Titolo nota 07/12/2016

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia

$X : \Omega \to \mathbb{R}$ v.a. discreta

$X(\Omega) = \{x_i\}_{i \in J}$ $J = \{1 \_\_ n\}$ o $\mathbb{N}$

Fisso $B \in \mathcal{E}$ $A \in \mathcal{B}(\mathbb{R})$

$\mathbb{P}(B \cap \{X \in A\}) = ?$

$$B \cap \{X \in A\} = B \cap \bigcup_{x_i \in A} \{X = x_i\} = \bigcup_{x_i \in A} (B \cap \{X = x_i\})$$

$$\mathbb{P}(B \cap \{X \in A\}) = \sum_{x_i \in A} \mathbb{P}(B \cap \{X = x_i\}) =$$

$$= \sum_{x_i \in A} \mathbb{P}(B \mid X = x_i)\, \mathbb{P}(X = x_i) =$$

$$= \sum_{x_i \in A} \mathbb{P}(B \mid X = x_i)\, \mathbb{P}_X(\{x_i\})$$

Pongo $\varphi_B : x \in \mathbb{R} \longmapsto \begin{cases} \mathbb{P}(B \mid X = x) & \mathbb{P}(X = x) > 0 \\ 0 & \mathbb{P}(X = x) = 0 \end{cases}$

$$\mathbb{P}(B \cap \{X \in A\}) = \int_A \varphi_B(x)\, \mathbb{P}_X(dx)$$

— o —

$X, Y : \Omega \to \mathbb{R}$ t.c. $\mathbb{P}_{X,Y} = f(x,y)\, dx\, dy$

$D \in \mathcal{B}(\mathbb{R})$ $B := Y^{-1}(D) \in \mathcal{E}$

$$\mathbb{P}(B \cap \{X \in A\}) = \mathbb{P}(\{Y \in D\} \cap \{X \in A\}) = \mathbb{P}\big((X,Y) \in A \times D\big)$$

$$= \int_{A\times D} f(x,y)\,dx\,dy = \int_A \left(\int_D f(x,y)\,dy \right) dx = (*)$$

$$\mathbb{P}_X = f_X(x)\,dx \qquad f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$(*)\ \int_A \underbrace{\left(\int_D \frac{f(x,y)}{f_X(x)}\,dy \right)}_{=:\ \varphi_D(x)} f_X(x)\,dx = \int_A \varphi_D(x)\ \mathbb{P}_X(dx)$$

$$S := \{ x \in \mathbb{R} : f_X(x) = 0 \} \qquad \mathbb{P}_X(S) = \int_S f_X(x)\,dx = 0$$

$$\tilde{S} = \{ (x,y) \in \mathbb{R}^2 : f_X(x) = 0 \} \qquad \mathbb{P}_{X,Y}(\tilde{S})$$

$$\mathbb{P}_{X,Y}(\tilde{S}) = \int_{\tilde{S}} f(x,y)\,dx\,dy \qquad \tilde{S} = S \times \mathbb{R}$$

$$= \int_S \left(\int_{\mathbb{R}} f(x,y)\,dy \right) dx = \int_S f_X(x)\,dx = 0$$

$\dfrac{f(x,y)}{f_X(x)}$ è $\mathbb{P}_{X,Y}$-qc ben definita

si chiama DENSITÀ DI Y DATO CHE $X = x$

$h(y|x)$

— o —

$(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

$(\Omega', \mathcal{E}')$ spazio misurabile

$X : \Omega \to \Omega'$ si dice $(\mathcal{E}, \mathcal{E}')$-misurabile se

$\forall A \in \mathcal{E}'$ si ha che $X^{-1}(A) \in \mathcal{E}$

Per ogni $A \in \mathcal{E}'$ definisco $\mathbb{P}_X(A) := \mathbb{P}(X \in A)$

Allora la terna $(\Omega', \mathcal{E}', \mathbb{P}_X)$ è uno spazio probabilizzato

TEO Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato, sia $(\Omega', \mathcal{E}')$ spazio misurabile

