ES. 2 A K Q J 10 9 8 7 8 carte, 4 semi $\Rightarrow$ 32 carte

Ci sono $\binom{32}{5} = \frac{32 \cdot 31 \cdot \cancel{30} \cdot 29 \cdot \cancel{28}\, 7}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 32 \cdot 31 \cdot 29 \cdot 7 = 201376$

Ci sono 1 · 28 "poker d'assi serviti" $\Rightarrow$ ci sono 8 · 28 "poker serviti"

La prob. di avere un poker servito è $\frac{\cancel{8} \cdot \cancel{28}}{\cancel{32}_4 \cdot 31 \cdot 29 \cdot \cancel{7}} = \frac{1}{31 \cdot 29} = \frac{1}{899}$

ES 3

10 bianche 8 rosse 12 nere	Si estraggono 4 palline	1 bianca 2 rosse 1 nera //

SENZA REIMBUSSOLAMENTO

$\binom{30}{4}$ estrazioni possibili

$\underbrace{B_1 - - - - - B_{10}}_{1}\ \underbrace{R_1 \text{———} R_8}_{2}\ \underbrace{N_1 \text{————} N_{12}}_{1}$

$\binom{10}{1} = 10 \qquad \binom{8}{2} \qquad \binom{12}{1} = 12$

Le estrazioni favorevoli sono $10 \cdot 12 \binom{8}{2} = 10 \cdot 12 \cdot \frac{\cancel{8}^{4} \cdot 7}{\cancel{2}} = 3360$

$\Rightarrow$ La prob. di estrarre 1B, 2R e 1N senza reimbussol.

è di $\frac{10 \cdot 12 \cdot \binom{8}{2}}{\binom{30}{4}} = \frac{\cancel{10} \cdot \cancel{12}^{4} \cdot \cancel{4} \cdot \cancel{7} \cdot 4 \cdot \cancel{3} \cdot 2}{\cancel{30} \cdot 29 \cdot \cancel{28} \cdot \cancel{27}_9} = \frac{32}{261}$

Con reimbussolamento: BRRN nell'ordine $\frac{10}{30} \cdot \frac{8}{30} \cdot \frac{8}{30} \cdot \frac{12}{30}$

Ci sono $\frac{4!}{2!} = 6$ ordini possibili.

$\Rightarrow$ la probabilità di estrarre 1B, 2R e 1N è $6 \cdot \frac{10}{30} \cdot \frac{8}{30} \cdot \frac{8}{30} \cdot \frac{12}{30}$

ES 9

Tennis 70% T $\qquad\Omega$ = totale iscritti

Golf 60% G $\qquad T \cup G \cup A = \Omega$

Atletica 40% A $\qquad T \cap G \cap A = \emptyset$

$$\mathbb{P}\Big(\underbrace{(T\cap G)\cup(G\cap A)\cup(T\cap A)}\Big)$$

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(T\cup G\cup A) = \mathbb{P}(T\cup G) + \mathbb{P}(A) - \mathbb{P}\big((T\cup G)\cap A\big) =$$

$$= \underline{\mathbb{P}(T)} + \underline{\mathbb{P}(G)} - \mathbb{P}(T\cap G) + \underline{\mathbb{P}(A)} - \mathbb{P}\big((T\cap A)\cup(G\cap A)\big) =$$

$$= \underset{70\%}{\mathbb{P}(T)} + \underset{60\%}{\mathbb{P}(G)} + \underset{40\%}{\mathbb{P}(A)} - \big(\underbrace{\mathbb{P}(T\cap G) + \mathbb{P}(T\cap A) + \mathbb{P}(G\cap A)}\big)$$

$\mathbb{P}$(degli iscritti che praticano 2 sport) =

$$= \mathbb{P}\big((T\cap G)\cup(G\cap A)\cup(T\cap A)\big) = \frac{70}{100} + \cancel{\frac{60}{100}} + \cancel{\frac{40}{100}} - \cancel{1} = 70\%$$

ES 8 $\quad(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato $\quad\mathbb{P}(B)>0$

