Calcolo delle Probabilità –2016-2017

Secondo Appello – 8 febbraio 2017

Domanda 1) **(8 punti)** Si hanno due urne, ciascuna delle quali contiene due palline bianche e due palline verdi.
Si estraggono (senza reimbussolamento) due palline dalla prima urna e le si inseriscono nella seconda.
Si estraggono (senza reimbussolamento) due palline dalla seconda urna. Calcolare la probabilità di estrarre due, una o nessuna pallina bianca dalla seconda urna.
Sapendo di aver estratto due palline bianche dalla seconda urna, calcolare la probabilità di aver estratto due, una o nessuna pallina bianca dalla prima urna.

$\frac{19}{90}$; $\frac{26}{45}$; $\frac{19}{90}$, $\frac{6}{19}$; $\frac{12}{19}$; $\frac{1}{19}$

Svolgimento

A_k = estraggo k palline bianche dalla 1ª urna, $k=0,1,2$

B_k = estraggo k palline bianche dalla 2ª urna, $k=0,1,2$

$$\mathbb{P}(A_k) = \frac{\binom{2}{k}\binom{2}{2-k}}{\binom{4}{2}} \quad ; \quad \mathbb{P}(B_k) = \sum_{i=0}^{2} \mathbb{P}(B_k|A_i)\mathbb{P}(A_i)$$

$$\mathbb{P}(B_k|A_i) = \frac{\binom{2+i}{k}\binom{2+2-i}{2-k}}{\binom{6}{2}} = \frac{1}{15}\binom{2+i}{k}\binom{4-i}{2-k}$$

$$\mathbb{P}(B_0) = \sum_{i=0}^{2} \frac{1}{15}\binom{4-i}{2}\binom{2}{i}\binom{2}{2-i}\frac{1}{6} = \frac{1}{90}\left\{6+3\cdot 2\cdot 2+1\right\} = \frac{19}{90}$$

$$\mathbb{P}(B_1) = \sum_{i=0}^{2} \frac{1}{15}\binom{2+i}{1}\binom{4-i}{1}\frac{1}{6}\binom{2}{i}\binom{2}{2-i} = \frac{1}{90}\left\{2\cdot 4+3\cdot 3\cdot 2\cdot 2+4\cdot 2\right\} = \frac{26}{45}$$

$$\mathbb{P}(B_2) = 1-(\mathbb{P}(B_0)+\mathbb{P}(B_1)) = \frac{90-(19+52)}{90} = \frac{19}{90}$$

$$\mathbb{P}(A_i|B_2) = \frac{\mathbb{P}(B_2|A_i)\mathbb{P}(A_i)}{\mathbb{P}(B_2)} = \frac{90}{19}\,\frac{1}{15}\binom{2+i}{2}\binom{4-i}{0}\binom{2}{i}\binom{2}{2-i}\frac{1}{6} =$$

$$= \frac{1}{19}\,\frac{(2+i)!}{\cancel{2!}\,i!}\,\frac{\cancel{2!}}{i!\,(2-i)!}\,\frac{2!}{i!\,(2-i)!}$$

$i=0 \quad \mathbb{P}(A_0|B_2) = \frac{1}{19} \qquad i=1 \quad \mathbb{P}(A_1|B_2) = \frac{1}{19}\cdot 3\cdot 2\cdot 2 = \frac{12}{19}$

$i=2 \quad \mathbb{P}(A_2|B_2) = 1-(\mathbb{P}(A_0|B_2)+\mathbb{P}(A_1|B_1)) = \frac{19-13}{19} = \frac{6}{19}$

Domanda 2) (12 punti) La v.a. vettoriale (X,Y) ha distribuzione assolutamente continua con densità

$$f(x,y) = C(x+y)\exp\left(\frac{-x^2-y^2}{2}\right)\mathbb{1}_{(0,+\infty)^2}(x,y)$$

Calcolare il valore della costante C, le densità marginali f_X e f_Y, il valore atteso e la varianza di X e Y e la loro covarianza.

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Svolgimento $1 = \int_{\mathbb{R}^2} f(x,y)\,dx\,dy = C\int_0^{+\infty}\int_0^{+\infty}(x+y)\exp\left(-\frac{x^2}{2}\right)\exp\left(-\frac{y^2}{2}\right)dx\,dy$

$$= 2C\underbrace{\int_0^{+\infty} x\exp\left(-\frac{x^2}{2}\right)dx}_{=1}\ \underbrace{\int_0^{+\infty}\exp\left(-\frac{y^2}{2}\right)dy}_{\sqrt{\frac{\pi}{2}}} = C\sqrt{2\pi} \Rightarrow C = \frac{1}{\sqrt{2\pi}}$$

$f_X(x) = 0 \qquad x \le 0$

Per $x>0$: $f_X(x) = \int_0^{+\infty}\frac{1}{\sqrt{2\pi}}(x+y)\exp\left(\frac{-x^2}{2}\right)\exp\left(\frac{-y^2}{2}\right)dy$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\left(\int_0^{+\infty}\left\{x\exp\left(-\frac{y^2}{2}\right) + y\exp\left(\frac{-y^2}{2}\right)\right\}dy\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\left(x\sqrt{\frac{\pi}{2}}+1\right)$$