Sia $X : \Omega \to \Omega'$ funzione $(\mathcal{E}, \mathcal{E}')$-misurabile

Sia $B \in \mathcal{E}$

Allora $\exists\, \varphi_B : \Omega' \to \mathbb{R}$ v.a. su $(\Omega', \mathcal{E}')$ t.c.

$$\mathbb{P}(B \cap \{X \in A\}) = \int_A \varphi_B(\omega')\, \mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

$$\begin{tikzcd}
(\Omega, \mathcal{E}) \arrow[r, "X"] & (\Omega', \mathcal{E}') \arrow[dl, dashed, "\varphi_B"] \\
(\mathbb{R}, \mathcal{B}(\mathbb{R})) &
\end{tikzcd}$$

DIM Considero $\mu_B : A \in \mathcal{E}' \longmapsto \mathbb{P}(B \cap \{X \in A\})$

$(\Omega', \mathcal{E}', \mu_B)$ è uno spazio di misura con μ_B misura finita.

$\mu_B(\emptyset) = \mathbb{P}(B \cap \{X \in \emptyset\}) = \mathbb{P}(\emptyset) = 0$

$\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}'$ A_i disgiunti 2 a 2

$$\mu_B\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(B \cap \left\{X \in \bigcup_{i=1}^{\infty} A_i\right\}\right) = \mathbb{P}\left(B \cap \left(\bigcup_{i=1}^{\infty} \{X \in A_i\}\right)\right)$$

$$= \mathbb{P}\left(\bigcup_{i=1}^{\infty} (B \cap \{X \in A_i\})\right) = \sum_{i=1}^{\infty} \underbrace{\mathbb{P}(B \cap \{X \in A_i\})}_{\mu_B(A_i)}$$

$\Rightarrow$ μ_B è una misura su $(\Omega', \mathcal{E}')$

$\mu_B(\Omega') = \mathbb{P}(B \cap \{X \in \Omega'\}) = \mathbb{P}(B \cap \Omega) = \mathbb{P}(B) \leq 1$

Se $\mathbb{P}_X(A) = 0$ cioè $\mathbb{P}(X \in A) = 0$ $\Rightarrow$

$\mu_B(A) = \mathbb{P}(B \cap \{X \in A\}) \leq \mathbb{P}(X \in A) = 0$

Applico Radon-Nikodým su $(\Omega', \mathcal{E}')$ con le misure μ_B e $\mathbb{P}_X$:

$\exists\, \varphi_B : \Omega' \to \mathbb{R}$ $\varphi_B \geq 0$ $\mathcal{E}'$-misurabile

$\varphi_B \in \mathcal{L}^1(\Omega', \mathbb{P}_X)$ t.c.

$$\mu_B(A) = \int_A \varphi_B(\omega')\, \mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

cioè
$$\mathbb{P}(B \cap \{X \in A\}) = \int_A f_B(\omega')\, \mathbb{P}_X(d\omega') \qquad \forall A \in \mathcal{E}'$$

Se esiste un'altra funzione $\varphi : \Omega' \to \mathbb{R}$ che gode di questa proprietà $\Rightarrow f_B = \varphi$ $\mathbb{P}_X$-q.c.

La $f_B(\omega')$ si indica $\mathbb{P}(B \mid X = \omega')$

— o —

TEO Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

Sia $(\Omega', \mathcal{E}')$ spazio misurabile

Sia $X : \Omega \to \Omega'$ $(\mathcal{E}, \mathcal{E}')$-misurabile

Sia $Y : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ con $Y \in \mathcal{L}^1(\Omega, \mathbb{P})$

$$\begin{tikzcd}
(\Omega, \mathcal{E}) \arrow[r, "X"] \arrow[d, "Y"'] & (\Omega', \mathcal{E}') \arrow[dl, "f_Y"] \\
(\mathbb{R}, \mathcal{B}(\mathbb{R})) &
\end{tikzcd}$$

Allora
$$\exists\, f_Y : \Omega' \to \mathbb{R} \text{ v.a}$$

t.c.
$$\int_A f_Y(\omega')\, \mathbb{P}_X(d\omega') = \int_{X^{-1}(A)} Y(\omega)\, \mathbb{P}(d\omega) \qquad \forall A \in \mathcal{E}'$$

Se φ è un'altra v.a. che gode delle stesse proprietà, allora $f_Y = \varphi$ $\mathbb{P}_X$-q.c.

