ABOUT THE SIGN OF ORIENTED FREDHOLM OPERATORS BETWEEN BANACH SPACES

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Abstract. We give conditions for an oriented family of bounded Fredholm operators of index zero between Banach spaces to have a sign jump. In particular, we discuss criteria for detecting the sign jump in some special situations. For instance, when a sort of Crandall-Rabinowitz condition for bifurcation is assumed or in the case of a family of Leray-Schauder type. Finally, some examples of ordinary differential operators are presented to illustrate the meaning of the abstract results.

1. Introduction

In [BF1] the first two authors introduced a fairly simple notion of orientation for linear Fredholm operators of index zero between real vector spaces. Any such operator, singular or nonsingular, admits exactly two orientations, each of them makes, by definition, the operator oriented. If an operator is invertible, one the two possible orientations is more relevant than the other, and for this reason called *natural* (see Definition 3.4 below). Thus it makes sense to assign to any oriented isomorphism a sign: 1 if the orientation is natural and -1 in the opposite case. For a noninvertible Fredholm operator of index zero no one of the two orientations is more relevant than the other.

In the finite dimensional context, the orientation of a linear operator turns out to be equivalent to the choice of a pair of orientations, one of the source space and one of the target space, up to an inversion of both of them. In this particular case the sign of an isomorphism agrees with the usual well known notion: 1 or -1 depending on whether the operator preserves or inverts the orientations of the spaces.

A crucial fact is that in the framework of Banach spaces the orientation has a sort of stability; in the sense that an orientation of an operator L induces, in a very natural way, an orientation to any operator which is sufficiently close to L. Using this fact, the notion of orientation was extended (in [BF1] and [BF2]) to the nonlinear case; namely, to the case of a C^1 Fredholm map of index zero between real Banach spaces (and Banach manifolds). Such an extension coincides (in the C^1 case) with the notion given by Dold in [Do, exercise 6, p. 271] for maps between finite dimensional manifolds and, in the most important cases, with the notion due to Fitzpatrick, Pejsachowicz and Rabier in [FPR2] for maps between Banach manifolds. The definition in [FPR2], however, does not agree completely in the finite dimensional case with that in [Do]. For example, a constant map whose domain is a nonorientable manifold is nonorientable according to Dold and

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orientable according to Fitzpatrick, Pejsachowicz and Rabier (but, peculiarly, with only one orientation).

In [BF1], by means of the concept of orientation, a degree theory for Fredholm maps between Banach manifolds was introduced. This degree agrees, in the most important cases, with that developed in [FP1], [FP2], [FPR1] and [FPR2]. The difference between the two theories, however, is mainly in the construction method and, in our opinion, in the simplicity of the definition given in [BF1] which easily leads to a homotopy invariance property in the C^1 case.

Let $f : M \to N$ be an oriented Fredholm map of index zero between two real Banach manifolds and let $y \in N$ be such that $f^{-1}(y)$ is compact. In this case the triple (f, M, y) is said to be admissible for the degree (see [BF1]). Assume first that (f, M, y) is generic; that is, y is a regular value for f. According to [BF1], the degree in this case is, as in the finite dimensional context, an algebraic count of the number of solutions of the equation $f(x) = y$. More precisely,

$$\deg(f, M, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(f'(x)),$$

where $f'(x) : T_xM \to T_yN$ is the Fréchet derivative of f at x, and $\operatorname{sign}(f'(x))$ denotes the sign of the oriented isomorphism $f'(x)$. In the general case the degree is defined by considering the restriction of f to a convenient neighborhood U of $f^{-1}(y)$ and putting

$$\deg(f, M, y) = \deg(f, U, z),$$

where z is a regular value for f which is sufficiently close to y.

To compute the degree of an admissible triple, the usual technique is the following: first, if necessary, deform (f, M, y), via a suitable homotopy, to a simpler admissible triple (g, M, z), where z is a regular value for g; then compute

$$\sum_{x \in g^{-1}(z)} \operatorname{sign}(g'(x)).$$

It is therefore of some interest to develop methods for evaluating the sign of an oriented isomorphism. That is, methods to decide whether or not the orientation of a given oriented isomorphism coincides with the natural one. This is, in fact, our purpose.

2. Preliminaries

Given two vector spaces E and F, we shall denote by $\mathrm{L}(E, F)$, or simply by $\mathrm{L}(E)$ when $F = E$, the vector space of all the linear operators from E into F. By $\mathrm{F}(E, F)$, or $\mathrm{F}(E)$ when $F = E$, we will mean the subspace of $\mathrm{L}(E, F)$ of the operators with finite dimensional image. The image of an element $L \in \mathrm{L}(E, F)$ will be denoted by $\operatorname{Im} L$, its kernel by $\operatorname{Ker} L$, and its cokernel (i.e. the quotient $F/\operatorname{Im} L$) by $\operatorname{coKer} L$. Other useful notations are $\operatorname{Iso}(E, F)$ for the set of all the isomorphisms from E onto F, and $\mathrm{GL}(E)$ for the set of the automorphisms of E (i.e. $\mathrm{GL}(E) = \operatorname{Iso}(E, E)$).

Let us recall some useful properties of Fredholm operators between vector spaces.

A linear operator $L : E \to F$ is (*algebraic*) *Fredholm* if $\operatorname{Ker} L$ and $\operatorname{coKer} L$ are finite dimensional. In this case, its *index* is the integer

$$\begin{aligned}\operatorname{ind}(L) &= \dim(\operatorname{Ker} L) - \operatorname{codim}(\operatorname{Im} L)\\ &= \dim(\operatorname{Ker} L) - \dim(\operatorname{coKer} L).\end{aligned}$$

The set of Fredholm operators of index n between E and F will be denoted by $\Phi_n(E, F)$.

Observe that, given a vector space E and a finite codimensional subspace E_1 of E, the inclusion $S : E_1 \to E$ is Fredholm with $\operatorname{ind}(S) = -\operatorname{codim}(E_1)$, and any projection $P : E \to E_1$ (onto E_1) is Fredholm with $\operatorname{ind}(S) = \operatorname{codim}(E_1)$.

Remark 2.1. If E and F are finite dimensional, then any linear operator from E into F is Fredholm of index $\dim(E) - \dim(F)$. This is a consequence of the well known linear algebra formula

$$\dim(E) = \dim(\operatorname{Ker} L) + \dim(\operatorname{Im} L).$$

The following useful result is a direct consequence of the definition of Fredholm operator. Recall first that, given a linear operator $L : E \to F$, a subspace F_1 of F is said to be *transverse* to L if $\operatorname{Im} L + F_1 = F$.

Lemma 2.2. *Let $L : E \to F$ be a Fredholm operator and let F_1 be a subspace of F which is transverse to L. Then the restriction*

$$L_1 : L^{-1}(F_1) \to F_1$$

of L is Fredholm with $\operatorname{ind}(L_1) = \operatorname{ind}(L)$.

The following lemma is useful for computing the index of a Fredholm operator written in block-matrix form. For its proof we refer to [KG, Part 1, Chap. 3, Sect. 1].

Lemma 2.3. *Assume $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$, and write a Fredholm operator $L : E \to F$ in the following block-matrix form:*

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

If $A : E_1 \to F_1$ is invertible, then $\operatorname{ind}(L) = \operatorname{ind}(D - CA^{-1}B)$.

The following result is the algebraic analogous of a well known one in the context of Banach spaces (see (ii) in Theorem 2.6 below). The simple proof is given for the sake of completeness.

Theorem 2.4. *Let $L : E \to F$ be Fredholm, and take $K \in \mathrm{F}(E, F)$. Then the operator $L + K$ is Fredholm and* $\operatorname{ind}(L + K) = \operatorname{ind}(L)$.

Proof. Let F_1 be a finite dimensional subspace of F containing $\operatorname{Im} K$ and transverse to L. Put $E_1 = L^{-1}(F_1)$. Clearly E_1 is finite dimensional. In fact, because of Lemma 2.2 and Remark 2.1, one has $\dim(E_1) - \dim(F_1) = \operatorname{ind}(L)$. Let E_0 be such that $E_0 \oplus E_1 = E$, and define $F_0 = L(E_0)$. It is easy to check, using the transversality of F_1, that $F = F_0 \oplus F_1$. Writing L and K in block-matrix form, we get

$$L = \begin{pmatrix} L_{00} & 0 \\ 0 & L_{11} \end{pmatrix} \quad \text{and} \quad L + K = \begin{pmatrix} L_{00} & 0 \\ K_{10} & L_{11} + K_{11} \end{pmatrix}$$

Thus $L+K$ is Fredholm, $L_{00} : E_0 \to F_0$ being an isomorphism. Lemma 2.3 implies the following two equalities:

$$\operatorname{ind}(L) = \operatorname{ind}(L_{11}), \quad \operatorname{ind}(L+K) = \operatorname{ind}(L_{11}+K_{11}).$$

The assertion now follows from Remark 2.1. □

Theorem 2.5. *Let $L_1 : E_1 \to E_2$ and $L_2 : E_2 \to E_3$ be Fredholm operators. Then L_2L_1 is Fredholm and* $\operatorname{ind}(L_2L_1) = \operatorname{ind}(L_1) + \operatorname{ind}(L_2)$.