$f_Y(x) = f_X(x)$ per simmetria $\Rightarrow$ $\mathbb{E}[Y] = \mathbb{E}[X]$
$\mathrm{Var}[Y] = \mathrm{Var}[X]$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\exp\left(\frac{-x^2}{2}\right)\left(x^2\sqrt{\frac{\pi}{2}}+x\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\sqrt{\frac{\pi}{2}}\int_0^{+\infty}(-x)\frac{d}{dx}\left(\exp\left(\frac{-x^2}{2}\right)\right)dx + 1\right\} =$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\sqrt{\frac{\pi}{2}}\int_0^{+\infty}\exp\left(-\frac{x^2}{2}\right)dx + 1\right\} = \frac{1}{\sqrt{2\pi}}\left\{\frac{\pi}{2}+1\right\}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f_X(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\exp\left(\frac{-x^2}{2}\right)\left(x^3\sqrt{\frac{\pi}{2}}+x^2\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \sqrt{\frac{\pi}{2}} \int_0^{+\infty} x^3 \exp\left(-\frac{x^2}{2}\right) dx + \int_0^{+\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \sqrt{\frac{\pi}{2}} \int_0^{+\infty} (-x^2) \frac{d}{dx} \exp\left(\frac{-x^2}{2}\right) dx + \sqrt{\frac{\pi}{2}} \right\} =$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \sqrt{\frac{\pi}{2}} \int_0^{+\infty} 2x \exp\left(\frac{-x^2}{2}\right) dx + \sqrt{\frac{\pi}{2}} \right\} = \frac{1}{\sqrt{2\pi}} 3\sqrt{\frac{\pi}{2}} = \frac{3}{2}$$

$$\mathrm{Var}[X] = \frac{3}{2} - \frac{1}{2\pi}\left(\frac{\pi}{2}+1\right)^2$$

$$\mathbb{E}[XY] = \int_{\mathbb{R}^2} xy f(x,y)\, dx dy = \frac{1}{\sqrt{2\pi}} \int_{(0,+\infty)^2} (x^2y + xy^2) \exp\left(\frac{-x^2-y^2}{2}\right) dx dy$$

$$= \sqrt{\frac{2}{\pi}} \int_{(0,+\infty)^2} x^2 y \exp\left(-\frac{x^2-y^2}{2}\right) dx dy = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} x^2 \exp\left(\frac{-x^2}{2}\right) dx \int_0^{+\infty} y \exp\left(\frac{-y^2}{2}\right) dy$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{\pi}{2}} \cdot 1 = 1$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 1 - \frac{1}{2\pi}\left(\frac{\pi}{2}+1\right)^2$$

Domanda 3) **(4 punti)** Siano X e Y due v.a. sullo stesso spazio probabilizzato $(\Omega, \mathcal{E}, \mathbb{P})$ tali che $\mathbb{P}(X = Y) = 1$. Provare che X e Y seguono la stessa legge.

Svolgimento

$F_X(t) - F_Y(t) = \mathbb{P}(X \le t) - \mathbb{P}(Y \le t)$

Ma $\mathbb{P}(Y \le t) \ge \mathbb{P}(X \le t, Y \le t) \Rightarrow$

$$F_X(t) - F_Y(t) \le \mathbb{P}(X \le t) - \mathbb{P}(X \le t, Y \le t) = \mathbb{P}(X \le t, Y > t)$$
$$\le \mathbb{P}(X \ne Y) = 0$$

Dunque $F_X(t) \le F_Y(t)$

Rovesciando il ruolo di X e Y troviamo la disuguaglianza opposta e dunque $F_X(t) = F_Y(t)$

Domanda 4) **(8 punti)** X e Y sono v.a. indipendenti con $\mathbb{P}_X = G'(p)$, $\mathbb{P}_Y = G'(q)$, dove $p, q \in (0,1)$. Nei due casi $p = q$ e $p \neq q$, calcolare la probabilità che la v.a. $Z := X + Y$ sia pari.

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Svolgimento $\mathbb{P}(X=k) = p(1-p)^k \quad \mathbb{P}(Y=k) = q(1-q)^k \quad k \in \mathbb{N}_0$

$$\mathbb{P}(Z \text{ pari}) = \sum_{n=0}^{\infty} \mathbb{P}(X+Y=2n) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \mathbb{P}(X=k)\mathbb{P}(Y=2n-k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{2n} p(1-p)^k q(1-q)^{2n-k}$$

(1) $p=q$

$$\mathbb{P}(Z \text{ pari}) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} p^2(1-p)^{2n} = p^2 \sum_{n=0}^{\infty} (2n+1)(1-p)^{2n} =$$

$$= p^2 \sum_{n=0}^{\infty} (2n+1)x^{2n} \Big|_{x=1-p} = p^2 \frac{d}{dx} \sum_{n=0}^{\infty} x^{2n+1} \Big|_{x=1-p}$$

$$= p^2 \frac{d}{dx}\left(x \frac{1}{1-x^2}\right)\Big|_{x=1-p} = p^2 \frac{1+x^2}{(1-x^2)^2}\Big|_{x=1-p} =$$

$$= \frac{p^2(1+(1-p)^2)}{(1-(1-p)^2)^2}$$

(3) $p \neq q$

$$\mathbb{P}(Z \text{ pari}) = \sum_{n=0}^{\infty} pq(1-q)^{2n} \sum_{k=0}^{2n} \left(\frac{1-p}{1-q}\right)^k =$$

$$= pq \sum_{n=0}^{\infty} (1-q)^{2n} \frac{1-\left(\frac{1-p}{1-q}\right)^{2n+1}}{1-\frac{1-p}{1-q}} = \frac{pq(1-q)}{p-q} \sum_{n=0}^{\infty} \left((1-q)^{2n} - \frac{(1-p)^{2n+1}}{1-q} \right)$$

$$= \frac{pq(1-q)}{p-q} \left\{ \frac{1}{1-(1-q)^2} - \frac{1-p}{1-q} \frac{1}{1-(1-p)^2} \right\}$$