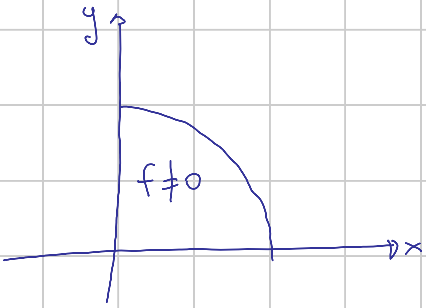
Alcuni esercizi dal foglio 4

Note Title 13/11/2017

Esercizio 18

$\mathbb{P}_{X,Y} = f(x,y)\,dx\,dy \qquad f(x,y) = \begin{cases} 2xy\,(x^2+y^2)^{-3/2} & x\geq 0,\ y\geq 0,\ x^2+y^2\leq 1 \\ 0 & \text{altrimenti} \end{cases}$

$g(x)$ densità di X $\qquad g(x) = \int_{\mathbb{R}} f(x,y)\,dx\,dy$



$x<0 \vee x\geq 1 \Rightarrow g(x) = 0$

$x\in[0,1) \quad g(x) = \int_0^{\sqrt{1-x^2}} 2xy\,(x^2+y^2)^{-3/2}\,dy =$

$= x\,\frac{1}{-1/2}\,(x^2+y^2)^{-1/2}\Big|_{y=0}^{y=\sqrt{1-x^2}}$

$= -2x\left(1-\frac{1}{x}\right) = -2x\,\frac{x-1}{x} = 2(1-x)$

$\Rightarrow \quad g(x) = 2(1-x)\,\mathbb{1}_{[0,1)}(x)$

$\mathbb{E}[X] = \int_{\mathbb{R}} x\,g(x)\,dx = \int_0^1 2x(1-x)\,dx = \int_0^1 (2x-2x^2)\,dx$

$= \left(x^2-\frac{2}{3}x^3\right)\Big|_{x=0}^{x=1} = 1-\frac{2}{3} = \frac{1}{3}$

$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2\,g(x)\,dx = \int_0^1 2x^2(1-x)\,dx = \int_0^1 (2x^2-2x^3)\,dx =$

$= \left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\Big|_{x=0}^{x=1} = \frac{2}{3}-\frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$

$\mathrm{Var}[X] = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{3-2}{18} = \frac{1}{18}$

Es 17

$\mathbb{P}_X = G(p) \qquad \mathbb{P}(X=k) = p(1-p)^{k-1} \quad k=1,2,\ldots$

$\mathbb{P}(Y \le t \mid X=k) = t^k \qquad \forall t \in [0,1) \quad \forall k=1,2,\ldots$

$\mathbb{P}(Y \le t) = 0$ per $t<0$ $\qquad \mathbb{P}(Y \le t) = 1$ per $t \geq 1$

$t \in [0,1)$:

$$F_Y(t) = \mathbb{P}(Y \le t) = \sum_{k=1}^{\infty} \mathbb{P}(Y \le t \mid X=k)\,\mathbb{P}(X=k) =$$

$$= \sum_{k=1}^{\infty} t^k p(1-p)^{k-1} = \sum_{k=1}^{\infty} pt\, x^{k-1} \qquad x = t(1-p)$$

$$j = k-1$$

$$= pt \sum_{j=0}^{\infty} x^j = pt \left.\frac{1}{1-x}\right|_{x=t(1-p)}$$

$$= \frac{pt}{1-t(1-p)}$$

N.B. $F_Y(0) = \left.\frac{pt}{1-t(1-p)}\right|_{t=0} = 0 = F_Y(0^-)$

$$F_Y(1^-) = \left.\frac{pt}{1-t(1-p)}\right|_{t=1} = \frac{p}{p} = 1 = F_Y(1)$$

$\Rightarrow$ F_Y è continua su $\mathbb{R}$ e C^∞ a tratti $\Rightarrow$ $\mathbb{P}_Y$ è AC e la densità è $f_Y(t) = (F_Y(t))'$