DIM 1° CASO $Y \geq 0$

$A \in \mathcal{E}'$ $\quad \mu(A) = \int_{X^{-1}(A)} Y(\omega)\, \mathbb{P}(d\omega)$

$(\Omega', \mathcal{E}', \mu)$ è uno spazio di misura con μ misura finita.

$$\mu(\emptyset) = \int_\emptyset Y(\omega)\, \mathbb{P}(d\omega) = 0$$

$\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}'$ $\quad A_i$ disgiunti due a due

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \int_{X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)} Y(\omega)\,\mathbb{P}(d\omega) = \int_{\bigcup_{i=1}^{\infty} X^{-1}(A_i)} Y(\omega)\,\mathbb{P}(d\omega) =$$

$$= \int_\Omega Y(\omega)\,\mathbb{1}_{\bigcup_{i=1}^{\infty} X^{-1}(A_i)}(\omega)\,\mathbb{P}(d\omega) = \int_\Omega Y(\omega)\sum_{i=1}^{\infty}\mathbb{1}_{X^{-1}(A_i)}(\omega)\,\mathbb{P}(d\omega)$$

$$= \sum_{i=1}^{\infty}\int_\Omega Y(\omega)\,\mathbb{1}_{X^{-1}(A_i)}(\omega)\,\mathbb{P}(d\omega) = \sum_{i=1}^{\infty}\int_{X^{-1}(A_i)} Y(\omega)\,\mathbb{P}(d\omega) = \sum_{i=1}^{\infty}\mu(A_i)$$

$$\mu(\Omega') = \int_{X^{-1}(\Omega')} Y(\omega)\,\mathbb{P}(d\omega) = \int_\Omega Y(\omega)\,\mathbb{P}(d\omega) = \mathbb{E}[Y]$$

$\mu \ll \mathbb{P}_X$

$A \in \mathcal{E}'$ $\quad \mathbb{P}_X(A) = 0 \quad \mathbb{P}(X\in A) = 0$

$$\mu(A) = \int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \int_\Omega Y(\omega)\,\underbrace{\mathbb{1}_{\{X\in A\}}(\omega)}_{=0\ \mathbb{P}\text{-q.c.}}\,\mathbb{P}(d\omega) = 0$$

Applico Radon-Nikodým a $(\Omega', \mathcal{E}')$ e alle misure μ e $\mathbb{P}_X$

$\exists\, f_Y : \Omega' \to \mathbb{R} \qquad f_Y \geq 0 \qquad f_Y \in \mathcal{L}^1(\Omega', \mathbb{P}_X)$

t.c.

$$\mu(A) = \int_A f_Y(\omega')\,\mathbb{P}_X(d\omega') \qquad \forall A \in \mathcal{E}'$$

$$\| $$

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega)$$

Se φ è un'altra funzione che soddisfa queste proprietà, allora $\varphi = f_Y$ $\mathbb{P}_X$-q.c.

2° CASO Y di segno variabile.

esistono finiti anche $\mathbb{E}[Y^+]$ e $\mathbb{E}[Y^-]$

$\exists\ \varphi^+, \varphi^- : \Omega' \to \mathbb{R}$ v.a. su $(\Omega', \mathcal{E}', \mathbb{P}_X)$ t.c.

$$\int_{X^{-1}(A)} Y^+(\omega)\mathbb{P}(d\omega) = \int_A \varphi^+(\omega')\mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

$$\int_{X^{-1}(A)} Y^-(\omega)\mathbb{P}(d\omega) = \int_A \varphi^-(\omega')\mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

$\varphi^+, \varphi^- : \mathcal{E}' \to \mathbb{R}$ $\mathcal{E}'$-misurabili in $\mathcal{L}^1(\Omega', \mathbb{P}_X)$

