DISTRIBUZIONI A.C. - 2

Note Title 18/11/2016

Per $\alpha > 0$ $\displaystyle\int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx =: \Gamma(\alpha)$

$$\Gamma(1) = \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{x\to+\infty} 0-(-1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} x^{-1/2} e^{-x}\,dx \qquad x = t^2 \quad dx = 2t\,dt$$

$$t = \sqrt{x}$$

$$= \int_0^{+\infty} \cancel{t^{-1}} \exp(-t^2)\, 2\cancel{t}\,dt = 2\int_0^{+\infty} \exp(-t^2)\,dt$$

$$= \int_{\mathbb{R}} \exp(-t^2)\,dt = \sqrt{\pi}$$

$$\Gamma(\alpha+1) = \int_0^{+\infty} x^{\alpha+1-1} e^{-x}\,dx = \int_0^{+\infty} \underbrace{x^{\alpha}}_{f}\underbrace{e^{-x}}_{g'}\,dx =$$

$$= x^{\alpha}(-e^{-x})\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} \alpha x^{\alpha-1} e^{-x}\,dx$$

$$= 0 + \alpha\Gamma(\alpha)$$

$\forall \alpha > 0$ $\qquad \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$\Gamma(1) = 1$

$\Gamma(2) = 1\cdot\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2\cdot 1$ $\qquad \underline{\Gamma(n) = (n-1)!}$

$\Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5\cdot 3\cdot 1}{2^3}\sqrt{\pi}$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \quad \text{(DIM. PER INDUZIONE)}$$

DISTRIBUZIONE GAMMA DI PARAMETRI α E λ

Per α, λ reali positivi, è la distribuzione $\Gamma(\alpha,\lambda)$ associata alla densità

$$f(t) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$\int_{\mathbb{R}} f(t)\,dt = 1 \iff \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t}\,dt = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

$$\int_0^{+\infty} t^{\alpha-1} e^{-\lambda t}\,dt \qquad x = \lambda t \qquad t = \frac{x}{\lambda} \qquad dt = \frac{1}{\lambda}dx$$

$$= \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda^{\alpha-\cancel{1}}} e^{-x} \frac{1}{\cancel{\lambda}}\,dx = \frac{1}{\lambda^\alpha}\int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

$$\boxed{\int_0^{+\infty} t^{\alpha-1} e^{-\lambda t}\,dt = \frac{\Gamma(\alpha)}{\lambda^\alpha} \qquad \forall \alpha,\lambda > 0}$$

Sia X v.a. t.c. $P_X = \Gamma(\alpha,\lambda)$

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^{+\infty} x\cdot x^{\alpha-1} e^{-\lambda x}\,dx =$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^{+\infty} x^{(\alpha+1)-1} e^{-\lambda x}\,dx$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\cancel{\alpha}+1}} = \frac{\alpha \cancel{\Gamma(\alpha)}}{\lambda \cancel{\Gamma(\alpha)}} = \frac{\alpha}{\lambda}$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^2 x^{\alpha-1} e^{-\lambda x}\, dx =$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{(\alpha+2)-1} e^{-\lambda x}\, dx =$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\lambda^{\cancel{\alpha}+2}} = \frac{(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\lambda^2 \cancel{\Gamma(\alpha)}}$$

$$Var[X] = \frac{(\alpha+1)\alpha}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha^2+\alpha-\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

DISTRIBUZIONE BETA DI PARAMETRI q E r

Per q, r reali positivi è la distribuzione $B(q,r)$ associata alla densità

$$f(x) = \begin{cases} \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} x^{q-1}(1-x)^{r-1} & x \in (0,1) \\ 0 & \text{altrimenti} \end{cases}$$

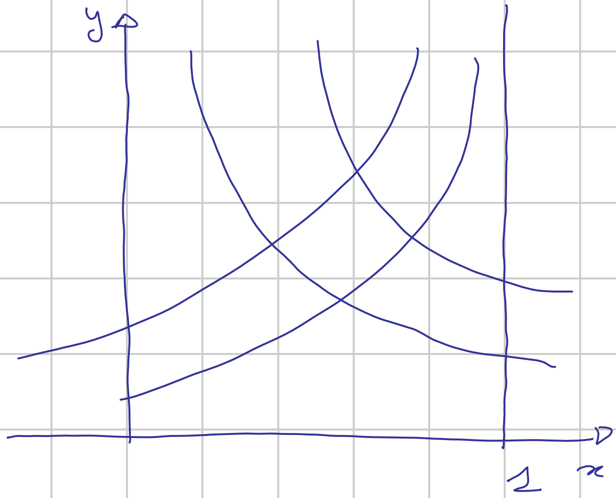
$$\int_{\mathbb{R}} f(x)\, dx = 1 \quad \text{SSE} \quad \Gamma(q+r)\int_0^1 x^{q-1}(1-x)^{r-1}\, dx = \Gamma(q)\Gamma(r)$$