Proof. See e.g. the proof of Theorem 13.1 in [TL, Chap. IV], which is still valid in the purely algebraic context. □

We pass now to recall some useful properties of Fredholm operators in the framework of Banach spaces. In this category, the morphisms are bounded linear operators. Therefore, from now on, all the linear operators between Banach spaces will be assumed continuous. In this particular context, the spaces $\mathrm{L}(E,F)$ and $\mathrm{F}(E,F)$, and the sets $\mathrm{Iso}(E,F)$, $\mathrm{GL}(E)$ and $\Phi_n(E,F)$ will be considered made up of bounded operators only. Any direct sum in a Banach space is assumed to be topological (besides being algebraic); that is, the summands are closed subspaces of the containing space.

Fredholm operators between Banach spaces enjoy many properties and are studied in detail by several authors (see e.g. [TL]). We state two important properties that are meaningful in this context.

Theorem 2.6. *Let E and F be Banach spaces. The following properties hold:*

(i) *The set $\Phi_n(E,F)$ is open in the Banach space* $\mathrm{L}(E,F)$.
(ii) *If $K \in \mathrm{L}(E,F)$ is compact and $L \in \Phi_n(E,F)$, then $L+K \in \Phi_n(E,F)$.*

We conclude this section by recalling that if $L : E \to F$ is a Fredholm operator between Banach spaces, then its image is necessarily a closed subspace of F.

3. Oriented Fredholm operators and oriented maps

In this section we first summarize (with some minor changes) the notion of orientation for a Fredholm operator of index zero between real vector spaces introduced in [BF1]. After that we show how such a notion becomes stable in the context of Banach spaces. Then we give a brief review of the concept of orientability for a continuous family of (bounded) Fredholm operators of index zero between Banach spaces. We refer to [BF1], [BF2] and [BF3] for more details.

The concept of orientation for a Fredholm operator of index zero is purely algebraic. The basic tool to define such a notion is a simple extension of the determinant to the infinite dimensional context. Let us recall first this extension.

Let $T : E \to E$ be an endomorphism of a (not necessarily real) vector space and denote by I the identity operator on E. Assume that the image of the operator $K = I - T$ is contained in a finite dimensional subspace E_1 of E. Thus T maps E_1 into itself and, consequently, the determinant of its restriction $T_1 : E_1 \to E_1$ is well defined. As shown in [BF1], this determinant does not depend on the space E_1 containing $\operatorname{Im} K$. Therefore, it makes sense to call *determinant of T* this common value, and to denote it by $\det(T)$. From Theorem 2.4 it follows that T is Fredholm of index zero. Moreover, one can easily check that, as in the case when E is finite dimensional, T is invertible if and only if $\det(T) \neq 0$.

From now on, given a vector space E, the affine subspace of $\mathrm{L}(E)$ of the operators which are *admissible for the determinant* will be denoted by $\Psi(E)$. Namely,

$$\Psi(E) = \{T \in \mathrm{L}(E) : I - T \in \mathrm{F}(E)\}.$$

The proof of the following useful result is a routine task.

Proposition 3.1. *The determinant has the following fundamental properties:*

(1) *If $T \in \Psi(E)$ and E_1 is any subspace of E containing $\mathrm{Im}(I - T)$, then the restriction T_1 of T to E_1 belongs to $\Psi(E_1)$ and $\det(T) = \det(T_1)$.*
(2) *If $T_1, T_2 \in \Psi(E)$, then $T_2T_1 \in \Psi(E)$ and $\det(T_2T_1) = \det(T_2)\det(T_1)$.*
(3) *If $S : F \to E$ is an isomorphism and $T \in \Psi(E)$, then $S^{-1}TS \in \Psi(F)$ and $\det(S^{-1}TS) = \det(T)$.*
(4) *If $T_1 \in \Psi(E_1)$ and $T_2 \in \Psi(E_2)$, then $T_1 \times T_2 \in \Psi(E_1 \times E_2)$ and $\det(T_1 \times T_2) = \det(T_1)\det(T_2)$.*

In many cases, a practical method for computing the determinant of an operator $T \in \Psi(E)$ is given by the following consequence of Proposition 3.1.

Corollary 3.2. *Let $T \in \mathrm{L}(E)$ and let $E = E_0 \oplus E_1$. Assume that, with this decomposition of E, the matrix representation of T is of the type*

$$\begin{pmatrix} I_0 & U \\ V & S \end{pmatrix}$$

where I_0 is the identity operator on E_0. If $\dim(E_1) < +\infty$ (or, more generally, if $S \in \Psi(E_1)$ and the operators U and V have finite dimensional image), then $T \in \Psi(E)$ and

$$\det(T) = \det(S - VU). \tag{3.1}$$

Proof. Recall that, given the decomposition $E = E_0 \oplus E_1$, the entries of the matrix representation of an operator $R \in \mathrm{L}(E)$ are defined as P_jRS_i, $i, j = 0, 1$, where $S_0 : E_0 \to E$ and $S_1 : E_1 \to E$ denote the inclusions, and $P_0 : E \to E_0$ and $P_1 : E \to E_1$ are the projections associated with the decomposition. Taking into account this, it is easy to show that $T \in \Psi(E)$. Hence $\det(T)$ makes sense.

Observe first that the equality (3.1) is true in the particular case of a lower triangular matrix; that is, when U is the trivial operator. In this case, in fact, $I - T$ maps E into E_1, therefore $\det(T) = \det(S)$, since S is the restriction of T to E_1 (use property 1 of Proposition 3.1 if $\dim(E_1) = +\infty$).

The assertion is also true for the upper triangular matrix

$$R = \begin{pmatrix} I_0 & -U \\ 0 & I_1 \end{pmatrix}$$

where I_1 denotes the identity on E_1. To show this it is enough to consider the restriction of R to any finite dimensional space containing $E_1 + \mathrm{Im}\, U$.

To prove the assertion in the general case, consider the composite operator TR. Given $x_0 \in E_0$ and $x_1 \in E_1$, with a standard computation one gets

$$TR(x_0 + x_1) = x_0 + (Vx_0 + Sx_1 - VUx_1).$$

Therefore TR can be represented by the triangular matrix

$$\begin{pmatrix} I_0 & 0 \\ V & S - VU \end{pmatrix}$$

Thus, $\det(TR) = \det(S - VU)$. On the other hand, property 2 of Proposition 3.1 implies $\det(TR) = \det(T)\det(R)$ and, consequently, $\det(T) = \det(S - VU)$, as claimed. □

Let now E and F be two *real* vector spaces and let $L \in \Phi_0(E, F)$. A linear operator $A : E \to F$ is a *corrector* of L if its image is finite dimensional and $L + A$ is an isomorphism. Denote by $\mathcal{C}(L)$ the set of correctors of L. It is easy to see that $\mathcal{C}(L)$ is nonempty. Indeed, any linear operator $A \in \mathrm{L}(E, F)$ such that $\operatorname{Ker} A \oplus \operatorname{Ker} L = E$ and $\operatorname{Im} A \oplus \operatorname{Im} L = F$ is a corrector of L. In particular, if L is an isomorphism, the trivial operator belongs to $\mathcal{C}(L)$.

We define an equivalence relation on $\mathcal{C}(L)$ as follows. Let $A, B \in \mathcal{C}(L)$ and consider the following endomorphism of E:

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)\,.$$

Clearly, T is an invertible finite dimensional perturbation of the identity. Thus, its determinant is defined and nonzero. We say that A is *equivalent* to B or, more precisely, A is L-*equivalent* to B, if

$$\det\big((L + B)^{-1}(L + A)\big) > 0.$$

In [BF1] it is shown that this is an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes. A simpler proof of this could be carried out as a straightforward consequence of Proposition 3.1.

Definition 3.3. Let L be a Fredholm operator of index zero between real vector spaces. An *orientation* of L is one of the two equivalence classes of $\mathcal{C}(L)$. The operator L is *oriented* when one of its orientations is selected.

Formally, an oriented operator is a pair $L = (L', \omega)$, where $L' \in \Phi_0(E, F)$ and ω is one of the two equivalence classes of $\mathcal{C}(L')$. However, for the sake of simplicity, unless required for the understanding, we shall not use different symbols to discern between an oriented operator L and its underlying nonoriented part L'.

As in [BF2], the set of all oriented operators from E into F will be denoted by $\widehat{\Phi}_0(E, F)$, or simply by $\widehat{\Phi}_0(E)$ when $F = E$. Observe that there is a natural projection

$$p : \widehat{\Phi}_0(E, F) \to \Phi_0(E, F)$$

defined by forgetting the orientation.

Given $L \in \widehat{\Phi}_0(E, F)$, its orientation is denoted by $\mathcal{C}_+(L)$ and called *class of positive correctors* of L. The other equivalence class is the *opposite orientation* of L and denoted by $\mathcal{C}_-(L)$. The elements of $\mathcal{C}_-(L)$ are the *negative correctors* of L.

As regards the orientation of an operator L, two particular (possibly coexisting) cases deserve a specific attention: the first one is when L is invertible; the second when L is an endomorphism of a finite dimensional space. As shown in the following two definitions, in both cases a distinguished orientation emerges.