$A \in \mathcal{E} \qquad \mathbb{P}_B(A) := \mathbb{P}(A|B)$

$(\Omega, \mathcal{E}, \mathbb{P}_B)$

$$\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{E} \qquad A_i\cap A_j = \emptyset \quad i\neq j \qquad \mathbb{P}_B\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \overset{?}{=} \sum_{i\in\mathbb{N}} \mathbb{P}_B(A_i)$$

$$\mathbb{P}_B(\Omega) \overset{?}{=} 1$$

$$\mathbb{P}_B\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \mathbb{P}\Big(\bigcup_{i\in\mathbb{N}} A_i \Big| B\Big) = \frac{\mathbb{P}\Big(\big(\bigcup_{i\in\mathbb{N}} A_i\big)\cap B\Big)}{\mathbb{P}(B)} =$$

$$= \frac{1}{\mathbb{P}(B)}\,\mathbb{P}\Big(\bigcup_{i\in\mathbb{N}}(A_i\cap B)\Big) = \qquad A_i\cap B \subset A_i$$

$$\Downarrow$$

$$= \frac{1}{\mathbb{P}(B)}\sum_{i\in\mathbb{N}} \mathbb{P}(A_i\cap B) = \qquad (A_i\cap B)\cap(A_j\cap B) = \emptyset \quad i\neq j$$

$$= \sum_{i\in\mathbb{N}} \frac{\mathbb{P}(A_i\cap B)}{\mathbb{P}(B)} = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i|B) = \sum_{i\in\mathbb{N}} \mathbb{P}_B(A_i)$$

$$\mathbb{P}_B(\Omega) = \mathbb{P}(\Omega|B) = \frac{\mathbb{P}(\Omega\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$$

ES 12 U_1 : 4B, 6R U_2 : 5G, 5V

Si estraggono (senza reimbussolamento) 2 palline da U_1)
Se hanno lo stesso colore $\Rightarrow$ estraggo 2 palline da U_2
se hanno colore diverso $\Rightarrow$ estraggo 4 palline da U_2

Calcolare la prob. che da U_2 venga estratta 1! pallina gialla

G = estraggo 1! pallina gialla da U_2
S = estraggo 2 palline dello stesso colore da U_1
D = estraggo 2 palline di colore diverso da U_1

$$\mathbb{P}(G) = \underline{\mathbb{P}(G|S)}\,\underline{\mathbb{P}(S)} + \underline{\mathbb{P}(G|D)}\,\underline{\mathbb{P}(D)}$$

$\mathbb{P}(G|S)$ 5 Gialle, 5 Verdi $\quad \dfrac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}} = \dfrac{5\cdot5\cdot2}{10\cdot9} = \dfrac{5}{9}$

$$\mathbb{P}(G|D) = \frac{\binom{5}{1}\binom{5}{3}}{\binom{10}{4}} = \cancel{5}\cdot\frac{5\cdot\cancel{4}}{\cancel{2}}\cdot\frac{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{10}\cdot\cancel{9}_3\cdot\cancel{8}\cdot7} = \frac{5}{21}$$

$$\mathbb{P}(S) = \mathbb{P}(2\text{ bianche}) + \mathbb{P}(2\text{ rosse}) = ?$$

$$\mathbb{P}(2\text{ bianche}) = \frac{\binom{4}{2}\binom{6}{0}}{\binom{10}{2}} = \frac{\cancel{4}^{2}\cdot\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{10}_5\cdot\cancel{9}_3} = \frac{2}{15}$$

$$\mathbb{P}(2\text{ rosse}) = \frac{\binom{6}{2}\binom{4}{0}}{\binom{10}{2}} = \frac{\cancel{6}\cdot\cancel{5}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{10}\cdot\cancel{9}_3} = \frac{1}{3}$$

$$\Rightarrow \mathbb{P}(S) = \frac{2}{15} + \frac{1}{3} = \frac{2+5}{15} = \frac{7}{15}$$

$$\mathbb{P}(D) = \frac{\binom{4}{1}\binom{6}{1}}{\binom{10}{2}} = \frac{4\cdot\cancel{6}^{2}\cdot\cancel{2}}{\cancel{10}_5\cdot\cancel{9}_3} = \frac{8}{15}$$