$$\Gamma(q+r)\int_0^1 x^{q-1}(1-x)^{r-1}\, dx =$$

$$= \left(\int_0^{+\infty} y^{q+r-1} e^{-y}\, dy\right)\left(\int_0^1 x^{q-1}(1-x)^{r-1}\, dx\right)$$

$$\int_R x^{q-1}(1-x)^{r-1} y^{q+r-1} e^{-y}\, dy$$

$$\begin{aligned} w &= xy \\ z &= (1-x)y \end{aligned} \qquad y = \frac{z}{1-x}$$



$$w + z = y$$

② $y = w + z$ ① $x = \dfrac{w}{w+z}$

$$J = \begin{pmatrix} \dfrac{\cancel{w} + z - \cancel{w}}{(w+z)^2} & \dfrac{-w}{(w+z)^2} \\ 1 & 1 \end{pmatrix}$$

$$\det J = \frac{z+w}{(w+z)^2} = \frac{1}{w+z} > 0 \quad \text{su } \mathbb{R}$$

$$\int_{(0,+\infty)\times(0,+\infty)} \left(\frac{w}{\cancel{w+z}}\right)^{q-1} \left(\frac{z}{\cancel{w+z}}\right)^{r-1} (w+z)^{\overbrace{\cancel{q+r-1}}^{1}} e^{-w-z} \frac{1}{\cancel{w+z}}\, dw\, dz$$

$$= \left(\int_{(0,+\infty)} w^{q-1} e^{-w}\, dw\right)\left(\int_{(0,+\infty)} z^{r-1} e^{-z}\, dz\right) = \Gamma(q)\,\Gamma(r)$$

$$\forall q, r > 0 \qquad \int_0^1 x^{q-1}(1-x)^{r-1}\, dx = \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)}$$

Sia X v.a. t.c. $\mathbb{P}_X = B(q,r)$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\, dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x\, x^{q-1}(1-x)^{r-1}\, dx$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^{(q+1)-1}(1-x)^{r-1}\, dx$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{\Gamma(q+1)\Gamma(r)}{\Gamma(q+1+r)} =$$

$$= \frac{\cancel{\Gamma(q+r)}}{\cancel{\Gamma(q)}\cancel{\Gamma(r)}} \frac{q\,\cancel{\Gamma(q)}\,\cancel{\Gamma(r)}}{(q+r)\,\cancel{\Gamma(q+r)}} = \frac{q}{q+r}$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^2 x^{q-1} (1-x)^{r-1}\,dx$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^{(q+2)-1} (1-x)^{r-1}\,dx =$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{\Gamma(q+2)\Gamma(r)}{\Gamma(q+2+r)} =$$

$$= \frac{\cancel{\Gamma(q+r)}}{\cancel{\Gamma(q)}\cancel{\Gamma(r)}} \frac{(q+1)\,q\,\cancel{\Gamma(q)}\,\cancel{\Gamma(r)}}{(q+r+1)(q+r)\cancel{\Gamma(q+r)}} = \frac{q(q+1)}{(q+r)(q+r+1)}$$

$$Var[X] = \frac{q(q+1)}{(q+r)(q+r+1)} - \left(\frac{q}{q+r}\right)^2 =$$

$$= \frac{q\left\{(\cancel{q}+1)(q+r) - q(\cancel{q+r}+1)\right\}}{(q+r)^2(q+r+1)} =$$

$$= \frac{q\{q+r-q\}}{(q+r)^2(q+r+1)} = \frac{qr}{(q+r)^2(q+r+1)}$$

V.A. VETTORIALI

$(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

$X, Y : \Omega \to \mathbb{R}$ v.a. finite (entrambe funzioni semplici)