Definition 3.4 (Natural orientation)**.** An oriented isomorphism $L : E \to F$ has the *natural orientation* if the trivial operator is a positive corrector of L.

Definition 3.5 (Canonical orientation)**.** Let $L : E \to E$ be an endomorphism of a finite dimensional real vector space (or, more generally, assume $L \in \Psi(E)$). The operator L is oriented with the *canonical orientation* if for a positive corrector A of L (or, equivalently, for any positive corrector A of L) one has $\det(L + A) > 0$.

If $L : E \to E$ is an automorphism of a finite dimensional real vector space (or, more generally, an invertible finite dimensional perturbation of the identity), then both the natural and the canonical orientations are defined. We point out that they coincide if and only if $\det(L) > 0$.

Let $L \in \Phi_0(E, F)$. Given an isomorphism $S : G \to E$, from property 3 of Proposition 3.1 one can immediately deduce that two correctors A and B of L are L-equivalent if and only if AS and BS are LS-equivalent. An analogous assertion holds for a composition of the type SL, with $S \in \mathrm{Iso}(F, G)$. This justifies the following definition.

Definition 3.6. Let $L_1 \in \widehat{\Phi}_0(E_1, E_2)$ and $L_2 \in \widehat{\Phi}_0(E_2, E_3)$ be two oriented operators, and denote by L_1' and L_2' the corresponding nonoriented operators. The *oriented composition* L_2L_1 is the composition $L_2'L_1'$ with the orientation obtained by choosing as a positive corrector any operator $K \in \mathrm{F}(E_1, E_3)$ of the type

$$K = (L_2' + A_2)(L_1' + A_1) - L_2'L_1',$$

with $A_1 \in \mathcal{C}_+(L_1)$ and $A_2 \in \mathcal{C}_+(L_2)$.

Notice that the oriented composition is associative. Thus, it makes sense to consider the oriented composition of three, or even more, oriented operators. Unless otherwise specified, from now on, the composition of oriented operators will be regarded as an oriented composition.

The orientation of an operator $L \in \Phi_0(E, F)$ can be regarded as the orientation of the restriction of L to any suitable pair of subspaces of E and F. Precisely, let F_1 be a subspace of F which is transverse to L and let $E_1 = L^{-1}(F_1)$. Consider any direct complement E_0 to E_1 in E and split E and F as follows: $E = E_0 \oplus E_1$, $F = L(E_0) \oplus F_1$. With this decomposition, L can be represented by a matrix

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}$$

Clearly, L_0 is an isomorphism and $L_1 \in \Phi_0(E_1, F_1)$. Thus, any linear operator $A : E \to F$ of the type

$$\begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix} \tag{3.2}$$

is a corrector of L if and only if A_1 is a corrector of L_1. From Proposition 3.1 one can easily deduce that two correctors of L_1 are L_1-equivalent if and only if the corresponding correctors of L (according to (3.2)) are L-equivalent. This shows that the following definition is well posed.

Definition 3.7. Given $L \in \Phi_0(E, F)$, let F_1 be a subspace of F which is transverse to L, and denote by L_1 the restriction of L to $L^{-1}(F_1)$ as domain and to F_1 as codomain. Two orientations, of L and L_1 respectively, are said to be *correlated* (or *one induced by the other*) if there exists a projection P of E onto $L^{-1}(F_1)$ and a positive corrector A_1 of L_1 such that the operator $A = SA_1P$ is a positive corrector of L, where $S : F_1 \to F$ is the inclusion. If L and L_1 are oriented with correlated orientations, L_1 is called the *oriented restriction* of L and L the *oriented extension* of L_1.

We define now the sign of an oriented operator.

Definition 3.8. Given $L \in \widehat{\Phi}_0(E,F)$, its *sign* is the integer

$$\operatorname{sign}(L) = \begin{cases} +1 & \text{if } L \text{ is invertible and naturally oriented,} \\ -1 & \text{if } L \text{ is invertible and not naturally oriented,} \\ 0 & \text{if } L \text{ is not invertible.} \end{cases}$$

Remark 3.9. Let $L \in \Phi_0(E,F)$ and let A be a corrector of L. We have

$$(L+A)^{-1}L = I - (L+A)^{-1}A\,.$$

Thus $(L+A)^{-1}L$ is a finite dimensional perturbation of the identity and, consequently, its determinant is well defined. One can easily check that if L is oriented and A is a positive corrector of L, then

$$\operatorname{sign}(L) = \operatorname{sign}\left(\det\left((L+A)^{-1}L\right)\right).$$

From the notion of correlated orientations (Definition 3.7) one can easily deduce the following property of the sign.

Proposition 3.10 (Reduction property)**.** *Let $L \in \Phi_0(E,F)$ and let F_1 be a subspace of F which is transverse to L. Denote by L_1 the restriction of L to the spaces $L^{-1}(F_1)$ and F_1. If L and L_1 are oriented with correlated orientations, then*

$$\operatorname{sign}(L) = \operatorname{sign}(L_1).$$

Another useful property of the sign is expounded by the following proposition, whose proof is immediate.

Proposition 3.11 (Invariance property)**.** *Let $L \in \widehat{\Phi}_0(E,F)$, and let $S : G \to E$ and $T : F \to H$ be two naturally oriented isomorphisms. Then*

$$\operatorname{sign}(TLS) = \operatorname{sign}(L).$$

From now on, throughout the paper, unless otherwise specified all the vector spaces considered are real Banach spaces, and all the linear operators between them are assumed to be bounded. Therefore, given any (bounded) $L \in \Phi_0(E,F)$, by a corrector of L we shall actually mean a bounded corrector. By abuse of notation, the set of correctors of L will be still denoted by $\mathcal{C}(L)$, even if it is a subset of the set considered in the purely algebraic case. As a consequence of the Hahn-Banach Theorem, the set $\mathcal{C}(L)$ is nonempty also in this enriched context, and an orientation of L can be regarded as an equivalence class of bounded correctors of L. As in the algebraic case, the set of oriented operators from a Banach space E into a Banach space F will be denoted by $\widehat{\Phi}_0(E,F)$, even if, we recall, all the operators are assumed to be bounded.

The salient fact that distinguishes the orientation in the context of Banach spaces from the orientation in the algebraic case is, loosely speaking, the influence exerted by an oriented operator over its neighbors. More precisely, due to the fact that $\operatorname{Iso}(E,F)$ is open in $\mathrm{L}(E,F)$, any corrector of a given $L \in \Phi_0(E,F)$ remains a corrector for all the operators in a neighborhood of L. This means that an oriented operator induces, by a sort of stability, an orientation on any sufficiently close operator. Thus, it is evident how to endow $\widehat{\Phi}_0(E,F)$ with a topology which makes the natural projection

$$p : \widehat{\Phi}_0(E,F) \to \Phi_0(E,F)$$

a two-fold covering space (see [BF2] for a formal definition).

In the context of Banach spaces one may ask whether or not the determinant function $\det : \Psi(E) \to \mathbb{R}$ is continuous. The following example provides a negative answer.

Example 3.12. For any $n \in \mathbb{N}$, let $T_n : \ell^2 \to \ell^2$ be the operator obtained by multiplying the first n coordinates of any $x \in \ell^2$ by $1+1/n$ (and leaving unchanged all the others). Clearly $T_n \in \Psi(\ell^2)$. Moreover

$$\det(T_n) = \left(1 + \frac{1}{n}\right)^n,$$

since $1 + 1/n$ is the only eigenvalue different from 1 and its multiplicity is n. Thus $\det(T_n) \to e$. On the other hand $T_n \to I$ and $\det(I) = 1$.

Notice that, if we fix a finite dimensional subspace E_1 of E, as a consequence of the definition, the restriction of the determinant to the affine subspace

$$\Psi(E, E_1) = \{T \in \mathrm{L}(E) : I - T \in \mathrm{F}(E, E_1)\}$$

of $\Psi(E)$ is continuous.

The following useful result shows that, in spite of fact that the determinant function can be discontinuous, its sign, if nonzero, is stable.

Proposition 3.13. *Let E be a Banach space. Then the restriction of the function $T \mapsto \operatorname{sign}(\det(T))$ to the open subset $\mathrm{GL}(E) \cap \Psi(E)$ of $\Psi(E)$ is continuous.*

Proof. Let $T_0 \in \mathrm{GL}(E) \cap \Psi(E)$. Since $\mathrm{GL}(E)$ is open, there exists a convex neighborhood U of T_0 in $\Psi(E)$ contained in $\mathrm{GL}(E)$. Thus $\det(T) \neq 0$ for all $T \in U$. Let us show that

$$\operatorname{sign}(\det(T)) = \operatorname{sign}(\det(T_0))$$

for all $T \in U$. Given any $T_1 \in U$, let E_1 be a finite dimensional subspace of E containing both $\mathrm{Im}(I - T_0)$ and $\mathrm{Im}(I - T_1)$. Thus, for any $\lambda \in [0, 1]$, the operator

$$T_\lambda = (1 - \lambda)T_0 + \lambda T_1$$

belongs to the affine subspace $\Psi(E, E_1)$ of $\Psi(E)$. This implies that the function $\lambda \mapsto \det(T_\lambda)$ is continuous in $[0, 1]$. The result now follows recalling that $\det(T_\lambda) \neq 0$ for all $\lambda \in [0, 1]$. □

We are ready to introduce the concept of orientation for a continuous family of Fredholm operators of index zero.

