VALORE ATTESO

Note Title 16/10/2017

X v.e. su $(\Omega, \mathcal{E}, \mathbb{P})$, $\mathbb{P}_X$, $F_X : \mathbb{R} \to \mathbb{R}$
$F_X : t \mapsto \mathbb{P}(X \leq t)$

1. V.a. con distribuzione discreta $\{t_j\}_{j \in J}$
f di Borel $: \mathbb{R} \to \mathbb{R}$ $\quad \int_{\mathbb{R}} f(t)\, \mathbb{P}_X(dt) = \sum_{j \in J} f(t_j)\, \mathbb{P}_X(\{t_j\})$

2. V.e. con distribuzione A.C. $\mathbb{P}_X = f(x)\,dx$
f di Borel $: \mathbb{R} \to \mathbb{R}$ $\quad \int_{\mathbb{R}} f(t)\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} f(t) f(t)\, dt$

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$\int_{\Omega} X(\omega)\, \mathbb{P}(d\omega)$ VALORE ATTESO DI X $\quad \mathbb{E}[X]$
expected value

$$\int_{\Omega} X(\omega)\, \mathbb{P}(d\omega) = \frac{1}{\mathbb{P}(\Omega)} \int_{\Omega} X(\omega)\, \mathbb{P}(d\omega)$$

$\mathbb{E}[X] := \mathbb{E}[X^+] - \mathbb{E}[X^-]$
$\mathbb{E}[X]$ è finito sse $\mathbb{E}[X^+]$ e $\mathbb{E}[X^-]$ sono entrambi finiti. D'altra parte $|X| = X^+ + X^-$
$\Rightarrow \mathbb{E}[X]$ è finito sse $\mathbb{E}[|X|]$ è finito.

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$X : \Omega \to \mathbb{R}$ v.e. su $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato
e sia $f : \mathbb{R} \to \mathbb{R}$ funzione di Borel.

PROPRIETÀ $\quad f \circ X : \Omega \to \mathbb{R}$ è ancora una v.e. su $(\Omega, \mathcal{E}, \mathbb{P})$

DIM $\quad t \in \mathbb{R}$ $\quad \{f \circ X \leq t\} = \{\omega \in \Omega : f \circ X(\omega) \in (-\infty, t]\}$

$$= \{\omega \in \Omega : X(\omega) \in \underbrace{f^{-1}((-\infty, t])}_{\in \mathcal{B}(\mathbb{R})}\}$$

$$= \{X \in f^{-1}((-\infty, t])\} \in \mathcal{E}$$

$\mathcal{E}$

TEO Sia $X:\Omega \to \mathbb{R}$ v.a. su $(\Omega,\mathcal{E},\mathbb{P})$ e sia $f:\mathbb{R}\to\mathbb{R}$ funzione di Borel nonnegativa.

$$\mathbb{E}[f\circ X] := \int_\Omega (f\circ X)(\omega)\,\mathbb{P}(d\omega) = \int_{\mathbb{R}} f(t)\,\mathbb{P}_X(dt)$$

1° CASO f funzione semplice $f(t)=\sum_{i=1}^n c_i \mathbb{1}_{E_i}(t)$

$$\int_{\mathbb{R}} f(t)\,\mathbb{P}_X(dt) = \sum_{i=1}^n c_i\, \mathbb{P}_X(E_i) =$$

$$= \sum_{i=1}^n c_i\, \mathbb{P}(X\in E_i)$$ $\qquad X\in E_i \iff f\circ X = c_i$

$$= \sum_{i=1}^n c_i\, \mathbb{P}(f\circ X = c_i) = \int_\Omega (f\circ X)(\omega)\,\mathbb{P}(d\omega) = \mathbb{E}[f\circ X]$$

2° CASO $f:\mathbb{R}\to\mathbb{R}$ funzione di Borel nonnegativa.

