda,p)\in[0,\infty)\times M$ is a *starting point* (for T-periodic solutions) if the Cauchy problem

$$\begin{cases}\dot{x}=g(x)+\lambda f(t,x)\\ x(0)=p\end{cases}\tag{5}$$

admits a T-periodic solution. A starting point (λ,p) is called *trivial* if $\lambda=0$ and $p\in g^{-1}(0)$. The set of all the starting points will be denoted by S. With the inclusions in diagram (4), $g^{-1}(0)$ can be regarded as the set of trivial T-pairs, and $S\setminus g^{-1}(0)$ stands for the set $S\setminus\big[\{0\}\times g^{-1}(0)\big]$ of nontrivial starting points of (3).

It is well known (see e.g. [3]) that, if g is C^1 and f is assumed to satisfy (C1) – (C4), then the Cauchy problem (5) admits a unique (local) solution for any $p\in M$, provided that the following assumption is satisfied:

(C5) for each compact subset K of M, there exists a L^1_T function α_K such that
$$|f(t,p_1)-f(t,p_2)|\le\alpha_K(t)\,|p_2-p_1|\,,$$
for a.a. $t\in\mathbf{R}$ and for any $p_1,p_2\in K$.

Moreover, under the assumptions (C1) – (C5), the set $D\subset[0,\infty)\times M$ of all the pairs (λ,p) such that the solution of (5) is defined in $[0,T]$ is open, thus locally compact. Obviously the set S of all the starting points of (3) is a closed subset of D, even if it could be not so in $[0,\infty)\times M$. This implies that S is locally compact. If U is an open subset of D, the set $S\cap U$ is open in S, thus it is locally compact as well.

Theorem 3.2. *Let M, g, f, S and D be as above. If U is an open subset of D such that* $\deg(g,U\cap M)$ *is well defined and nonzero, then* $(S\cap U)\setminus g^{-1}(0)$ *admits a connected subset whose closure in U meets $g^{-1}(0)$ and is not compact.*

Proof. It follows the outline of the proof of Theorem 3.1 in [7] which was given for the special case of C^1 tangent vector fields. ∎

It is important to notice that, in the Carathéodory setting, the property (C5) is, in some sense, generic. More precisely:

Remark 3.3. *Assume f satisfies (C1) – (C4), then there exists a sequence $\{f_n\}$ of equi-Carathéodory T-periodic tangent vector fields satisfying (C5) such that if $p_n \to p_0$ then, for a.a. $t \in \mathbf{R}$, $f_n(t, p_n) \to f(t, p_0)$. Namely (compare* [6]*):*

$$f_n(t,p) = \pi_p \left(\int_M \varphi_n(p,q) f(t,q)\, dq \right),$$

where $\pi_p : \mathbf{R}^k \to T_pM$ is the orthogonal projection and $\varphi_n : M \times M \to \mathbf{R}$ is a smooth function such that:

1. *$\varphi(p,q) \geq 0$ for $(p,q) \in M \times M$;*
2. *$\varphi(p,q) = 0$ whenever $|p - q| > 1/n$;*
3. *$\int_M \varphi_n(p,q)\, dq = 1$ for any $p \in M$.*

The last preliminary result that is needed for the proof of Theorem 3.1 is the following global connectivity result of [5].

Lemma 3.4. *Let Y be a locally compact Hausdorff space and let Y_0 be a compact subset of Y. Assume that no open compact subset of Y contains Y_0. Then $Y \backslash Y_0$ contains a not relatively compact component whose closure intersects Y_0.*

Proof of Theorem 3.1. Let X denote the set of T-pairs of (3). Since X is closed, it is enough to show that Ω contains a connected set Γ of nontrivial T-pairs whose closure in $X \cap \Omega$ meets $g^{-1}(0)$ and is not compact.