Sottraggo membro a membro

$$\int_{X^{-1}(A)} Y(\omega)\mathbb{P}(d\omega) = \int_A \underbrace{(\varphi^+ - \varphi^-)}_{\varphi_Y}(\omega')\mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

Sia ψ un'altra funzione che soddisfa queste proprietà

$$\int_A \psi(\omega')\mathbb{P}_X(d\omega') = \int_{X^{-1}(A)} Y(\omega)\mathbb{P}(d\omega) = \int_A \varphi_Y(\omega')\mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

Scelgo $A = \{\psi > \varphi_Y\}$

$$\int_A (\psi - \varphi_Y)(\omega')\mathbb{P}_X(d\omega') = 0$$

$$\int_{\Omega'} (\psi - \varphi_Y)(\omega')\mathbb{1}_A(\omega')\mathbb{P}_X(d\omega') = 0$$

$\Rightarrow \mathbb{P}_X(A) = 0$

Analogamente scelgo $\tilde{A} = \{\varphi_Y > \psi\}$

e Trovo $\mathbb{P}_X(\tilde{A}) = 0$ $\quad = 0 \quad \psi = f_Y \quad \mathbb{P}_X$-q.c.

La funzione $f_Y(\omega')$ si indica $\mathbb{E}[Y | X = \omega']$
e si chiama SPERANZA CONDIZIONATA DI Y
DATO L'EVENTO $\{X = \omega'\}$

—o—

$(\Omega, \mathcal{E}, \mathbb{P})$
$(\Omega', \mathcal{E}')$

$$\begin{tikzcd}
(\Omega, \mathcal{E}) \arrow[r, "X"] \arrow[dr, "Y"'] & \underbrace{(\Omega', \mathcal{E}')}_{=(\mathbb{R}, \mathcal{B}(\mathbb{R}))} \arrow[d, dashed, "f_Y"] \\
& (\mathbb{R}, \mathcal{B}(\mathbb{R}))
\end{tikzcd}$$

$X : \Omega \to \mathbb{R}$ discreta
$Y : \Omega \to \mathbb{R}$ v.a. $\mathcal{L}^1(\Omega, \mathbb{P})$

$\forall A \in \mathcal{B}(\mathbb{R})$
$$\int_{X^{-1}(A)} Y(\omega)\, \mathbb{P}(d\omega) = \int_A f_Y(\omega')\, \mathbb{P}_X(d\omega')$$

$X(\Omega) = \{x_i\}_{i \in J} \qquad J = \{1 \dots n\}$ o $J = \mathbb{N}$

$$X^{-1}(A) = \bigcup_{x_i \in A} \{X = x_i\}$$

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \int_{\bigcup_{x_i \in A}\{X = x_i\}} Y(\omega)\,\mathbb{P}(d\omega) = \int_\Omega Y(\omega)\, \mathbb{1}_{\bigcup_{x_i \in A}\{X = x_i\}}(\omega)\,\mathbb{P}(d\omega)$$

$$= \sum_{x_i \in A} \int_\Omega Y(\omega)\, \mathbb{1}_{\{X = x_i\}}(\omega)\,\mathbb{P}(d\omega) =$$

$$= \sum_{\substack{x_i \in A \\ \mathbb{P}(X = x_i) > 0}} \int_\Omega Y(\omega)\, \mathbb{1}_{\{X = x_i\}}(\omega)\,\mathbb{P}(d\omega) =$$

$$= \sum_{\substack{x_i \in A \\ \mathbb{P}(X = x_i) > 0}} \int_{\{X = x_i\}} Y(\omega)\,\mathbb{P}(d\omega) =$$

$$= \sum_{x_i \in A : \mathbb{P}(X = x_i) > 0} \left(\frac{1}{\mathbb{P}(X = x_i)} \int_{\{X = x_i\}} Y(\omega)\,\mathbb{P}(d\omega) \right) \mathbb{P}(X = x_i)$$