$\exists \{f_n\}_{n\in\mathbb{N}}$ $f_n:\mathbb{R}\to\mathbb{R}$ semplici di Borel nonnegative t.c. $0\le f_n(x)\le f_{n+1}(x)$, $\lim_{n\to\infty} f_n(x)=f(x)$ $\forall x\in\mathbb{R}$,

$$\int_{\mathbb{R}} f(t)\,\mathbb{P}_X(dt) = \lim_{n\to\infty}\int_{\mathbb{R}} f_n(t)\,\mathbb{P}_X(dt) =$$

$$= \lim_{n\to\infty}\int_\Omega (f_n\circ X)(\omega)\,\mathbb{P}(d\omega)$$

$$= \int_\Omega f\circ X(\omega)\,\mathbb{P}(d\omega) = \mathbb{E}[f\circ X]$$

$f_n\circ X(\omega)\le f_{n+1}\circ X(\omega)$

$\lim_{n\to\infty} f_n\circ X(\omega) = f\circ X(\omega)$ $\forall \omega\in\Omega$

COROLLARIO Sia X v.a. su $(\Omega,\mathcal{E},\mathbb{P})$. Allora $\mathbb{E}[X]$ esiste se e solo se esiste $\int_{\mathbb{R}} t\,\mathbb{P}_X(dt)$ e in tal caso sono uguali.

Inoltre $\mathbb{E}[|X|] = \int_{\mathbb{R}} |t|\,\mathbb{P}_X(dt)$

DIM $f(t) = t^+ := \max\{t,0\}$

$f\circ X = X^+$

(A) $\mathbb{E}[X^+] = \mathbb{E}[f \circ X] = \int_{\mathbb{R}} f(t)\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} t^+\, \mathbb{P}_X(dt)$

$$\tilde{f}(t) = t^- := \max\{-t, 0\}$$
$$\tilde{f} \circ X = X^-$$

(B) $\mathbb{E}[X^-] = \mathbb{E}[\tilde{f} \circ X] = \int_{\mathbb{R}} \tilde{f}(t)\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} t^-\, \mathbb{P}_X(dt)$

Sommando membro e membro ottengo

$$\mathbb{E}[|X|] = \mathbb{E}[X^+] + \mathbb{E}[X^-] = \int_{\mathbb{R}} t^+\, \mathbb{P}_X(dt) + \int_{\mathbb{R}} t^-\, \mathbb{P}_X(dt)$$
$$= \int_{\mathbb{R}} (t^+ + t^-)\, \mathbb{P}_X(dt)$$
$$= \int_{\mathbb{R}} |t|\, \mathbb{P}_X(dt)$$

$$\underbrace{\mathbb{E}[X^+] - \mathbb{E}[X^-]}_{\mathbb{E}[X]} = \int_{\mathbb{R}} t^+\, \mathbb{P}_X(dt) - \int_{\mathbb{R}} t^-\, \mathbb{P}_X(dt)$$
$$= \int_{\mathbb{R}} (t^+ - t^-)\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} t\, \mathbb{P}_X(dt)$$

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FORMULA DI COMPOSIZIONE

Sia $X : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ –

$f : \mathbb{R} \to \mathbb{R}$ funzione di Borel

$\psi : \mathbb{R} \to \mathbb{R}$ funzione di Borel non negativa.

Allora
$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{f \circ X}(dt) = \int_{\mathbb{R}} (\psi \circ f)(s)\, \mathbb{P}_X(ds)$$

DIM
$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{f \circ X}(dt) = \int_{\Omega} \psi \circ (f \circ X)(\omega)\, \mathbb{P}(d\omega) =$$
$$= \int_{\Omega} (\psi \circ f) \circ X(\omega)\, \mathbb{P}(d\omega) = \int_{\mathbb{R}} (\psi \circ f)(s)\, \mathbb{P}_X(ds)$$

ES 1 foglio 2

Noto F_X ; $Y := X^2$, calcolare F_Y in funzione di F_X

$t \in \mathbb{R}$ $\quad F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(X^2 \le t) = \begin{cases} 0 & t < 0 \\ ? & \end{cases}$

$t \geq 0$ $\quad F_Y(t) = \mathbb{P}(X^2 \le t) = \mathbb{P}(-\sqrt{t} \le X \le \sqrt{t}) = (\star)$

$$\{-\sqrt{t} \le X \le \sqrt{t}\} = \{X \le \sqrt{t}\} \setminus \{X < -\sqrt{t}\}$$

$(\star) = \mathbb{P}(X \le \sqrt{t}) - \mathbb{P}(X < -\sqrt{t})$

$\{X < -\sqrt{t}\} = \{X \le -\sqrt{t}\} \setminus \{X = -\sqrt{t}\}$

$= \mathbb{P}(X \le \sqrt{t}) - \left(\mathbb{P}(X \le -\sqrt{t}) - \mathbb{P}(X = -\sqrt{t})\right)$

$= F_X(\sqrt{t}) - F_X(-\sqrt{t}) + \mathbb{P}(X = -\sqrt{t})$

1° MODO $\quad \mathbb{P}_X = f(x)\,dx \quad \Rightarrow \quad \mathbb{P}(X = -\sqrt{t}) = 0$