questo perché $\{D, S\}$ è una partizione dell'evento certo

$$\Rightarrow \mathbb{P}(G) = \mathbb{P}(G|S)\,\mathbb{P}(S) + \mathbb{P}(G|D)\,\mathbb{P}(D) =$$

$$= \frac{\cancel{5}^{1}}{9} \cdot \frac{7}{\cancel{15}_{3}} + \frac{\cancel{5}^{1}}{\underset{7\cdot 3}{\underset{\parallel}{21}}} \cdot \frac{8}{\cancel{15}_{3}} = \frac{49+24}{3^3 \cdot 7} = \frac{73}{189}$$

$$\mathbb{P}(\text{2 brioche} \mid G) = \frac{\mathbb{P}(G|\text{2 brioche})\,\mathbb{P}(\text{2 brioche})}{\mathbb{P}(G)} =$$

$$= \frac{\mathbb{P}(G|S)\,\mathbb{P}(\text{2 brioche})}{\mathbb{P}(G)} = \frac{\cancel{5}^{1}}{\cancel{9}} \cdot \frac{2}{\cancel{15}_{\cancel{3}}} \cdot \frac{\cancel{3}^{\cancel{3}} \cdot 7}{73} = \frac{14}{73}$$

$(\Omega, \mathcal{E}, \mathbb{P})$ $\quad X : \Omega \to \overline{\mathbb{R}}$ v.a.

$\forall t \in \mathbb{R} \quad \underbrace{\{\omega \in \Omega : X(\omega) \leq t\}}_{\{X \leq t\}} \in \mathcal{E}$

$$\begin{cases} \{X = +\infty\}, \{X = -\infty\} \in \mathcal{E} \\ \forall A \in \mathcal{B}(\mathbb{R}) \quad \{X \in A\} = X^{-1}(A) \in \mathcal{E} \end{cases}$$

$(\mathbb{R}, \mathcal{B}(\mathbb{R})) \qquad \mathbb{P}_X : A \in \mathcal{B}(\mathbb{R}) \mapsto \mathbb{P}(X \in A) \in \mathbb{R}$

$\mathbb{P}_X$ è una misura su $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (finita)

$\mathbb{P}_X$ è una probabilità su $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ SSE $\mathbb{P}(X = +\infty) = \mathbb{P}(X = -\infty) = 0$

— o —

$X : \Omega \to \overline{\mathbb{R}}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ LEGGE DI X

$F_X : t \in \mathbb{R} \mapsto \mathbb{P}(X \leq t) \in \mathbb{R}$ FUNZIONE DI RIPARTIZIONE

FUNZIONE DI DISTRIBUZIONE CUMULATIVA

$$\begin{aligned} F_X(t) &= \mathbb{P}(X \leq t) = \mathbb{P}(X \in [-\infty, t]) = \\ &= \mathbb{P}(X = -\infty) + \mathbb{P}(X \in (-\infty, t]) \\ &= \mathbb{P}(X = -\infty) + \mathbb{P}_X((-\infty, t]) \end{aligned}$$

$a < b$

$\mathbb{P}_X((a,b]) = \mathbb{P}(X \in (a,b]) = \mathbb{P}(a < X \leq b) \overset{(*)}{=}$

$\{a < X \leq b\} = \{X \leq b\} \setminus \{X \leq a\}$

$\{X \leq a\} \subseteq \{X \leq b\}$

$(*) \quad \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F_X(b) - F_X(a)$

PROPOSIZIONE Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato.
Sia $X : \Omega \to \overline{\mathbb{R}}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ e sia
$F_X : t \in \mathbb{R} \mapsto \mathbb{P}(X \leq t) \in \mathbb{R}$ la sua legge.
Allora F_X gode delle seguenti proprietà

1) F_X è monotona non decrescente

DIM $s < t \Rightarrow F_X(s) \leq F_X(t)$

$$F_X(s) = \mathbb{P}(X \leq s) \qquad \{X \leq s\} \subseteq \{X \leq t\}$$
$$\leq \mathbb{P}(X \leq t) = F_X(t)$$