Definition 3.14. Let Λ be a topological space and $h : \Lambda \to \Phi_0(E, F)$ be a continuous map. An *orientation* of h is a continuous choice of an orientation $\omega(\lambda)$ of $h(\lambda)$ for each $\lambda \in \Lambda$; where "continuous" means that for any $\lambda \in \Lambda$ there exists $A_\lambda \in \omega(\lambda)$ such that $A_\lambda \in \omega(\mu)$ for all μ in a neighborhood of λ. The map h is *orientable* when it admits an orientation and *oriented* when an orientation has been chosen. In particular, a subset $\mathcal{A}$ of $\Phi_0(E, F)$ is said to be *orientable* (or *oriented*) if so is the inclusion $i : \mathcal{A} \to \Phi_0(E, F)$.

Notice that if $\mathcal{A} \subseteq \Phi_0(E, F)$ is orientable, then so is any subset of $\mathcal{A}$ and, more generally, any continuous map $h : \Lambda \to \mathcal{A}$. In fact, an orientation of $\mathcal{A}$ induces an orientation on any map $h : \Lambda \to \mathcal{A}$.

An outstanding (and somehow surprising) result of Kuiper (see [Ku]) asserts that the subset $\mathrm{GL}(\ell^2)$ of $\Phi_0(\ell^2)$ is contractible. In particular, it is connected. As shown in [BF2], this implies that $\mathrm{GL}(\ell^2)$ is not orientable.

Perhaps the simplest example of a nonconstant orientable map $h : \Lambda \to \Phi_0(E, F)$ is when any $h(\lambda)$ is invertible. To show this, endow h with the *natural orientation*, namely the orientation given by choosing the trivial operator as a positive corrector of $h(\lambda)$ for any $\lambda \in \Lambda$. Observe that in this case one has $\operatorname{sign}(h(\lambda)) = 1$ for all $\lambda \in \Lambda$.

Clearly any orientable map h admits at least two orientations. In fact, if h is oriented by ω, reverting this orientation at any $\lambda \in \Lambda$, one gets what we call the *opposite orientation* ω_- of h.

Remark 3.15. As a consequence of Proposition 3.13 one can easily deduce that if A and B are two equivalent correctors of a given $L_0 \in \Phi_0(E, F)$, then they remain L-equivalent for any L in a neighborhood of L_0. This implies that the notion of "continuous choice of an orientation" in Definition 3.14 is equivalent to the following:

- *for any $\lambda \in \Lambda$ **and any** $A_\lambda \in \omega(\lambda)$, there exists a neighborhood U of λ such that $A_\lambda \in \omega(\mu)$ for all $\mu \in U$.*

Remark 3.15 implies that the set in which two orientations of h coincide is open in Λ, and for the same reason it is open the set in which two orientations of h are opposite one to the other. Therefore, if Λ is connected, the map h, if orientable, admits exactly two orientations. As a straightforward consequence of this argument (or, if one prefers, of Proposition 3.13) one obtains the following result, which is a sort of intermediate value theorem.

Proposition 3.16. *Let $h : \Lambda \to \Phi_0(E, F)$ be an oriented map defined on a connected space Λ. Assume there are two points, $\lambda_1, \lambda_2 \in \Lambda$, such that*

$$\operatorname{sign}(h(\lambda_1)) \operatorname{sign}(h(\lambda_2)) < 0.$$

Then there exists $\lambda_0 \in \Lambda$ such that $\operatorname{sign}(h(\lambda_0)) = 0$.

We observe that the notion of "continuous choice of an orientation" in Definition 3.14 becomes the usual concept of continuity if one regards the orientation of the map $h : \Lambda \to \Phi_0(E, F)$ as a lifting of h in the covering space

$$p : \widehat{\Phi}_0(E, F) \to \Phi_0(E, F).$$

The advantage of this point of view is that one could use known results on covering space theory to deduce important properties of orientable maps, such as the homotopy invariance of the orientability, the transport of the orientation by means of a homotopy, the orientability and nonorientability of some maps (see [BF2] for details).

In particular, from covering space theory one gets that if Λ is simply connected and locally path connected, then any continuous map $h : \Lambda \to \Phi_0(E, F)$ is orientable (see [BF2]). Hence, in this case, h admits exactly two orientations, each of them is uniquely determined by the orientation of h at any chosen $\lambda_0 \in \Lambda$.

As a consequence of this, any convex subset of $\Phi_0(E, F)$ is orientable. More generally, given $L \in \Phi_0(E, F)$, let $\operatorname{Star} L$ denote the union of all convex subsets of $\Phi_0(E, F)$ containing L. It is easy to check that $\operatorname{Star} L$ is an open, star-shaped subset of $\Phi_0(E, F)$. Thus, in particular, it is orientable with just two orientations. Taking into account this, the following definition provides a practical way of assigning an orientation to a large class of orientable subsets of $\Phi_0(E, F)$.

Definition 3.17. Let $\mathcal{A} \subseteq \Phi_0(E,F)$ and $L \in \mathrm{Iso}(E,F)$ be such that $\mathcal{A} \subseteq \operatorname{Star} L$. The set $\mathcal{A}$ is *oriented by* L (or L*-oriented*, for short) if it is oriented according to the orientation of $\operatorname{Star} L$ determined by the natural orientation at L. More generally, an orientation of a map $h : \Lambda \to \mathcal{A}$ is an L*-orientation*, or h *is oriented by* L, if any $h(\lambda)$ is oriented according to the L-orientation of $\mathcal{A}$.

The following is a simple and important example of a set oriented by the identity.

Example 3.18 (Leray-Schauder case)**.** Let $\Psi_K(E)$ denote the *compact hull* of the identity operator in a Banach space E; that is,

$$\Psi_K(E) = \big\{T \in \mathrm{L}(E) : I - T \in \mathrm{K}(E)\big\},$$

where $\mathrm{K}(E)$ is the subspace of $\mathrm{L}(E)$ of the compact linear operators. Then $\Psi_K(E)$ is a convex subset of $\operatorname{Star} I$ (actually, an affine subspace of $\mathrm{L}(E)$), and, therefore, can be (and will be, unless otherwise specified) oriented by the identity.

Definition 3.19. Given a Banach space E, by the *canonical orientation* of an operator $L \in \operatorname{Star} I$ we mean the orientation at L of the I-oriented set $\operatorname{Star} I$.

Observe that, in the context of Banach spaces, this new concept of canonical orientation extends that given in Definition 3.5, since in both cases the sign of the identity is 1 (and $\operatorname{Star} I = \mathrm{L}(E)$ when $\dim(E) < +\infty$).

We will show (see Corollary 5.2 below) that, given any canonically oriented isomorphism $T \in \Psi_K(E)$, its sign equals its Leray-Schauder degree (at 0 in any open ball centered at the origin).

4. Finite dimensional reduction

Let $J \subseteq \mathbb{R}$ be an open interval, and consider an *oriented family* $\{L_\lambda\}_{\lambda\in J}$ of Fredholm operators of index zero between two Banach spaces E and F; that is, an oriented map from J into $\Phi_0(E,F)$. Assume there exists $\lambda_0 \in J$ such that

$$L_\lambda \text{ is an isomorphism for any } \lambda \in J \setminus \{\lambda_0\}.$$

We are interested in sufficient (and necessary) conditions for L_λ to have a sign jump at $\lambda = \lambda_0$. From now on, without loss of generality, we shall assume $\lambda_0 = 0$, and L_0 singular; the last assumption being necessary for the existence of a sign jump (because of Proposition 3.16).

Let F_0 be a Banach space and let $P : F \to F_0$ and $Q : F \to \mathbb{R}^n$ be bounded linear operators with the following properties:

(4.1a) $\quad P$ and Q are surjective;

(4.1b) $\quad \operatorname{Ker} Q \oplus \operatorname{Ker} P = F$;

(4.1c) $\quad PL_0$ is surjective.

We observe that such a pair of operators, P and Q, always exists. For example, one could take as P any projection of F onto $F_0 = \operatorname{Im} L_0$ (or onto a finite codimensional subspace F_0 of $\operatorname{Im} L_0$) and as Q the associated projection $I - P$ onto the space $\operatorname{Ker} P$, which can be regarded as $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Remark 4.1. Properties (4.1a) and (4.1b) above hold if and only if the operator

$$P \times Q : y \mapsto (Py, Qy)$$

is an isomorphism between F and $F_0 \times \mathbb{R}^n$.

Remark 4.2. Property (4.1c) holds if and only if P is surjective and L_0 is transverse to $\operatorname{Ker} P$.

Remark 4.3. Since Q is surjective, $\operatorname{Ker} Q$ has codimension n. Hence, from (4.1b) it follows $\dim \operatorname{Ker} P = n$. Thus P, being surjective, is Fredholm of index n. Therefore, PL_λ is Fredholm and

$$\operatorname{ind}(PL_\lambda) = \operatorname{ind}(P) + \operatorname{ind}(L_\lambda) = n.$$

Moreover, since PL_0 is surjective, so is PL_λ for λ small. Consequently, for these values of λ one has $\dim(\operatorname{Ker} PL_\lambda) = n$.