Assume first that g is C^1 and f satisfies (C5) in addition to (C1) – (C4). Denote by S the set of all starting points of (3), and let $\tilde{S}$ denote the set of the starting points (λ, p) such that the pair (λ, x), with x solution of (5), is contained in Ω. Obviously $\tilde{S}$ is an open subset of S, thus we can find an open subset U of D such that $S \cap U = \tilde{S}$ (recall that D is the set of all the pairs (λ, p) such that the solution of (5) is defined in $[0, T]$). We have that

$$g^{-1}(0) \cap \Omega = g^{-1}(0) \cap \tilde{S} = g^{-1}(0) \cap U,$$

thus $\deg(g, U \cap M) = \deg(g, \Omega \cap M) \neq 0$. Applying Theorem 3.2, we get the existence of a connected set $\Sigma \subset (S \cap U) \setminus g^{-1}(0)$ whose closure in U is not compact and meets $g^{-1}(0)$. Let $h : X \to S$ be the map which assigns to any T-pair (λ, x) the starting point $(\lambda, x(0))$. Observe that h is continuous, onto and, by the assumptions on f and g, it is also one to one. Furthermore, by the continuous dependence on initial data, we get the continuity of $h^{-1} : S \to X$. Thus h maps $X \cap \Omega$ homeomorphically onto $S \cap U$, and the trivial T-pairs correspond to the trivial starting points under this homeomorphism. This implies that $\Gamma = h^{-1}(\Sigma)$ satisfies the requirements.

Let us remove the additional assumptions on g and f. Take $Y_0 = g^{-1}(0) \cap \Omega$ and $Y = X \cap \Omega$. We have only to prove that the pair (Y, Y_0) satisfies the hypothesis of Lemma 3.4. Assume the contrary. We can find a relatively open compact subset C of Y containing Y_0. Thus there exists an open subset W of Ω with closure $\overline{W}$ contained in Ω and such that $W \cap Y = C$, $\partial W \cap Y = \emptyset$. Since C is compact and $[0, \infty) \times M$ is locally compact, we can choose W in such a way that the set

$$\{(\lambda\,,\, x(t)) \in [0, \infty) \times M : (\lambda, x) \in W,\ t \in [0, T]\}$$

is contained in a compact subset K of $[0, \infty) \times M$. This implies that W is bounded with complete closure in Ω and $W \cap M$ is a relatively compact subset of $\Omega \cap M$. In particular g is nonzero on the boundary of $W \cap M$ (relative to M). By well known approximation results on manifolds, we can find a sequence $\{g_i\}$ of smooth tangent vector fields uniformly approximating g. For $i \in \mathbf{N}$ large enough, we get

$$\deg(g_i, W \cap M) = \deg(g, W \cap M).$$

Furthermore, by excision,

$$\begin{aligned} \deg(g, W \cap M) &= \deg(g, \Omega \cap M), \\ \deg(g_i, \Omega \cap M) &= \deg(g_i, W \cap M). \end{aligned}$$

Hence $\deg(g_i, \Omega \cap M) \neq 0$. Therefore, given i large enough, the first part of the proof can be applied to the equation

$$\dot{x} = g_i(x) + \lambda f_i(t, x), \tag{6}$$

where $\{f_i\}$ is a sequence of equi-Carathéodory T-periodic tangent vector fields as in Remark 3.3.

Let X_i denote the set of T-pairs of (6). There exists a connected subset Γ_i of $\Omega \cap X_i$ whose closure in Ω meets $g_i^{-1}(0) \cap W$ and is not contained in any compact subset of Ω. Since W is bounded with complete closure, as in the proof of Lemma 2.2 $X_i \cap \overline{W}$ is compact. Thus, for i large enough, there exists a T-pair $(\lambda_i, x_i) \in \Gamma_i \cap \partial W$ of (6). Therefore, since for any $i \in \mathbf{N}$ and $t \in \mathbf{R}$ the pair $(\lambda_i, x_i(t))$ belongs to the compact subset K of $[0, \infty) \times M$ introduced above, there exists a function $\gamma \in L^1_T(\mathbf{R})$ such that

$$|\dot{x}_i(t)| = |g_i\left(x_i(t)\right) + \lambda_i f_i\left(t, x_i(t)\right)| \leq \gamma(t) \text{ for a.a. } t \in \mathbf{R} \text{ and all } i \in \mathbf{N}.$$

As a consequence, the sequence $\{x_i\}$ is equicontinuous and by Ascoli's theorem we may assume that $x_i \to x_0$ in $C_T(M)$. Moreover, without loss of generality, $\lambda_i \to \lambda_0$ and, consequently, $(\lambda_0, x_0) \in \partial W$. Thus, by the assumptions on the sequences $\{g_i\}$ and $\{f_i\}$, for a.a. $t \in [0, T]$ we have $g_i\left(x_i(t)\right) \to g\left(x_0(t)\right)$ and $f_i\left(t, x_i(t)\right) \to f\left(t, x_0(t)\right)$; therefore, by the dominated convergence theorem,

$$x_0(t) = x_0(0) + \int_0^t \left[g\left(x_0(s)\right) + \lambda_0 f\left(s, x_0(s)\right) \right] ds\,.$$

In other words (λ_0, x_0) is a T-pair in ∂W. This contradicts the assumption $\partial W \cap Y = \emptyset$.