2) $\lim_{t \to -\infty} F_X(t) = \mathbb{P}(X = -\infty)$

$$\{t_n\}_{n \in \mathbb{N}} \quad t_{n+1} < t_n \quad \lim_{n \to \infty} t_n = -\infty$$

$$\{X = -\infty\} = \bigcap_{n \in \mathbb{N}} \{X \leq t_n\}$$

$$\omega : X(\omega) = -\infty \Rightarrow X(\omega) \leq t_n \quad \forall n \in \mathbb{N}$$
$$\omega \in \{X \leq t_n\} \quad \forall n \in \mathbb{N}$$
$$\omega \in \bigcap_{n \in \mathbb{N}} \{X \leq t_n\}$$

cioè $\{X = -\infty\} \subset \bigcap_{n \in \mathbb{N}} \{X \leq t_n\}$

$$\omega \in \bigcap_{n \in \mathbb{N}} \{X \leq t_n\} \qquad X(\omega) \leq t_n \quad \forall n \in \mathbb{N}$$
$$X(\omega) \leq \lim_{n \to \infty} t_n = -\infty$$
$$\Downarrow$$
$$X(\omega) = -\infty$$

cioè $\bigcap_{n \in \mathbb{N}} \{X \leq t_n\} \subseteq \{X = -\infty\}$

$$\mathbb{P}(X = -\infty) = \mathbb{P}\left(\bigcap_{n \in \mathbb{N}} \{X \leq t_n\}\right) \qquad \{X \leq t_n\} \subseteq \{X \leq t_{n-1}\}$$

$$= \lim_{n \to \infty} \mathbb{P}(X \leq t_n) = \lim_{n \to \infty} F_X(t_n)$$

$$\Rightarrow \lim_{t \to -\infty} F_X(t) = \mathbb{P}(X = -\infty)$$

3) $\lim\limits_{t\to+\infty} F_X(t) = 1 - \mathbb{P}(X=+\infty)$

$\{t_n\} \quad t_n < t_{n+1} \quad \lim t_n = +\infty$

$\{X=+\infty\} = \bigcap\limits_{n\in\mathbb{N}} \{X > t_n\}$

$$\mathbb{P}(X=+\infty) = \mathbb{P}\Big(\bigcap_{n\in\mathbb{N}} \{X>t_n\}\Big) = \lim_{n\to\infty} \underbrace{\mathbb{P}(X>t_n)}$$

$\{X>t_n\} = \{X\le t_n\}^c$

$$= \lim_{n\to\infty} \big(1-\mathbb{P}(X\le t_n)\big) =$$

$$= \lim_{n\to\infty} \big(1 - F_X(t_n)\big) = 1 - \lim_{n\to\infty} F_X(t_n)$$

$\exists \lim\limits_{t\to+\infty} F_X(t) = 1 - \mathbb{P}(X=+\infty)$

4) F_X è continua da destra cioè

$\lim\limits_{s\to t^+} F_X(s) = F_X(t)$

Sia $\{s_n\} \quad s_{n+1} < s_n \quad \lim s_n = t$

$\{X\le t\} = \bigcap\limits_{n\in\mathbb{N}} \{X\le s_n\}$

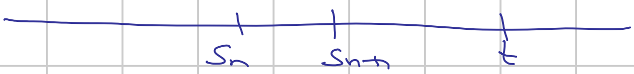
$$F_X(t) = \mathbb{P}(X\le t) = \mathbb{P}\Big(\bigcap_{n\in\mathbb{N}}\{X\le s_n\}\Big) =$$

$$= \lim_{n\to\infty} \mathbb{P}(X\le s_n) = \lim_{n\to\infty} F_X(s_n)$$

$\exists \lim\limits_{s\to t^+} F_X(s) = F_X(t)$

5) $F_X(t) - \lim\limits_{s\to t^-} F_X(s) = \mathbb{P}(X=t)$

$\{s_n\} \quad s_n < s_{n+1} \quad \lim\limits_{n\to\infty} s_n = t$



$$\{X \le s_n\} \subseteq \{X \le s_{n+1}\}$$

$$\bigcup_{n\in\mathbb{N}} \{X \le s_n\} = \{X < t\}$$ (dimostrare con la doppia inclusione)