Given P and Q as above, the space F can be regarded as the direct sum of two fixed subspaces: $\operatorname{Ker} Q$ and $\operatorname{Ker} P$. The fact that PL_0 is surjective allows us to define a λ-dependent splitting of E in two subspaces in such a way that L_λ can be represented, up to the composition with isomorphisms, as a triangular 2×2 block-matrix.

Let $A_\lambda : \mathbb{R}^n \to E$, with $\lambda \in J$ sufficiently small, be a continuous family of isomorphisms between $\mathbb{R}^n$ and $\operatorname{Ker} PL_\lambda$. (An explicit construction of such a family is given in Lemma 4.5 below.) Let E_0 be a direct complement to $\operatorname{Ker} PL_0$ in E. Clearly, for small values of λ, the map

$$B_\lambda : E_0 \times \mathbb{R}^n \to E$$

given by $(u_0, u_1) \mapsto u_0 + A_\lambda u_1$ is an isomorphism and depends continuously on λ. Let

$$C_\lambda : E_0 \times \mathbb{R}^n \to F_0 \times \mathbb{R}^n$$

be the composition $(P \times Q)L_\lambda B_\lambda$, which can be regarded as a representation of L_λ, up to the isomorphisms B_λ and $(P \times Q)$. Since $PL_\lambda A_\lambda = 0$, writing C_λ in block-matrix form, we get

$$C_\lambda = \begin{pmatrix} PL_\lambda|_{E_0} & 0 \\ QL_\lambda|_{E_0} & QL_\lambda A_\lambda \end{pmatrix}$$

Remark 4.4. Since L_λ is an isomorphism for small $\lambda \neq 0$, so is the endomorphism $QL_\lambda A_\lambda$ of $\mathbb{R}^n$.

The following result, applied to $M_\lambda = PL_\lambda$, ensures the existence of the family A_λ required in the above construction.

Lemma 4.5. *Let $M_\lambda : E \to F_0$, $\lambda \in J$, be a continuous family of linear operators between Banach spaces. Assume that M_0 is surjective with n-dimensional kernel ($n \in \mathbb{N}$). Then, for λ small enough, there exists a continuous family of injective linear operators $A_\lambda : \mathbb{R}^n \to E$ such that any A_λ maps $\mathbb{R}^n$ onto* $\operatorname{Ker} M_\lambda$.

Proof. Let $A_0 : \mathbb{R}^n \to E$ be an isomorphism between $\mathbb{R}^n$ and $\operatorname{Ker} M_0$, and let E_0 be a (closed) complement to $\operatorname{Ker} M_0$ in E. The operator from $E_0 \times \mathbb{R}^n$ to E defined by $(u_0, u_1) \mapsto u_0 + A_0 u_1$, is an isomorphism. Let $u \in \operatorname{Ker} M_\lambda$ be given and let $u_0 \in E_0$ and $u_1 \in \mathbb{R}^n$ be such that $u = u_0 + A_0 u_1$. One has

$$0 = M_\lambda u = M_\lambda u_0 + M_\lambda A_0 u_1. \tag{4.2}$$

Since $M_0|_{E_0}$ is an isomorphism between E_0 and F_0, the same is true for $M_\lambda|_{E_0}$ when λ is small. Thus, by (4.2),

$$u_0 = -\left(M_\lambda|_{E_0}\right)^{-1} M_\lambda A_0 u_1,$$

and

$$u = A_0 u_1 - \left(M_\lambda|_{E_0}\right)^{-1} M_\lambda A_0 u_1. \tag{4.3}$$

For small λ, define $A_\lambda : \mathbb{R}^n \to E$ by

$$A_\lambda x = A_0 x - \left(M_\lambda|_{E_0}\right)^{-1} M_\lambda A_0 x.$$

Clearly A_λ depends continuously on λ, and the notation is consistent with the initial choice of A_0, since $M_0 A_0 = 0$.

From (4.2) and (4.3) one gets $\operatorname{Ker} M_\lambda \subseteq \operatorname{Im} A_\lambda$. Clearly, $\dim \operatorname{Im} A_\lambda \leq n$. Since, for λ small, M_λ is onto and Fredholm of index n, one has $\dim \operatorname{Ker} M_\lambda = n$. Thus $\operatorname{Ker} M_\lambda = \operatorname{Im} A_\lambda$, which implies the assertion. □

The following is the main result of this section.

Theorem 4.6. *Let $\{L_\lambda\}_{\lambda \in J}$, $P : F \to F_0$ and $Q : F \to \mathbb{R}^n$ be as above. Let $A_\lambda : \mathbb{R}^n \to E$ be a continuous family of injective linear operators such that $\operatorname{Im} A_\lambda = \operatorname{Ker} PL_\lambda$. Then L_λ has a sign jump at $\lambda = 0$ if and only if the same is true for the function $\lambda \mapsto \det(QL_\lambda A_\lambda)$.*

Proof. We will deduce the assertion from the reduction and invariance properties of oriented operators (see Propositions 3.10 and 3.11).

As above, let E_0 be such that $E = E_0 \oplus \operatorname{Ker} PL_0$ and define $B_\lambda : E_0 \times \mathbb{R}^n \to E$ by $B_\lambda(u_0, u_1) = u_0 + A_\lambda u_1$. Endow the isomorphisms B_λ and $P \times Q$ with the natural orientation. Recalling that $\{L_\lambda\}_{\lambda \in J}$ is an oriented family, consider the oriented composition

$$C_\lambda = (P \times Q) L_\lambda B_\lambda : E_0 \times \mathbb{R}^n \to F_0 \times \mathbb{R}^n.$$

By the invariance of the sign,

$$\operatorname{sign}(C_\lambda) = \operatorname{sign}(L_\lambda). \tag{4.4}$$

Since PL_0 is surjective, so is PL_λ for λ small. Therefore, for these values of λ, C_λ is transverse to the subspace $\{0\} \times \mathbb{R}^n$ of $F_0 \times \mathbb{R}^n$.

Now, observe that $C_\lambda^{-1}(\{0\} \times \mathbb{R}^n)$ coincides, for any λ, with the subspace $\{0\} \times \mathbb{R}^n$ of $E_0 \times \mathbb{R}^n$. Consequently, by the reduction property, the sign of C_λ, for λ small, coincides with the sign of the oriented restriction $\hat{C}_\lambda$ of C_λ to the subspace $\{0\} \times \mathbb{R}^n$ of $E_0 \times \mathbb{R}^n$, as domain, and to the subspace $\{0\} \times \mathbb{R}^n$ of $F_0 \times \mathbb{R}^n$, as codomain.

Clearly, the operator $\hat{C}_\lambda$ can be canonically identified with the endomorphism $QL_\lambda A_\lambda$ of $\mathbb{R}^n$, whose induced orientation, without loss of generality, can be assumed to be the canonical one. Indeed, the existence of a sign jump does not depend on the chosen orientation of $\{L_\lambda\}_{\lambda \in J}$. Thus, with this assumption, one has

$$\operatorname{sign}(C_\lambda) = \operatorname{sign}(QL_\lambda A_\lambda) = \operatorname{sign}(\det(QL_\lambda A_\lambda)),$$

and the assertion follows from (4.4). □

In some applications it may happen that the choice of one of the operators, P or Q, is quite natural. It is useful to have criteria which allow, given P, to construct Q so that the (4.1) are satisfied, and vice versa.

Assume we are given $P : F \to F_0$ such that

(1) PL_0 is surjective;
(2) $\operatorname{Ker} P$ has finite dimension, say n.

In this case, we can define Q by composing any projection onto $\operatorname{Ker} P$ with any isomorphism from $\operatorname{Ker} P$ to $\mathbb{R}^n$.

Assume now we are given $Q : F \to \mathbb{R}^n$ such that

(1) Q is surjective;
(2) There exists a subspace F_1 of F such that $\operatorname{Ker} Q \oplus F_1 = F$ and F_1 is transverse to $\operatorname{Im} L_0$. (For example, this is true when $\operatorname{Ker} Q \subseteq \operatorname{Im} L_0$.)

In this case we can take P as the projector onto $F_0 = \operatorname{Ker} Q$ associated with the decomposition $\operatorname{Ker} Q \oplus F_1 = F$.

We observe that, given an oriented family $\{L_\lambda\}_{\lambda\in J}$ as above, one can always split both the spaces E and F in two subspaces, say $E_0, E_1 \subseteq E$ and $F_0, F_1 \subseteq F$, in order to express L_λ as

$$\begin{pmatrix} T_\lambda & U_\lambda \\ V_\lambda & S_\lambda \end{pmatrix}$$

with E_1 and F_1 finite dimensional, and T_0 invertible. For example take E_0 a (closed) complement to $\operatorname{Ker} L_0$, $E_1 = \operatorname{Ker} L_0$, $F_0 = \operatorname{Im} L_0$, and F_1 a complement to F_0. Since E_1 and F_1 must have the same dimension (say n), up to isomorphisms we may suppose $E_1 = F_1 = \mathbb{R}^n$. With this in mind, we state the following consequence of Theorem 4.6.