It remains to prove the last assertion. Let M be closed and let $\Gamma \subset \Omega$ be the connected set of nontrivial T-pairs obtained above. Assume, by contradiction, Γ contained in a bounded and complete subset H of Ω. In this case, it is easily verified that also the closure $\bar{\Gamma}$ of Γ in $[0, \infty) \times C_T(M)$ is contained in H. Thus, being closed and bounded, $\bar{\Gamma}$ is compact by Lemma 2.2; a contradiction. ■

Consider, for example, the case when $M = \mathbf{R}^k$. If $g^{-1}(0)$ is compact and $\deg(g, M) \neq 0$, then there exists an unbounded connected set Γ of T-pairs of (3) in $[0, \infty) \times C_T(M)$ which meets $g^{-1}(0)$. By Theorem 3.3 in [8] the existence of this unbounded branch cannot be destroyed by the choice of a *continuous* f. Theorem 3.1 above implies that even the choice of a *Carathéodory* tangent vector field f cannot affect the existence of Γ. However, as shown by simple examples, Γ may be contained in the slice $\{0\} \times C_T(M)$.

4. Applications

Most of the following results are related to (and inspired by) those of section 3 in [6] which are valid in the case when $g = 0$.

When the manifold M is closed in $\mathbf{R}^k$, from Theorem 3.1 we deduce the following "geometric" feature of the set of T-pairs of (3).

Theorem 4.1. *Let M, g and f be as in Theorem 3.1 and assume in addition M is closed in $\mathbf{R}^k$. Let U be an open subset of M. If $\deg(g, U)$ is well defined and nonzero, then (3) admits a connected set Γ of nontrivial T-pairs whose closure $\overline{\Gamma}$ meets $U \cap g^{-1}(0)$ and satisfies at least one of the following properties:*

(1) *$\overline{\Gamma}$ is unbounded;*
(2) *$\overline{\Gamma}$ meets $g^{-1}(0)$ outside U.*

In particular, if $g^{-1}(0) \subset U$, then (1) holds.

Proof. Define $\Omega = ([0, \infty) \times C_T(M)) \setminus (g^{-1}(0) \setminus U)$. Since $\Omega \cap g^{-1}(0) = U \cap g^{-1}(0)$, by the excision property of the degree,

$$\deg(g, \Omega \cap M) = \deg(g, U) \neq 0 .$$

Thus, by Theorem 3.1 there exists a connected set of nontrivial T-pairs whose closure $\bar{\Gamma}$ in $[0, \infty) \times C_T(M)$ is not contained in a compact subset of Ω.

Assume that $\bar{\Gamma} \cap \left(g^{-1}(0) \setminus U\right) = \emptyset$. In this case $\bar{\Gamma} \subset \Omega$. Since M is assumed closed, by Lemma 2.2, $\bar{\Gamma}$ cannot be both bounded and complete. On the other hand $\bar{\Gamma}$, being a closed subset of the complete metric space $[0, \infty) \times C_T(M)$, is complete as well. Hence $\bar{\Gamma}$ must be unbounded. ■

Corollary 4.2. *Let M be a compact boundaryless manifold with $\chi(M) \neq 0$ and let g and f as in Theorem 3.1. Then there exists an unbounded connected set G of T-pairs which meets $g^{-1}(0)$ and such that $\pi_1(G) = [0, \infty)$, where π_1 denotes the projection onto the first factor of $[0, \infty) \times C_T(M)$.*

Proof. By the Poincaré-Hopf theorem $\deg(g, M) = \chi(M) \neq 0$. Hence by Theorem 4.1 there exists an unbounded connected set Γ of nontrivial T-pairs whose closure $\bar{\Gamma}$ meets $g^{-1}(0)$. Take $G = \bar{\Gamma}$; since M is bounded and G is unbounded and connected, it meets $\{\lambda\} \times C_T(M)$, for any $\lambda \geq 0$, and the assertion follows. ■

Another application of Theorem 3.1 is the following continuation principle. It extends (but only in the context of differentiable manifolds) a result proved in [2] for locally Lipschitzian vector fields on closed flow-invariant ENR's.