$$\mathbb{P}(X<t) = \mathbb{P}\Big(\bigcup_{n\in\mathbb{N}} \{X \le s_n\}\Big) = \lim_{n\to\infty} \mathbb{P}(X \le s_n)$$

$$= \lim_{n\to\infty} F_X(s_n)$$

$$\exists \lim_{s\to t^-} F_X(s) = \mathbb{P}(X<t) = \mathbb{P}(X\le t) - \mathbb{P}(X=t)$$
$$= F_X(t) - \mathbb{P}(X=t)$$

<u>COROLLARIO</u> I pti $t\in\mathbb{R}$: $\mathbb{P}(X=t)>0$ cioè i pti $t\in\mathbb{R}$ t.c. $\mathbb{P}_X(\{t\})>0$ sono al più una infinità numerabile.

— o —

X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ $\quad X:\Omega\to\mathbb{R}$

$A\in\mathcal{B}(\mathbb{R})$ $\quad X^{-1}(A) = \{\omega\in\Omega : X(\omega)\in A\} \in \mathcal{E}$

$$\Longrightarrow \{X^{-1}(A) : A\in\mathcal{B}(\mathbb{R})\} \subseteq \mathcal{E}$$

σ-ALGEBRA DEGLI EVENTI RILEVATI DA X

$\mathcal{E}(X)$ o $\sigma(X)$

<u>ESEMPIO</u> $\quad E\in\mathcal{E}$ $\quad X = \mathbb{1}_E$ $\quad X(\omega) = \begin{cases} 1 & \omega\in E \\ 0 & \omega\notin E \end{cases}$

$A\in\mathcal{B}(\mathbb{R})$ $\quad X^{-1}(A) = ?$

$A\ni\{0,1\}$ $\quad X^{-1}(A) = \Omega$

$A\ni 0,\ A\not\ni 1$ $\quad X^{-1}(A) = E^c$

$A\ni 1,\ A\not\ni 0$ $\quad X^{-1}(A) = E$

$A\not\ni 1,\ A\not\ni 0$ $\quad X^{-1}(A) = \emptyset$

$$\sigma(X) = \{\emptyset, \Omega, E, E^c\}$$

CLASSI TIPICHE DI V.A.

1) V.A. con distribuzione discreta

2) V.A. con distribuzione assolutamente continua

→ CONTROESEMPIO DI UNA V.A. che non appartiene a nessuna delle due classi.

1) V.A. con distribuzione discreta

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $X: \Omega \to \overline{\mathbb{R}}$ v.a.

Dico che X ha distribuzione discreta se

$\exists \{t_j\}_{j \in J}$ $\quad J = \{1 \dots N\}$ o $J = \mathbb{N}$ $\quad \{t_j\}_{j \in J} \subseteq \mathbb{R}$

t.c. $\mathbb{P}_X(A) = 0 \quad \forall A \in \mathcal{B}(\mathbb{R})$ t.c. $A \cap \{t_j\} = \emptyset$

$A \in \mathcal{B}(\mathbb{R}) \qquad \mathbb{P}_X(A) = ? \qquad T = \bigcup_{j \in J} \{t_j\}$

$$A = (A \cap T) \cup \underbrace{(A \cap T^c)}_{\mathbb{P}_X(A \cap T^c) = 0}$$

$$\mathbb{P}_X(A) = \mathbb{P}_X(A \cap T) = \mathbb{P}_X\Big(A \cap \bigcup_{j \in J}\{t_j\}\Big) =$$

$$= \mathbb{P}_X\Big(\bigcup_{j \in J}\big(A \cap \{t_j\}\big)\Big) = \sum_{j \in J} \mathbb{P}_X(A \cap \{t_j\})$$

$$= \sum_{j : t_j \in A} \mathbb{P}_X(\{t_j\})$$

$t \in \mathbb{R} \qquad F_X(t) \qquad A = (-\infty, t]$

$$F_X(t) = \sum_{j : t_j \le t} \mathbb{P}_X(\{t_j\})$$

$j \in J \qquad p_j := \mathbb{P}_X(\{t_j\}) = \mathbb{P}(X = t_j)$

p_j si dice DENSITÀ DISCRETA DI X IN t_j

$p_j \in [0, 1]$

$\mathbb{P}(X \in \mathbb{R}) = 1 \qquad \sum_{j \in J} p_j = \sum_{j : t_j \in \mathbb{R}} p_j = \mathbb{P}(X \in \mathbb{R}) = 1$