$$\sum_{\substack{x_i \in A : \\ \mathbb{P}(X=x_i)>0}} \left(\frac{1}{\mathbb{P}(X=x_i)} \int_{\{X=x_i\}} Y(\omega)\,\mathbb{P}(d\omega) \right) \mathbb{P}_X(\{x_i\})$$

$$f_Y : t \in \mathbb{R} \longmapsto \begin{cases} \dfrac{1}{\mathbb{P}(X=t)} \displaystyle\int_{\{X=t\}} Y(\omega)\,\mathbb{P}(d\omega) & \mathbb{P}(X=t)>0 \\ 0 & \text{altrimenti} \end{cases}$$

$$\mathbb{E}[Y \mid X=t]$$

$$= \int_{\mathbb{R}} f_Y(t)\, \mathbb{P}_X(dt)$$

$(X,Y) : \Omega \to \mathbb{R}^2$ v.a. con $\mathbb{P}_{X,Y} = f(x,y)\,dx\,dy$

```tikzcd
(\Omega, \mathcal{E}) \arrow[r, "X"] \arrow[dr, "Y"'] & \underbrace{(\Omega', \mathcal{E}')}_{(\mathbb{R}, \mathcal{B}(\mathbb{R}))} \arrow[d, dashed, "f_Y"] \\
 & (\mathbb{R}, \mathcal{B}(\mathbb{R}))
\end{tikzcd}
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$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \int_A f_Y(\omega')\, \mathbb{P}_X(d\omega') \qquad \forall A \in \mathcal{E}' = \mathcal{B}(\mathbb{R})$$

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) =$$

$$X^{-1}(A) = \{X \in A\} = \{X \in A,\ Y \in \mathbb{R}\} = \{(X,Y) \in A \times \mathbb{R}\}$$

$$= \int_{(X,Y) \in A \times \mathbb{R}} Y(\omega)\,\mathbb{P}(d\omega) =$$

$$\pi : (x,y) \to y$$

$$Y = \pi \circ (X,Y)$$

$$= \int_{A \times \mathbb{R}} y\, \mathbb{P}_{X,Y}(dx\,dy) =$$

$$= \int_{A\times\mathbb{R}} y f(x,y)\,dx\,dy = \int_{A\times\mathbb{R}} y \underbrace{\frac{f(x,y)}{f_X(x)}}_{f(y|x)} f_X(x)\,dx\,dy$$

$$= \int_{A\times\mathbb{R}} y f(y|x)\, f_X(x)\,dx\,dy =$$

$$= \int_A \underbrace{\left(\int_{\mathbb{R}} y f(y|x)\,dy\right)}_{f_Y(x) \,=\, \mathbb{E}[Y \mid X=x]} f_X(x)\,dx = \int_A f_Y(x)\,\mathbb{P}_X(dx)$$

EX 12.21

$$\mathbb{P}_{X,Y} = f(x,y)\,dx\,dy \qquad f(x,y) = \begin{cases} c x^2 y^2 & (x,y) \in [0,1]^2 \\ 0 & \text{altrimenti} \end{cases}$$

$$(3X, XY)$$

$$1 = \int_{\mathbb{R}^2} f(x,y)\,dx\,dy = c \int_0^1 \left(\int_0^1 x^2 y^2\, dy \right) dx =$$

$$= c \left(\int_0^1 x^2\,dx \right)\left(\int_0^1 y^2\,dy \right) = c \left(\frac{x^3}{3}\Big|_0^1 \right)\left(\frac{y^3}{3}\Big|_0^1 \right) = \frac{c}{9}$$

$$\Rightarrow \boxed{c = 9}$$

$$(3X, XY) = \varphi \circ (X,Y) \qquad \varphi : (x,y) \in \mathbb{R}^2 \longmapsto (3x, xy) \in \mathbb{R}^2$$

$\psi : \mathbb{R}^2 \to \mathbb{R}$ di Borel, nonnegativa