Theorem 4.7. *Let $L_\lambda : E \to F$, $\lambda \in J$, be as above, and assume $E = E_0 \times \mathbb{R}^n$ and $F = F_0 \times \mathbb{R}^n$, with E_0 and F_0 Banach spaces. Let*

$$\begin{pmatrix} T_\lambda & U_\lambda \\ V_\lambda & S_\lambda \end{pmatrix}$$

be the matrix representation of L_λ according to the above decompositions of E and F. If T_0 is invertible, then L_λ has a sign jump at $\lambda = 0$ if and only if the same happens to the real function $\lambda \mapsto \det(S_\lambda - V_\lambda T_\lambda^{-1} U_\lambda)$.

Proof. Let P and Q denote the two canonical projections of $F = F_0 \times \mathbb{R}^n$ onto the factors F_0 and $\mathbb{R}^n$, respectively. Observe that

$$\operatorname{Ker} PL_\lambda = \Big\{(x_0, x_1) \in E_0 \times \mathbb{R}^n : T_\lambda x_0 + U_\lambda x_1 = 0\Big\}$$

coincides with the image of the injective operator $A_\lambda : \mathbb{R}^n \to E_0 \times \mathbb{R}^n$ given by $A_\lambda x_1 = (-T_\lambda^{-1} U_\lambda x_1, x_1)$, which is defined for λ small because T_0 is invertible. Thus

$$QL_\lambda A_\lambda = S_\lambda - V_\lambda T_\lambda^{-1} U_\lambda,$$

and the assertion follows from Theorem 4.6. □

5. Special cases

In this section we discuss criteria for detecting a sign jump in particular situations. Although the cases considered look quite different, the construction presented in the previous section provides a general framework.

As before, E and F denote real Banach spaces, J is an open interval containing $\lambda = 0$, and $\{L_\lambda\}_{\lambda\in J}$ is an oriented family of Fredholm operators of index zero between them. Which one of the two orientations is chosen is not important for the detection of a sign jump at $\lambda = 0$, the only point of J where L_λ is assumed singular.

We are concerned first with the case when $L_\lambda = L_0 + \lambda K_\lambda$, where $K_\lambda : E \to F$ is a family of linear operators which depends continuously on $\lambda \in J$. As a consequence of Theorem 4.6 we have the following result, in which a sort of Crandall-Rabinowitz condition for bifurcation is considered (see [CR]).

Corollary 5.1. *Let $L_0 \in \Phi_0(E,F)$ and let $K_\lambda : E \to F$, $\lambda \in J$, be a continuous family of linear operators. Assume that the following Crandall-Rabinowitz type condition is satisfied:*

(CR) $u \in \operatorname{Ker} L_0$ *and* $K_0 u \in \operatorname{Im} L_0$ *imply* $u = 0$.

Then, $L_\lambda = L_0 + \lambda K_\lambda$ has a sign jump at $\lambda = 0$ if and only if $\operatorname{Ker} L_0$ is odd dimensional.

Proof. Let $n = \dim(\operatorname{Ker} L_0) > 0$ and let $P : F \to \operatorname{Im} L_0$ be any projection onto the closed subspace $F_0 = \operatorname{Im} L_0$ of F. Let $S : \operatorname{coKer} L_0 \to \mathbb{R}^n$ be any isomorphism and define $Q : F \to \mathbb{R}^n$ as the composition $Q = SR$, where $R : F \to \operatorname{coKer} L_0$ denotes the natural projection. In other words, consider any linear operator Q onto $\mathbb{R}^n$ such that $\operatorname{Ker} Q = \operatorname{Im} L_0$.

Clearly P and Q satisfy the assumptions (4.1), and therefore, as observed in Remark 4.3, PL_0 is Fredholm of index n.

Let $A_\lambda : \mathbb{R}^n \to E$ be as in Lemma 4.5, with $M_\lambda = PL_\lambda$. By Theorem 4.6 it is enough to prove that $\det(QL_\lambda A_\lambda)$ changes sign at $\lambda = 0$ if and only if n is odd. Since $QL_0 = 0$, we have

$$\begin{aligned} \det\Big(Q\big(L_0 + \lambda K_\lambda\big)A_\lambda\Big) &= \det\big(\lambda Q K_\lambda A_\lambda\big) \\ &= \lambda^n \det\big(QK_\lambda A_\lambda\big). \end{aligned}$$

Thus, the assertion follows if we show that $\det(QK_\lambda A_\lambda)$ is nonzero when λ is small. This is true, since assumption (CR) means that the operator

$$RK_0|_{\operatorname{Ker} L_0} : \operatorname{Ker} L_0 \to \operatorname{coKer} L_0$$

is injective and, consequently, one has $\det(QK_0A_0) \neq 0$. □

A very special and interesting case of a family L_λ is the one considered by Leray-Schauder in [LS], where $L_\lambda = I - (\alpha + \lambda)K$, with $K : E \to E$ a compact linear operator and $L_0 = I - \alpha K$ a singular operator (i.e., α^{-1} is an eigenvalue of K). They proved that the (Leray-Schauder) degree of L_λ (which is defined for λ in a pinched neighborhood of 0, α^{-1} being isolated in the spectrum of K) has a sign jump at $\lambda = 0$ if and only if the algebraic multiplicity of α^{-1} is odd. We recall that the algebraic multiplicity of an eigenvalue $\mu \neq 0$ of K is the dimension of the space

$$\bigcup_{k=1}^{\infty} \operatorname{Ker}(\mu I - K)^k\,,$$

which is well known to be finite, since K is compact.

Corollary 5.2 below, which will be deduced directly from Theorem 4.7, shows that for the above special family the same jump phenomenon occurs to the sign function. That is, L_λ has a sign jump at $\lambda = 0$ if and only if the algebraic multiplicity of α^{-1} is odd (hence, the same is true for the degree defined in [BF1]). Actually, in Corollary 5.2, a more general situation is considered, even if apparently different: the family L_λ is of the type $L_0 + \lambda I$, which formally does not include the case

$$I - (\alpha + \lambda)K \tag{5.1}$$

studied by Leray-Schauder. However, composing (5.1) with $(\alpha+\lambda)^{-1}I$ yields a new family which, as regards the possible existence of a sign jump, is equivalent to the previous one and, up to a reparametrization, is of the type $L_0 + \lambda I$. We observe that in this (Leray-Schauder) case L_0 is a very special Fredholm operator of index 0; namely $L_0 = \alpha^{-1} I - K$.

Corollary 5.2. *Let $L_0 : E \to E$ be a noninvertible Fredholm operator of index 0. Assume that*

$$\dim \left(\bigcup_{k=1}^{\infty} \operatorname{Ker} L_0^k \right) = n < \infty.$$

Then $L_\lambda = L_0 + \lambda I$ has a sign jump at $\lambda = 0$ if and only if n is odd.

Proof. By assumption, there exists $p \in \mathbb{N}$ such that $\operatorname{Ker} L_0^{p+i} = \operatorname{Ker} L_0^p$, for all $i \in \mathbb{N}$. Put $E_0 = \operatorname{Im} L_0^p$ and $E_1 = \operatorname{Ker} L_0^p$. It is known that $E = E_0 \oplus E_1$ (see for example [TL, Chap. V, Theorem 6.2]). Clearly, one has $L_0(E_0) \subseteq E_0$ and $L_0(E_1) \subseteq E_1$. Thus, L_λ can be represented as a diagonal block-matrix as follows:

$$L_\lambda = \begin{pmatrix} T_0 + \lambda I_0 & 0 \\ 0 & S_0 + \lambda I_1 \end{pmatrix}$$

where I_0 is the identity of E_0, I_1 is the identity of E_1, T_0 and S_0 are the restrictions of L_0 to E_0 and E_1, respectively. Observe now that $T_0 : E_0 \to E_0$ is Fredholm of index 0 and injective; therefore it is actually invertible. Thus, by Theorem 4.7, L_λ has a sign jump at $\lambda = 0$ if and only if the same is true for the function $\lambda \mapsto \det(S_0 + \lambda I_1)$. The assertion now follows from the fact that $\det(S_0 + \lambda I_1) = \lambda^n$, since the spectrum $\sigma(S_0 + \lambda I_1)$ of $S_0 + \lambda I_1$ is given by $\{\lambda\} + \sigma(S_0)$, which coincides with the singleton $\{\lambda\}$, S_0 being nilpotent. □

We now consider the situation in which one has a continuous family M_λ of surjective Fredholm operators of index $n \in \mathbb{N}$ between two Banach spaces G and F, and a surjective operator $C : G \to \mathbb{R}^n$. To understand how this situation can arise, we may think of M_λ as a family of differential operators associated with a parameter-dependent differential equation, and of C as expressing a set of boundary conditions (compare Examples 6.5 and 6.6 below). The following result is a consequence of Theorem 4.6.