$\varphi : \mathbb{R} \to \mathbb{R}$ funzione di Borel

$\forall t \in \mathbb{R} \quad \{\varphi \le t\} \in \mathcal{B}(\mathbb{R})$

$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \,?$

1° CASO φ funzione semplice $\varphi(t) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(t)$

$E_i \in \mathcal{B}(\mathbb{R}) \quad E_i := \{t \in \mathbb{R} : \varphi(t) = c_i\}$

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \sum_{i=1}^{n} c_i \mathbb{P}_X(E_i)$$

$$= \sum_{i=1}^{n} c_i \sum_{j : t_j \in E_i} \mathbb{P}_X(\{t_j\}) = \sum_{i=1}^{n} \sum_{j : t_j \in E_i} c_i \mathbb{P}_X(\{t_j\})$$

$$= \sum_{i=1}^{n} \sum_{j : t_j \in E_i} \varphi(t_j) \mathbb{P}_X(\{t_j\}) = \sum_{j \in J} \varphi(t_j) \underbrace{\mathbb{P}_X(\{t_j\})}_{p_j}$$

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \sum_{j \in J} \varphi(t_j) p_j$$

2) $\varphi : \mathbb{R} \to \mathbb{R}$ funzione di Borel non negativa

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \sup \left\{ \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_X(dt) : \begin{array}{l} 0 \le \psi \le \varphi \\ \psi \text{ semplice} \end{array} \right\}$$

$$= \sup \left\{ \sum_{j \in J} \psi(t_j) p_j : 0 \le \psi \le \varphi,\ \psi \text{ semplice} \right\}$$

$$= \sum_{j \in J} \varphi(t_j) p_j$$

3) $\varphi : \mathbb{R} \to \mathbb{R}$ di segno variabile

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \underbrace{\int_{\mathbb{R}} \varphi^+(t)\, \mathbb{P}_X(dt) - \int_{\mathbb{R}} \varphi^-(t)\, \mathbb{P}_X(dt)}_{\text{se almeno uno di questi integrali è finito}}$$

$$= \sum_{j \in J} \varphi^+(t_j) p_j - \sum_{j \in J} \varphi^-(t_j) p_j = \qquad \varphi^+ := \max\{\varphi, 0\}$$

Se $\varphi(t_j) \geq 0 \Rightarrow \varphi^+(t_j) = \varphi(t_j)$

$\varphi(t_j) < 0 \Rightarrow \varphi^+(t_j) = 0$

$$\boxed{\sum_{j \in J} \varphi^+(t_j) p_j = \sum_{j : \varphi(t_j) \geq 0} \varphi(t_j) p_j}$$

$$\sum_{j \in J} \varphi^-(t_j) p_j \qquad \varphi^- = \max\{-\varphi, 0\}$$

$\varphi(t_j) \geq 0 \qquad \varphi^-(t_j) = 0$

$\varphi(t_j) < 0 \qquad \varphi^-(t_j) = -\varphi(t_j)$

$$\boxed{\sum_{j \in J} \varphi^-(t_j) p_j = \sum_{j : \varphi(t_j) < 0} -\varphi(t_j) p_j}$$

$$\int_{\mathbb{R}} \varphi(t) \mathbb{P}_X(dt) = \sum_{j \in J} \varphi(t_j) p_j$$ purché almeno uno tra $\sum_{j : \varphi(t_j) \geq 0} \varphi(t_j) p_j$ e $\sum_{j : \varphi(t_j) < 0} -\varphi(t_j) p_j$ converga

In particolare $\int_{\mathbb{R}} \varphi(t) \mathbb{P}_X(dt)$ esiste finito SSE $\sum_{j \in J} |\varphi(t_j)| p_j < +\infty$

2) v.A. con distribuzione assolutamente continua (A.C.)