Theorem 4.3. *Let g and f as in Theorem 3.1 and let Ω_0 be a bounded open subset of $C_T(M)$ with complete closure and such that*

(i) *for any $\bar{\lambda} > 0$ given, the family of functions*
$$\left\{t \mapsto |g\,(x(t)) + \lambda f\,(t, x(t))| : x \in \Omega_0 \;,\; \lambda \in [0, \bar{\lambda}]\right\}$$
is dominated by an $L^1_T(\mathbf{R})$ function;
(ii) *the degree $\deg(g, M \cap \Omega_0)$ is well defined and nonzero.*

Then the equation (3) admits in $[0, \infty) \times \Omega_0$ a connected set of T-pairs whose closure in $[0, \infty) \times C_T(M)$ meets $\Omega_0 \cap g^{-1}(0)$ and either intersects $[0, \infty) \times \partial\Omega_0$ or meets $\{\lambda\} \times \Omega_0$ for any $\lambda \geq 0$.

In particular, if in addition

(iii) *the set $[0, 1] \times \partial\Omega_0$ does not contain any T-pair of (3),*

then the equation $\dot{x} = g(x) + f(t, x)$ has a T-periodic solution in Ω_0.

Proof. Take $\Omega = [0,\infty) \times \Omega_0$, assumption (ii) implies that $\deg(g, \Omega \cap M) \neq 0$. Hence, by Theorem 3.1, there exists a connected set Γ of nontrivial T-pairs in Ω whose closure $\bar{\Gamma}$ in $[0,\infty) \times C_T(M)$ meets $\Omega \cap g^{-1}(0) = \Omega_0 \cap g^{-1}(0)$ and is not relatively compact in Ω.

Assume that $\bar{\Gamma} \cap \big([0,\infty) \times \partial\Omega_0\big) = \emptyset$. In this case $\bar{\Gamma}$ must be unbounded, since otherwise, arguing as in the proof of Lemma 2.2, from assumption (i) it would follow the compactness of $\bar{\Gamma}$. Hence, given $\lambda \geq 0$, $\bar{\Gamma}$ cannot be contained in $[0,\lambda] \times \Omega_0$. The assertion follows by the connectedness of $\bar{\Gamma}$.

The last statement is proved by an analogous argument. ∎

Remark 4.4. *If M is a closed manifold, then in Theorem 4.3 the assumption (i) is always fulfilled. Indeed, if Z denotes the closure in $\mathbf{R}^k$ of the set $\{x(t) : x \in \Omega_0\ ,\ t \in [0,T]\}$ then, Ω_0 being bounded, Z is a compact subset of M, because M is closed in $\mathbf{R}^k$. Hence, by the Carathéodory assumption on f, there exists a $L^1_T(\mathbf{R})$ function γ_Z such that $|f(t,p)| \leq \gamma_Z(t)$ for a.a. $t \in \mathbf{R}$ and all $p \in Z$. Thus, for any $x \in \Omega_0$ and $\lambda \in [0,\bar{\lambda}]$,*

$$|g\,(x(t)) + \lambda f\,(t, x(t))| \leq \max_{p \in Z} |g(p)| + \gamma_Z(t)\ ,\ \textit{a.e. in } \mathbf{R}.$$

As a consequence of Theorem 3.1 we get the following continuation principle that, in the flat case, i.e. when $M = \mathbf{R}^k$, reduces to Theorem 2 in [2].

Corollary 4.5. *Let g, f and M be as in Theorem 4.3 and assume in addition M to be closed in $\mathbf{R}^k$. Let Ω_0 be an open bounded subset of $C_T(M)$ such that*

(i) *the degree $\deg(g, M \cap \Omega_0)$ is well defined and nonzero;*
(ii) *the set $[0,1] \times \partial\Omega_0$ does not contain any T-pair of (3).*

Then the equation $\dot{x} = g(x) + f(t,x)$ has a T-periodic solution in Ω_0.

Proof. Since M is closed, $C_T(M)$ is a complete metric space; hence Ω_0 has complete closure. The assertion follows from Remark 4.4 and Theorem 4.3. ∎

A straightforward consequence of Corollary 4.5 is the following.