$$f_Y(t) = \begin{cases} \dfrac{p}{(1-t(1-p))^2} & t \in (0,1) \\ 0 & \text{altrimenti} \end{cases}$$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} t f_Y(t)\,dt = \int_0^1 \frac{pt}{(t(1-p)-1)^2}\,dt =$$

$$= \frac{p}{1-p} \int_0^1 \frac{t(1-p)-1+1}{(t(1-p)-1)^2}\,dt =$$

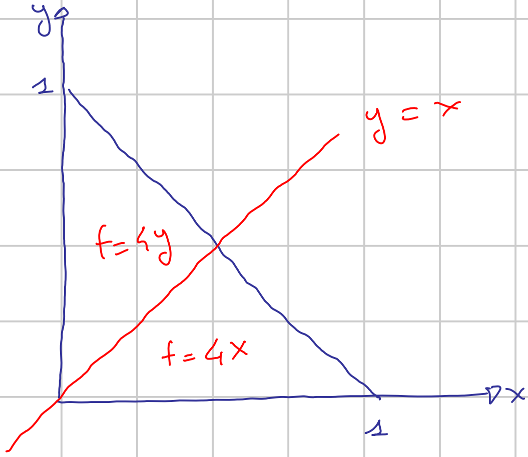
$$= \frac{p}{1-p} \int_0^1 \left((t(1-p)-1)^{-1} + (t(1-p)-1)^{-2} \right) dt =$$

$$= \frac{p}{1-p}\left(\frac{1}{1-p}\log\left|t(1-p)-1\right| - \frac{1}{1-p}\left(t(1-p)-1\right)^{-1}\right)\Bigg|_{t=0}^{t=1} =$$

$$= \frac{p}{(1-p)^2}\left(\log p - \left(-\frac{1}{p}+1\right)\right) = \frac{p}{(1-p)^2}\left(\frac{1}{p} - 1 + \log p\right)$$

Esercizio 15

$$f(x,y) = \begin{cases} 4\max(x,y) & (x,y) \in T \\ 0 & \text{altrimenti.} \end{cases}$$



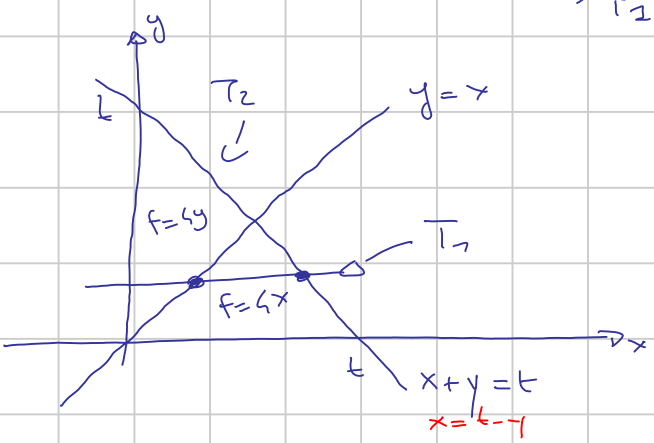
$$Z = X + Y$$

$$F_Z(t) = \mathbb{P}(Z \le t) = \mathbb{P}(X+Y \le t)$$

$$= \int_{\{x+y \le t\}} f(x,y)\,dx\,dy$$

(1) $t < 0 \quad F_Z(t) = 0$ (2) $t \geq 1 \quad F_Z(t) = 1$

(3) $t \in [0,1) \quad F_Z(t) = \int_{T_1} 4x\,dx\,dy + \int_{T_2} 4y\,dx\,dy$



$$= 2\int_{T_1} 4x\,dx\,dy \quad \text{per simmetria}$$

$$= \int_0^{t/2}\left(\int_y^{t-y} 8x\,dx\right)dy =$$

$$= \int_0^{t/2} 4x^2\Big|_{x=y}^{x=t-y}\,dy =$$

$$= \int_0^{t/2} 4\left((t-y)^2 - y^2\right)dy = \int_0^{t/2} 4\left(t^2 - 2ty\right)dy = 4\left(t^2 y - t y^2\right)\Big|_{y=0}^{y=t/2}$$