Sono le v.a. X $\quad \mathbb{P}(X \in \mathbb{R}) = 1$ t.c.

$\exists f : \mathbb{R} \to [0, +\infty]$ funzione di Borel t.c.

$\forall A \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}_X(A) = \mathbb{P}(X \in A) = \int_A f(x)\, dx$$

$$F_X(t) = \mathbb{P}(X \leq t) = \underbrace{\mathbb{P}(X = -\infty)}_{=0 \text{ per ipotesi}} + \mathbb{P}(X \in (-\infty, t]) =$$

$$= \int_{-\infty}^{t} f(x)\,dx \implies F_X \text{ è finalmente continua}$$

Se f è continua in $\bar{t} \implies F_X'(\bar{t}) = f(\bar{t})$

La funzione f si dice DENSITÀ DELLA DISTRIBUZIONE $\mathbb{P}_X$ o DENSITÀ DELLA V.A. X.

$\varphi : \mathbb{R} \to \mathbb{R}$ di Borel

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = ?$$

1) φ funzione semplice $\varphi(t) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(t)$

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \sum_{i=1}^{n} c_i \mathbb{P}_X(E_i) = \sum_{i=1}^{n} c_i \int_{E_i} f(x)\,dx =$$

$$= \sum_{i=1}^{n} \int_{E_i} c_i f(x)\,dx = \sum_{i=1}^{n} \int_{E_i} \varphi(x) f(x)\,dx = \int_{\mathbb{R}} \varphi(x) f(x)\,dx$$

2) φ di Borel nonnegativa

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \sup\left\{ \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_X(dt) : 0 \leq \psi \leq \varphi, \ \psi \text{ semplice} \right\} =$$

$$= \sup\left\{ \int_{\mathbb{R}} \psi(t) f(t)\,dt : 0 \leq \psi \leq \varphi : \psi \text{ semplice} \right\}$$

$$= \int_{\mathbb{R}} \varphi(t) f(t)\,dt \quad \text{per Beppo Levi}$$

3) $\varphi : \mathbb{R} \to \mathbb{R}$ di Borel di segno variabile

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt) = \underbrace{\int_{\mathbb{R}} \varphi^+(t)\, \mathbb{P}_X(dt) - \int_{\mathbb{R}} \varphi^-(t)\, \mathbb{P}_X(dt)}_{\text{purché almeno uno tra i due sia finito}} =$$

$$\overset{*}{=} \int_{\mathbb{R}} \varphi^+(t) f(t)\,dt - \int_{\mathbb{R}} \varphi^-(t) f(t)\,dt$$

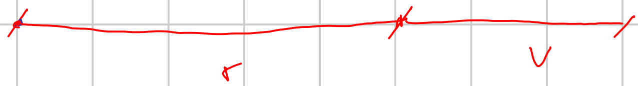
$\varphi^+ f = (\varphi f)^+$

$\varphi^- f = (\varphi f)^-$

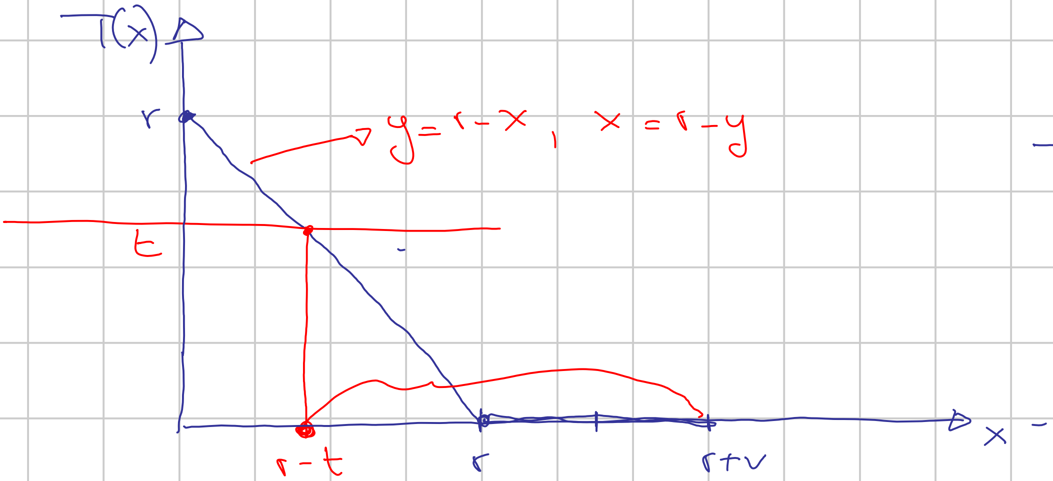
$$= \int_{\mathbb{R}} \varphi(t) f(t)\, dt$$