Corollary 5.3. *Let $M_\lambda : G \to F$ be continuous a family of surjective Fredholm operators of index $n \in \mathbb{N}$, and let $A_\lambda : \mathbb{R}^n \to G$ be a continuous family of injective linear operators such that $\operatorname{Im} A_\lambda = \operatorname{Ker} M_\lambda$. Let $C : G \to \mathbb{R}^n$ be linear and surjective, put $E = \operatorname{Ker} C$ and define $L_\lambda : E \to F$ by $L_\lambda = M_\lambda|_E$. Then L_λ has a sign jump at $\lambda = 0$ if and only if the real function $\lambda \mapsto \det(CA_\lambda)$ changes sign crossing $\lambda = 0$.*

Proof. Define $N_\lambda : G \to F \times \mathbb{R}^n$ by $N_\lambda x = (M_\lambda x, Cx)$. Theorems 2.4 and 2.5 imply that N_λ is Fredholm of index 0, since $N_\lambda = SM_\lambda + K$, where S is the inclusion of F into $F \times \mathbb{R}^n$ and $K : G \to F \times \mathbb{R}^n$ is the map $x \mapsto (0, Cx)$.

Let P and Q denote the projections of $F \times \mathbb{R}^n$ onto the factors F and $\mathbb{R}^n$, respectively. One can easily verify that P and Q satisfy properties (4.1), that $PN_\lambda = M_\lambda$ and that $QN_\lambda A_\lambda = CA_\lambda$. Therefore, Theorem 4.6 implies that N_λ has a sign jump at $\lambda = 0$ if and only if the same is true for the real function $\det(CA_\lambda)$.

The assertion now follows from the fact that, by the reduction property of the sign (Proposition 3.10), we have $\operatorname{sign}(L_\lambda) = \operatorname{sign}(N_\lambda)$, for any $\lambda \in J$. (Observe that the subspace $F \times \{0\}$ of $F \times \mathbb{R}^n$ is transverse to N_λ and $N_\lambda^{-1}(F \times \{0\}) = E$.) □

6. Examples

We provide now some simple examples of ordinary differential operators in order to illustrate the meaning of the abstract results obtained in the previous sections.

Given a compact interval $[a,b]$, as usual $C^0([a,b])$, or simply $C([a,b])$, stands for the Banach space of the continuous real functions defined on $[a,b]$, with the norm

$$\|x\|_0 = \sup_{t\in[a,b]} |x(t)|$$

of the uniform convergence. More generally, given $n \in \mathbb{N}$, $C^n([a,b])$ denotes the Banach space of the C^n real functions on $[a,b]$ with the norm

$$\|x\|_n = \|x\|_0 + \|x'\|_0 + \cdots + \|x^{(n)}\|_0,$$

or any equivalent norm (such as $\|x\|_n = \|x\|_0 + \|x^{(n)}\|_0$).

We now show how Theorem 4.6 can be applied in some concrete cases.

Example 6.1. Take $E = C^2([0,\pi])$ and $F = C^0([0,\pi]) \times \mathbb{R}^2$, and, for λ in a neighborhood J of 0, consider the family of bounded linear operators $L_\lambda : E \to F$ given by

$$L_\lambda(x) = \big(\ddot{x} + x + \lambda x, x(0), x(\pi)\big).$$

Clearly, given λ, the operator L_λ is associated with the following boundary value problem:

$$\begin{cases} \ddot{x} + x + \lambda x = y(t) \\ x(0) = a \\ x(\pi) = b \end{cases}$$

where $y \in C^0([0,\pi])$ and $a, b \in \mathbb{R}$. Let us show first that L_λ is Fredholm of index 0. To this end we will use an argument that can be adapted to any (linear) boundary value problem for a linear ordinary differential equation (or system).

The differential operator

$$M_\lambda : C^2([0,\pi]) \to C^0([0,\pi]),$$

defined by $M_\lambda(x) = \ddot{x}+x+\lambda x$, is onto with a two-dimensional kernel. Consequently, it is Fredholm of index 2. Thus, because of Theorem 2.5, the composition $C_\lambda = SM_\lambda$ of M_λ with the inclusion

$$S : C^0([0,\pi]) \to C^0([0,\pi]) \times \mathbb{R}^2$$

is Fredholm of index zero. Now observe that $L_\lambda = C_\lambda + K$, with $K \in \mathrm{F}(E,F)$ given by $K(x) = (0, x(0), x(\pi))$, and apply Theorem 2.4 to show that $L_\lambda \in \Phi_0(E,F)$.

With the notation of Corollary 5.1, put $F_0 = C([0,\pi])$ and define $P : F \to F_0$ by $P(y,a,b) = y$ and $Q : F \to \mathbb{R}^2$ by $Q(y,a,b) = (a,b)$. As already pointed out, the differential operator $M_0 = PL_0$ is surjective and Fredholm of index 2. For $\lambda > -1$, the kernel of PL_λ is the subspace of $C^2([0,\pi])$ spanned by the functions $u_1^\lambda(t) = \cos(\sqrt{1+\lambda}\; t)$ and $u_2^\lambda(t) = \sin(\sqrt{1+\lambda}\; t)$. Thus, for λ small, we define $A_\lambda : \mathbb{R}^2 \to E$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 u_1^\lambda + \alpha_2 u_2^\lambda.$$

Hence, in (canonical) matrix form,

$$QL_\lambda A_\lambda = \begin{pmatrix} u_1^\lambda(0) & u_2^\lambda(0) \\ u_1^\lambda(\pi) & u_2^\lambda(\pi) \end{pmatrix}$$

Therefore, by Theorem 4.6, L_λ has a sign jump at $\lambda = 0$ since

$$\det(QL_\lambda A_\lambda) = \sin(\sqrt{1+\lambda}\;\pi).$$

Example 6.2. One may tackle the family $\{L_\lambda\}$ of Example 6.1 putting $F_0 = C^0([0,\pi]) \times \mathbb{R}$ and defining the operators $P : F \to F_0$ and $Q : F \to \mathbb{R}$ by $P(y,a,b) = (y,a)$ and $Q(y,a,b) = b$. With this choice, the kernel of PL_λ is one-dimensional and spanned by $u_\lambda(t) = \sin(\sqrt{1+\lambda}\;t)$. Therefore, given λ, one may define $A_\lambda : \mathbb{R} \to E$ by $A_\lambda(\alpha) = \alpha u_\lambda$. Thus, $\det(QL_\lambda A_\lambda) = u_\lambda(\pi)$, and the same conclusion as in Example 6.1 follows.

The following two examples illustrate how Corollary 5.1 applies.

Example 6.3. Take $E = \{x \in C^2([0,\pi]) : x(0) = x(\pi) = 0\}$ and $F = C^0([0,\pi])$, and consider the family of bounded linear operators $L_\lambda : E \to F$ given by $L_\lambda x = \ddot{x} + x + \lambda x$. Clearly these operators differ from those of Examples 6.1 and 6.2 but are related to the same boundary value problem. In fact, because of the surjectivity of the boundary operator $x \mapsto (x(0), x(\pi))$, this problem can be transformed into an equivalent one with the homogeneous boundary conditions $x(0) = 0$ and $x(\pi) = 0$.

Clearly, for any given λ, L_λ is Fredholm of index zero, since it is the composition of the inclusion $S : E \to C^2([0,\pi])$ with the differential operator $M_\lambda : C^2([0,\pi]) \to C^0([0,\pi])$ defined by $M_\lambda x = \ddot{x} + x + \lambda x$ (see Theorem 2.5).

We can write $L_\lambda = L_0 + \lambda K$, where K is the inclusion of E into F. In order to apply Corollary 5.1, we need to show that condition (CR) is satisfied.

Since $\operatorname{Ker} L_0$ is one-dimensional (precisely the vector space spanned by $u(t) = \sin t$), so is $\operatorname{coKer} L_0$. Therefore, there exists a (unique up to a multiplicative constant) nontrivial linear functional $Q : F \to \mathbb{R}$ such that $\operatorname{Ker} Q = \operatorname{Im} L_0$. Let us try to find Q of the type

$$Q(y) = \int_0^\pi y(t)\varphi(t)\,dt\,,$$

where φ is a (sufficiently regular) nontrivial function. Since $\operatorname{Im} L_0$ and $\operatorname{Ker} Q$ have the same (finite) codimension (which is one, in this case), any of the inclusions

$$\operatorname{Im} L_0 \subseteq \operatorname{Ker} Q \quad \text{or} \quad \operatorname{Im} L_0 \supseteq \operatorname{Ker} Q$$

implies the equality $\operatorname{Im} L_0 = \operatorname{Ker} Q$. It is therefore sufficient to find φ such that

$$\int_0^\pi \big(\ddot{x}(t) + x(t)\big)\varphi(t)\,dt = 0, \quad \forall x \in E, \tag{6.1}$$

which means $\operatorname{Im} L_0 \subseteq \operatorname{Ker} Q$. Integrating twice (6.1) by parts one can check that $\varphi(t) = \sin t$ satisfies this requirement. Thus, condition (CR) of Corollary 5.1 is verified since

$$\int_0^\pi \sin^2 t\,dt \neq 0.$$

Finally, since the dimension of $\operatorname{Ker} L_0$ is odd, L_λ has a sign jump at $\lambda = 0$.