$t \geq 0$ $\quad F_Y(t) = F_X(\sqrt{t}) - F_X(-\sqrt{t}) = \int_{-\infty}^{\sqrt{t}} f(x)\,dx - \int_{-\infty}^{-\sqrt{t}} f(x)\,dx$

$= \int_{-\sqrt{t}}^{\sqrt{t}} f(x)\,dx$ $\quad \cancel{x = \sqrt{y}}$

$= \int_{-\sqrt{t}}^{0} f(x)\,dx + \int_{0}^{\sqrt{t}} f(x)\,dx$

$x = -\sqrt{y}$ $\quad x = 0 \quad y = 0$ $\qquad\qquad x = \sqrt{y}$ $\quad x = 0 \quad y = 0$

$\qquad x = -\sqrt{t} \quad y = t$ $\qquad\qquad\qquad x = \sqrt{t} \quad y = t$

$dx = \frac{-1}{2\sqrt{y}}\,dy$ $\qquad\qquad dx = \frac{1}{2\sqrt{y}}\,dy$

$$= \int_{t}^{0} f(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}}\,dy + \int_{0}^{t} f(\sqrt{y})\,\frac{1}{2\sqrt{y}}\,dy$$

$$= \int_0^t f(-\sqrt{y}) \frac{1}{2\sqrt{y}} dy + \int_0^t f(\sqrt{y}) \frac{1}{2\sqrt{y}} dy =$$

$$= \int_0^t \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right) dy = \int_{-\infty}^t g(y)\, dy$$

$$g(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right) & y > 0 \end{cases}$$

<u>2° METODO</u> $Y = X^2 = \varphi \circ X \qquad \varphi : x \in \mathbb{R} \mapsto x^2 \in \mathbb{R}$

$\psi : \mathbb{R} \to \mathbb{R}$ funzione di Borel non negativa

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi \circ X}(dt) =$$

$$= \int_{\mathbb{R}} (\psi \circ \varphi)(x)\, \mathbb{P}_X(dx) = \int_{\mathbb{R}} \psi(\varphi(x)) f(x)\, dx =$$

$$= \int_{\mathbb{R}} \psi(x^2) f(x)\, dx \qquad x^2 = y$$

$$= \int_0^{+\infty} \psi(x^2) f(x)\, dx + \int_{-\infty}^0 \psi(x^2) f(x)\, dx$$

$y = x^2 \qquad x = \sqrt{y}$	$y = x^2 \qquad x = -\sqrt{y}$
$x = 0 \quad y = 0 \qquad dx = \frac{1}{2\sqrt{y}} dy$	$x = 0 \quad y = 0 \qquad dx = \frac{-1}{2\sqrt{y}} dy$
$x \to +\infty \quad y \to +\infty$	$x \to -\infty \quad y \to +\infty$

$$\int_0^{+\infty} \psi(y) f(\sqrt{y}) \frac{1}{2\sqrt{y}} dy + \int_{+\infty}^0 \psi(y) f(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} dy$$

$$= \int_0^{+\infty} \psi(y) f(\sqrt{y}) \frac{1}{2\sqrt{y}} dy + \int_0^{+\infty} \psi(y) f(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} dy$$

$$= \int_0^{+\infty} \varphi(y) \left(\frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right) \right) dy = \int_{\mathbb{R}} \varphi(y)\, g(y)\, dy$$

se $g(y) := \begin{cases} 0 & y \leq 0 \\ \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right) & y > 0 \end{cases}$

$$\int_{\mathbb{R}} \varphi(y)\, \mathbb{P}_Y(dy) = \int_{\mathbb{R}} \varphi(y)\, g(y)\, dy \quad \forall \varphi \text{ di Borel nonnegative}$$