$$= 4\left(\frac{t^3}{2} - \frac{t^3}{4}\right) = t^3$$

$$F_Z(t) = \begin{cases} 0 & t < 0 \\ t^3 & t \in [0,1) \\ 1 & t \geq 1 \end{cases}$$

$$f_Z(t) = \begin{cases} 3t^2 & t \in (0,1) \\ 0 & \text{altrimenti} \end{cases}$$

$$\mathbb{E}[Z] = \int_{\mathbb{R}} t f_Z(t)\,dt = \int_0^1 3t^3\,dt = \frac{3}{4} t^4 \Big|_{t=0}^{t=1} = \frac{3}{4}$$

$$\mathbb{E}[Z^2] = \int_{\mathbb{R}} t^2 f_Z(t)\,dt = \int_0^1 3t^4\,dt = \frac{3}{5} t^5 \Big|_{t=0}^{t=1} = \frac{3}{5}$$

$$\mathrm{Var}[Z] = \frac{3}{5} - \frac{9}{16} = 3\left(\frac{1}{5} - \frac{3}{16}\right) = 3\,\frac{16-15}{80} = \frac{3}{80}$$

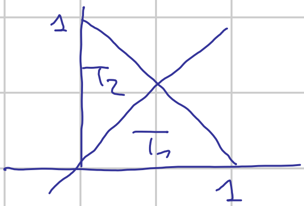
<u>OPPURE</u>

$$Z = \varphi \circ (X,Y) \qquad \varphi : (x,y) \in \mathbb{R}^2 \mapsto x+y \in \mathbb{R}$$

ψ funzione di Borel nonnegativa

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Z(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{X,Y}(dx\,dy) = \int_{\mathbb{R}^2} \psi(x+y) \cdot f(x,y)\,dx\,dy$$

$$= \int_{T_1} 4x\,\psi(x+y)\,dx\,dy + \int_{T_2} 4y\,\psi(x+y)\,dx\,dy =$$

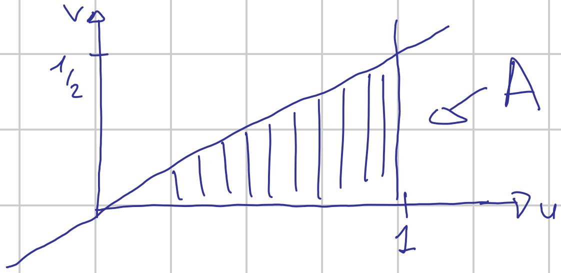


$$= 2\int_{T_1} 4x\,\psi(x+y)\,dx\,dy$$ per simmetria rispetto alla bisettrice $y=x$

$$T_1 : \begin{cases} 0 \leq y \leq \frac{1}{2} \\ y \leq x \leq 1-y \end{cases} \qquad \begin{cases} u = x+y \\ v = y \end{cases} \qquad \begin{cases} x = u-v \\ y = v \end{cases}$$

$$J = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad |\det J| = 1$$

$$\begin{cases} 0 \leq v \leq \frac{1}{2} \\ v \leq u-v \leq 1-v \end{cases} \qquad \begin{cases} 0 \leq v \leq \frac{1}{2} \\ 2v \leq u \leq 1 \end{cases}$$



$$\int_{\mathbb{R}} \varphi(t)\,\mathbb{P}_Z(dt) = \int_A 8(u-v)\,\varphi(u)\,du\,dv$$

$$= \int_0^1 \varphi(u)\left(\int_0^{u/2} 8(u-v)\,dv\right)du$$

$$= \int_0^1 \varphi(u)\left(8uv - 4v^2\right)\Big|_{v=0}^{v=u/2} du = \int_0^1 \varphi(u)\left(4u^2 - u^2\right)du =$$

$$= \int_0^1 \varphi(u)\,3u^2\,du = \int_{\mathbb{R}} \varphi(u)\,g(u)\,du \quad \text{se } g(u) = \begin{cases} 3u^2 & u \in (0,1) \\ 0 & \text{altrimenti} \end{cases}$$

$\Rightarrow \mathbb{P}_Z = g(u)\,du$ con $g(u)$ come sopra.