$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_X(dt)$ esiste finito sse $\int_{\mathbb{R}} |\varphi(t)|\, f(t)\, dt < +\infty$

— o —



$[0, r+v]$, $\mathcal{E} = \mathcal{B}(\mathbb{R})$, $\mathbb{P}$ uniforme $\qquad \mathbb{P}(A) = \dfrac{\mathcal{L}(A)}{r+v} \qquad A \in \mathcal{B}$



$$T(x) = \begin{cases} r - x & 0 \le x \le r \\ 0 & r < x \le r+v \end{cases}$$

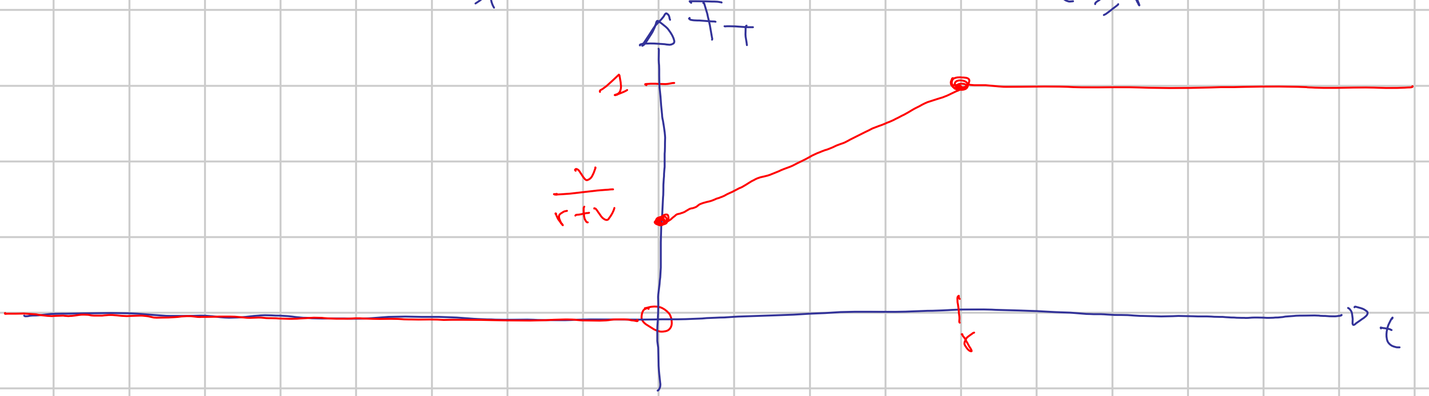
$F_T(t) = \mathbb{P}(T \le t)$

$t < 0 \qquad \{T \le t\} = \emptyset \qquad F_T(t) = 0$

$t \geq r \qquad \{T \le t\} = \Omega = [0, r+v] \qquad F_T(t) = 1$

$t \in [0, r) \qquad \{T \le t\} = [r - t, r+v] \qquad F_T(t) = \dfrac{t+v}{r+v}$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t+v}{r+v} & 0 \le t < r \\ 1 & t \geq r \end{cases}$$



$$F_X(t) = \sum_{j:\, t_j \leq t} \mathbb{P}(X = t_j) \qquad t : \mathbb{P}(X = t) > 0 \iff t = 0$$

$$= \sum_{0 \leq t} \mathbb{P}(X = 0) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$