Corollary 4.6. *Let g, f and M as in Theorem 3.1 and assume in addition M to be closed in $\mathbf{R}^k$. Let W be an open and bounded subset of M such that*

(i) *the degree $\deg(g, W)$ is well defined and nonzero;*
(ii) *for any $\lambda \in [0,1]$, if x is a T-periodic solution of (3) then $\partial W \cap x\,([0,T]) = \emptyset$.*

Then the equation $\dot{x} = g(x) + f(t,x)$ has a T-periodic solution x such that $x\,([0,T]) \subset W$.

Proof. Take $\Omega_0 = \{x \in C_T(M) : x\,([0,T]) \subset W\}$. Since $W = M \cap \Omega_0$, the degree $\deg(g, M \cap \Omega_0)$ is well defined and nonzero. The assertion follows applying Corollary 4.5 to Ω_0. ∎

As a final application of Theorem 3.1 we extend a result of [8] about forced oscillations of a pendulum-type equation. Consider the following second order differential equation

$$\ddot{\theta} = g(\theta) + \lambda f(t,\theta), \tag{7}$$

where $g : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are 2π-periodic in θ, g is continuous and f is Carathéodory, T-periodic in t. It is convenient to regard g and f as defined on S^1 and $\mathbf{R} \times S^1$ respectively.

Equivalently, (7) may be seen as a first order differential equation on the manifold $S^1 \times \mathbf{R}$ (the tangent bundle to S^1) as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g(x_1) + \lambda f(t, x_1)\,. \end{cases} \tag{8}$$

Assume that g takes both positive and negative values and it has exactly two zeros θ_1 and θ_2 on S^1. Hence $(\theta_1, 0)$ and $(\theta_2, 0)$, which we identify with θ_1 and θ_2, are the unique zeros of the vector field $(x_1, x_2) \mapsto (x_2, g(x_1))$ (which can be regarded as a tangent vector field on $S^1 \times \mathbf{R}$). In what follows, θ_1 and θ_2 will be also identified, respectively, with the trivial T-pairs $(0; \hat{\theta}_1, 0)$, $(0; \hat{\theta}_2, 0)$, where $\hat{\theta}_i$ is the constant map $t \mapsto \theta_i$ for $i \in \{1, 2\}$.

Theorem 4.7. *Let f and g be as above. Denote by C_1 and C_2 the connected components of the set of T-pairs of (8) containing θ_1 and θ_2 respectively. Then for any $\mu \geq 0$, the intersections of C_1 and C_2 with $[0, \mu] \times C_T(S^1 \times \mathbf{R})$ are bounded. Moreover, exactly one of the following alternatives holds:*

(1) *$C_1 = C_2$,*
(2) *C_1 and C_2 are disjoint and both unbounded.*

In particular, if the second alternative holds, there exist at least two distinct T-periodic solutions of (7) for each $\lambda \in [0, \infty)$.

Proof. The proof is analogous to that of Theorem 4.1 in [8] with only small changes due to the Carathéodory assumptions on f. ∎

The theorem above leads to the desired multiplicity result.

Corollary 4.8. *Let $g : \mathbf{R} \to \mathbf{R}$ be a 2π-periodic continuous function whose image contains 0 in its interior. Assume that g has exactly two zeros $\theta_1, \theta_2 \in [0, 2\pi)$. Then, given a Carathéodory function $(t, \theta) \mapsto f(t, \theta)$, T-periodic in $t \in \mathbf{R}$ and 2π-periodic in θ, there exists $\lambda_f > 0$ such that the equation (7) has at least two geometrically distinct T-periodic solutions for each $\lambda \in [0, \lambda_f]$.*

Proof. Let C_1 and C_2 be as in Theorem 4.7. It is enough to consider the case when $C_1 = C_2 = C$. Theorem 4.7 implies that the intersection C_0 of C with the slice $\{0\} \times C_T(S^1 \times \mathbf{R})$ is bounded, hence compact; in particular, any connected component of C_0 is compact. Proceeding as in the proof of Corollary 4.2 in [8] one can show that the T-pairs θ_1 with θ_2 belong to different connected components of C_0, say $C_{0,1}$ and $C_{0,2}$ respectively. By the local compactness of C and the compactness of $C_{0,1}$ and $C_{0,2}$ it follows that in $C_T(S^1 \times \mathbf{R})$ there exist open neighborhoods W_i, $i = 1, 2$, of $C_{0,i}$, with disjoint closures, and a positive number λ_f such that

$$C \cap \big\{ [0, \lambda_f] \times (\partial W_1 \cup \partial W_2) \big\} = \emptyset.$$

The assertion now follows from the connectedness of C. ∎

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