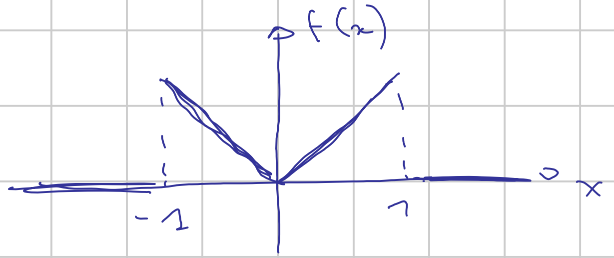
$\varphi(y) = \mathbb{1}_A(y) \quad A \in \mathcal{B}(\mathbb{R})$

$$\int_{\mathbb{R}} \mathbb{1}_A(y)\, \mathbb{P}_Y(dy) = \mathbb{P}_Y(A)$$

$$\mathbb{P}_Y(A) = \int_A g(y)\, dy$$

$$\int_{\mathbb{R}} \mathbb{1}_A(y)\, g(y)\, dy = \int_A g(y)\, dy$$

ES 7 $\quad \mathbb{P}_X = f(x)\, dx \qquad f(x) = \begin{cases} |x| & |x| < 1 \\ 0 & \text{altrimenti} \end{cases}$



$f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$$\int_{\mathbb{R}} f(x)\, dx = \mathbb{P}(X \in \mathbb{R}) = 1$$

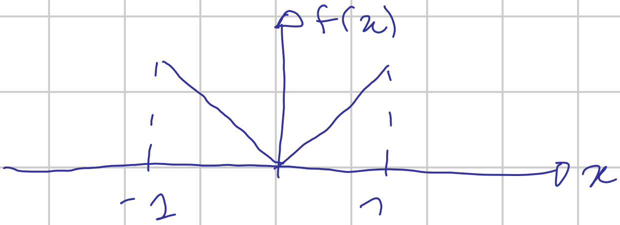
$Y = X^2$

$$g(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right) & y > 0 \end{cases}$$

$$f(\sqrt{y}) = \begin{cases} \sqrt{y} & \sqrt{y} \in (0,1) \\ 0 & \sqrt{y} \geq 1 \end{cases} = \begin{cases} \sqrt{y} & y \in (0,1) \\ 0 & y \geq 1 \end{cases}$$

$$f(-\sqrt{y}) = \begin{cases} \sqrt{y} & -\sqrt{y} \in (-1,0) \\ 0 & -\sqrt{y} \leq -1 \end{cases} = \begin{cases} \sqrt{y} & y \in (0,1) \\ 0 & y \geq 1 \end{cases}$$

$$g(y) = \begin{cases} 0 & y \le 0 \vee y \ge 1 \\ \frac{1}{2\sqrt{y}}\left(\sqrt{y}+\sqrt{y}\right) = 1 & y \in (0,1) \end{cases}$$



$$\int_{\mathbb{R}} x f(x)\,dx = \int_{-1}^{1} x f(x)\,dx = 0$$

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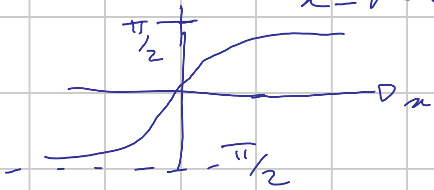
$$f(x) = \frac{1}{\pi(1+x^2)} \qquad x \in \mathbb{R} \qquad\qquad f(x) > 0 \quad \forall x \in \mathbb{R}$$

$$\int_{\mathbb{R}} f(x)\,dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\,\operatorname{arctg}(x)\Big|_{x\to-\infty}^{x\to+\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$



$$\mathbb{P}_X = f(x)\,dx$$

$$\mathbb{E}[X] = ? \qquad \underbrace{\int_{\mathbb{R}} x f(x)\,dx}_{} = \underbrace{\int_{\mathbb{R}} x^+ f(x)\,dx}_{} - \underbrace{\int_{\mathbb{R}} x^- f(x)\,dx}_{}$$

$$\int_{\mathbb{R}} \frac{1}{\pi}\left(\frac{x}{1+x^2}\right) dx = 0$$

$$\int_{\mathbb{R}} x^+ f(x)\,dx = \int_0^{+\infty} x f(x)\,dx = \int_0^{+\infty} \frac{1}{2\pi}\,\frac{2x}{1+x^2}\,dx =$$

$$= \frac{1}{2\pi}\log(1+x^2)\Big|_{x=0}^{x\to+\infty} = \frac{1}{2\pi}\left(+\infty - 0\right) = +\infty$$

$$\int_{\mathbb{R}} x^- f(x)\,dx = \int_{-\infty}^{0} (-x)\,\frac{1}{\pi(1+x^2)}\,dx \qquad y = -x$$

$$= \int_{+\infty}^{0} y\,\frac{1}{\pi(1+y^2)}\,(-1)\,dy = \int_0^{+\infty} \frac{y}{\pi(1+y^2)}\,dy = +\infty$$

$$\mathbb{E}[X^+] = \mathbb{E}[X^-] = +\infty$$