We point out that the existence of a sign jump of the differential operator L_λ in Example 6.3 could be deduced directly from Example 6.1 using the reduction property of the sign (Proposition 3.10). In fact, this operator is the restriction to the pair of spaces

$$\big\{x \in C^2([0,\pi]) : x(0) = x(\pi) = 0\big\} \ \text{ and } C^0([0,\pi]).$$

of the other operator L_λ in Example 6.2. Obviously, also the converse implication could be considered.

The method used to deduce a sign jump of the differential operator L_λ in Example 6.3 applies to a more general situation. Namely, to the case when the perturbation $\lambda K : E \to F$ is of the type $(\lambda K x)(t) = \lambda g(t)x(t)$, where g is a continuous real function defined on $[0, \pi]$. In this case, condition (CR) of Corollary 5.1 is fulfilled provided that

$$\int_0^\pi g(t)\sin^2 t\,dt \neq 0.$$

The following example shows that for a periodic problem the sign jump, in some sense, is not likely to occur. Again we apply Corollary 5.1.

Example 6.4. Consider the Banach spaces

$$E = \{x \in C^2([0, 2\pi]) : x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi)\} \text{ and } F = C^0([0, 2\pi]),$$

and the family $L_\lambda : E \to F$ of bounded linear operators given by

$$(L_\lambda x)(t) = \ddot{x}(t) + x(t) + \lambda g(t)x(t),$$

where g is a continuous function defined on $[0, 2\pi]$.

Clearly L_λ is related to the following problem with periodic boundary conditions:

$$\begin{cases} \ddot{x} + x + \lambda g(t)x = y(t) \\ x(0) = x(2\pi) \\ \dot{x}(0) = \dot{x}(2\pi) \end{cases}$$

As in the previous example, for any given λ, L_λ is Fredholm of index zero, since it is the composition of the inclusion $S : E \to C^2([0, \pi])$ with the differential operator $M_\lambda : C^2([0, \pi]) \to C^0([0, \pi])$ defined by $M_\lambda x = \ddot{x} + x + \lambda g x$.

Denote by L_0 and K the operators (from E to F) defined by $L_0 x = \ddot{x} + x$ and $Kx = gx$, so that $L_\lambda = L_0 + \lambda K$. In order to apply Corollary 5.1, we need to determine the spaces $\operatorname{Ker} L_0$ and $\operatorname{Im} L_0$. The subspace $\operatorname{Ker} L_0$ of E contains the functions $u_1(t) = \cos t$ and $u_2(t) = \sin t$, thus it is two-dimensional (L_0 being a second order differential operator). Consequently $\operatorname{Im} L_0$ has codimension two. With the same argument as in Example 6.3, one can check that y belongs to $\operatorname{Im} L_0$ only if (and consequently if)

$$\int_0^{2\pi} y(t)\cos t\,dt = \int_0^{2\pi} y(t)\sin t\,dt = 0.$$

Thus, if the determinant of the matrix

$$\begin{pmatrix} \int_0^{2\pi} g(t)\cos^2 t\,dt & \int_0^{2\pi} g(t)\sin t\,\cos t\,dt \\ \int_0^{2\pi} g(t)\cos t\,\sin t\,dt & \int_0^{2\pi} g(t)\sin^2 t\,dt \end{pmatrix} \tag{6.2}$$

is different from zero, condition (CR) of Corollary 5.1 is verified. Hence, in this case, L_λ has not a sign jump at $\lambda = 0$, the dimension of $\operatorname{Ker} L_0$ being even.

As regards the above example, and with the same notation, if the determinant of the matrix (6.2) is zero (e.g. when $g(t) = 1 + 2\sin 2t$), Corollary 5.1 does not apply. In this case, to verify whether L_λ has a sign jump at $\lambda = 0$ one can use another consequence of Theorem 4.6; namely, Corollary 5.3. In order to apply this result it is not necessary to determine $\operatorname{Im} L_0$, but it is not sufficient to find out just

$\operatorname{Ker} L_0$: we actually need to evaluate $\operatorname{Ker} M_\lambda$ for λ in a neighborhood of 0. This can be very hard with pencil and paper, and in most cases impossible. Numerical computations, however, may help to overcome these difficulties, as shown in the following example.

Example 6.5. Consider the Banach spaces

$$G = C^2([0, 2\pi]), \quad E = \{x \in E : x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi)\} \text{ and } F = C^0([0, 2\pi]),$$

and the family $L_\lambda : E \to F$ given by the restriction to E of the differential operators $M_\lambda : G \to F$ defined as

$$(M_\lambda x)(t) = \ddot{x}(t) + x(t) + \lambda(1 + 2\sin 2t)x(t).$$

To detect a possible sign jump of L_λ, the method employed in Example 6.4 does not work in this case, since the determinant of the matrix (6.2) is zero when $g(t) = 1 + 2\sin 2t$. However, we will use Corollary 5.3, in combination with numerical computations, to verify that L_λ has a sign jump at $\lambda = 0$. Here, the boundary operator $C : G \to \mathbb{R}^2$ is given by

$$Cx = (x(2\pi) - x(0), \dot{x}(2\pi) - \dot{x}(0)).$$

Given λ in a neighborhood of 0, let u_1^λ and u_2^λ denote the solutions of the differential equation

$$\ddot{x}(t) + x(t) + \lambda(1 + 2\sin 2t)x(t) = 0$$

satisfying the Cauchy conditions

$$u_1^\lambda(0) = 1,\ \dot{u}_1^\lambda(0) = 0; \quad u_2^\lambda(0) = 0,\ \dot{u}_2^\lambda(0) = 1.$$

Thus, for λ small, the kernel of M_λ is spanned by u_1^λ and u_2^λ, and a continuous family of injective operators $A_\lambda : \mathbb{R}^2 \to G$ such that $\operatorname{Im} A_\lambda = \operatorname{Ker} M_\lambda$ is, for example, $A_\lambda(\alpha, \beta) = \alpha u_1^\lambda + \beta u_2^\lambda$. With this choice of A_λ, the real function $\sigma(\lambda) = \det(CA_\lambda)$ as in Corollary 5.3 is

$$\sigma(\lambda) = \det \begin{pmatrix} u_1^\lambda(2\pi) - u_1^\lambda(0) & u_2^\lambda(2\pi) - u_2^\lambda(0) \\ \dot{u}_1^\lambda(2\pi) - \dot{u}_1^\lambda(0) & \dot{u}_2^\lambda(2\pi) - \dot{u}_2^\lambda(0) \end{pmatrix}$$

and a computer analysis of this function by a standard numerical ODE-solver shows that $\sigma(\lambda)$ changes sign at $\lambda = 0$. Thus, as claimed, L_λ has a sign jump at $\lambda = 0$.

The method used in Example 6.5 to detect a sign jump can be extended in order to include any linear boundary value problem for a family of first order differential systems. We close with the following abstract example, which illustrates this extension.

Example 6.6. Consider the following family of ordinary differential equations in $\mathbb{R}^n$:

$$\dot{x}(t) + (A(t) + \lambda B(t))x(t) = y(t), \quad \lambda \in \mathbb{R}, \tag{6.3}$$

where A and B are $n \times n$ matrices of continuous functions defined on a compact interval $[a, b]$, the assigned function $y : [a, b] \to \mathbb{R}^n$ is continuous, and the unknown function $x : [a, b] \to \mathbb{R}^n$ is of class C^1. Let G and F denote, respectively, the Banach spaces

$$C^1([a, b], \mathbb{R}^n) \cong (C^1([a, b]))^n \text{ and } C^0([a, b], \mathbb{R}^n) \cong (C^0([a, b]))^n.$$

Associated with (6.3) one has the operators $M_\lambda : G \to F$ defined by

$$(M_\lambda x)(t) = \dot{x}(t) + (A(t) + \lambda B(t))x(t),$$

which are surjective with n-dimensional kernel (because of well known results on differential equations). Let $Cx = 0$ be a (well-posed) linear boundary condition for (6.3); that is, C is a surjective linear operator from G to $\mathbb{R}^n$ (so that $\operatorname{Ker} C$ has codimension n in G). Put $E = \operatorname{Ker} C$ and define the family of differential operators $L_\lambda : E \to F$ by the restriction of M_λ to E. The operators L_λ are Fredholm of index 0, as composition of the inclusion $S : E \to G$ (whose index is $-n$) with M_λ (whose index is n).

Assuming that L_0 is singular, to detect a possible sign jump for L_λ (at $\lambda = 0$) we proceed as follows. Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. For any $i \in \{1, 2, \ldots, n\}$ let u_i^λ denote the (maximal) solution of (6.3) satisfying the Cauchy condition $u_i^\lambda(a) = e_i$. Clearly, because of the linearity of (6.3), any u_i^λ is defined on the whole interval $[a, b]$. Notice that, given λ, the matrix $Y_\lambda(t)$ whose columns are the solutions $u_i^\lambda(t)$ is a fundamental matrix of the system (6.3). Denote by CY_λ the $n \times n$ real matrix whose columns are the vectors Cu_i^λ. As in Example 6.5, Corollary 5.3 implies that if the function $\sigma(\lambda) = \det(CY_\lambda)$ has a sign jump at $\lambda = 0$, the same is true for the family L_λ.

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