$$\int_{\mathbb{R}^2} \psi(s,t)\, \mathbb{P}_{(3X,XY)}(ds\,dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{X,Y}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} \psi(3x, xy)\, f(x,y)\,dx\,dy = \int_{[0,1]^2} \psi(3x,xy)\, 9x^2y^2\,dx\,dy$$

$$\begin{aligned} s &= 3x \\ t &= xy \end{aligned} \qquad \begin{aligned} x &= \frac{s}{3} \\ y &= \frac{3t}{s} \end{aligned} \qquad J = \begin{pmatrix} \frac{1}{3} & 0 \\ \cancel{\alpha} & \frac{3}{s} \end{pmatrix}$$

$$|\det J| = \left| \frac{1}{3} \cdot \frac{3}{s} \right| = \left| \frac{1}{s} \right| = \frac{1}{s}$$

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \qquad \begin{cases} 0 < \frac{s}{3} < 1 \\ 0 < \frac{3t}{s} < 1 \end{cases} \qquad \begin{cases} 0 < s < 3 \\ 0 < t < \frac{s}{3} \end{cases}$$

$$= \int_T \varphi(s,t)\, 9 \left(\frac{\cancel{s}}{\cancel{3}}\right)^2 \left(\frac{\cancel{3}t}{\cancel{s}}\right)^2 \frac{1}{s}\, ds\, dt$$

$$= \int_{\mathbb{R}^2} \varphi(s,t)\, g(s,t)\, ds\, dt \qquad g(s,t) = \begin{cases} \frac{9t^2}{s} & (s,t) \in T \\ 0 & (s,t) \notin T \end{cases}$$

$$= \begin{cases} \frac{9t^2}{s} & 0<s<3, \quad 0<t<\frac{s}{3} \\ 0 & \text{altrimenti} \end{cases}$$

— o —

X, Y i.i.d. $\mathbb{P}_{X,Y}$ è distribuzione uniforme su $[0,1]^2$

$$f(x,y) = \begin{cases} 1 & (x,y) \in [0,1]^2 \\ 0 & \text{altrimenti} \end{cases}$$

$Z = XY$

$Z = \varphi \circ (X,Y) \qquad \varphi : (x,y) \in \mathbb{R}^2 \longmapsto xy \in \mathbb{R}$

ψ di Borel non negativa

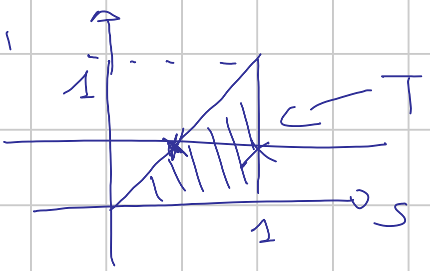
$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Z(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{X,Y}(dx\,dy)$$

$$= \int_{[0,1]^2} \psi(xy)\, 1 \cdot dx\,dy$$

$$\begin{cases} s = x \\ t = xy \end{cases} \qquad \begin{cases} x = s \\ y = \frac{t}{s} \end{cases} \qquad J = \begin{pmatrix} 1 & 0 \\ \cancel{\ast} & \frac{1}{s} \end{pmatrix}$$

$$\begin{cases} 0<x<1 \\ 0<y<1 \end{cases} \qquad \begin{cases} 0<s<1 \\ 0<\frac{t}{s}<1 \end{cases} \qquad |\det J| = \left|\frac{1}{s}\right| = \frac{1}{s}$$

$$\begin{cases} 0<s<1 \\ 0<t<s \end{cases}$$



$$\int_T \psi(t)\, \frac{1}{s}\, ds\,dt = \int_0^1 \psi(t) \left(\int_t^1 \frac{1}{s}\, ds \right) dt$$

$$= \int_0^1 \psi(t)\left(\log|s| \Big|_{s=t}^{s=1}\right) dt = \int_0^1 \psi(t)\left(-\log|t|\right) dt$$

$$= \int_0^1 \psi(t)\left(-\log t\right) dt = \int_{\mathbb{R}} \psi(t)\, g(t)\, dt$$

$$g(t) = \begin{cases} -\log(t) & t \in (0,1) \\ 0 & \text{altrimenti} \end{cases}$$