$X(\Omega) = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}$

$Y(\Omega) = \{y_1 \ldots y_m\} \subseteq \mathbb{R}$

$(X, Y) : \omega \in \Omega \mapsto (X(\omega), Y(\omega)) \in \mathbb{R}^2$

$(X, Y)(\Omega) \subseteq X(\Omega) \times Y(\Omega)$

$\{X = x_i, Y = y_j\}$ $\quad i = 1 \ldots n \quad j = 1 \ldots m$

$$\{X = x_i, Y = y_j\} = \underbrace{\{X = x_i\}}_{\in \mathcal{E}} \cap \underbrace{\{Y = y_j\}}_{\in \mathcal{E}} \in \mathcal{E}$$

$\mathbb{P}(X = x_i, Y = y_j)$

$p_i := \mathbb{P}(X = x_i)$ $\quad i = 1 \ldots n \qquad (p_1 \ldots p_n)$

$q_j := \mathbb{P}(Y = y_j)$ $\quad j = 1 \ldots m \qquad (q_1 \ldots q_m)$

$$\{X = x_i\} = \bigcup_{j=1}^{m} \{X = x_i, Y = y_j\} \quad , \quad \{Y = y_j\}_{j=1}^{m}$$

$$p_i = \mathbb{P}(X = x_i) = \sum_{j=1}^{m} \mathbb{P}(X = x_i, Y = y_j)$$

$$\{X = x_i\}_{i=1}^{n} \qquad \{Y = y_j\} = \bigcup_{i=1}^{n} \{X = x_i, Y = y_j\}$$

$$q_j = \mathbb{P}(Y = y_j) = \sum_{i=1}^{n} \mathbb{P}(X = x_i, Y = y_j)$$

— o —

$\Omega = [0,1] \qquad \mathcal{E} = \mathcal{B}([0,1]) \qquad \mathbb{P} = \mathcal{L}^1 \big|_{\mathcal{B}([0,1])}$

Fisso $E, F \in \mathcal{E}$

$X = \mathbb{1}_E \qquad Y = \mathbb{1}_F \qquad (X, Y)(\Omega) \subseteq \{0,1\}^2$

$p_0 = \mathbb{P}(X=0) = \mathcal{L}^1(E^c)$ $\qquad q_0 = \mathbb{P}(Y=0) = \mathcal{L}^1(F^c)$

$p_1 = \mathbb{P}(X=1) = \mathcal{L}^1(E)$ $\qquad q_1 = \mathbb{P}(Y=1) = \mathcal{L}^1(F)$

$\mathbb{P}(X=0, Y=0) = \mathcal{L}^1(E^c \cap F^c)$

$\mathbb{P}(X=0, Y=1) = \mathcal{L}^1(E^c \cap F)$

$\mathbb{P}(X=1, Y=0) = \mathcal{L}^1(E \cap F^c)$

$\mathbb{P}(X=1, Y=1) = \mathcal{L}^1(E \cap F)$

$E_1 = [0, \frac{1}{2})$ $\qquad p_0 = p_1 = \frac{1}{2}$

$F_1 = [0, \frac{1}{3}]$ $\qquad q_0 = \frac{2}{3} \quad q_1 = \frac{1}{3}$

$\mathbb{P}(X=0, Y=0) = \mathcal{L}^1((\frac{1}{2}, 1]) = \frac{1}{2}$

$\mathbb{P}(X=0, Y=1) = \mathcal{L}^1(\emptyset) = 0$

$\mathbb{P}(X=1, Y=0) = \mathcal{L}^1((\frac{1}{3}, \frac{1}{2})) = \frac{1}{6}$

$\mathbb{P}(X=1, Y=1) = \mathcal{L}^1([0, \frac{1}{3}]) = \frac{1}{3}$

$E_2 = [0, \frac{1}{2})$ $\qquad p_0 = p_1 = \frac{1}{2}$

$F_2 = [\frac{2}{3}, 1]$ $\qquad q_0 = \frac{2}{3} \quad q_1 = \frac{1}{3}$

$\mathbb{P}(X=0, Y=0) = \mathcal{L}^1(E^c \cap F^c) = \mathcal{L}^1((\frac{1}{2}, \frac{2}{3})) = \frac{1}{6} \neq \frac{1}{2}$

— o —

V.A. VETTORIALE

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio vettoriale.

Siano $X_1, X_2, \dots X_N : \Omega \to \mathbb{R}$ funzioni e sia

$X = (X_1, X_2, \dots, X_N) : \Omega \to \mathbb{R}^N$.

Se $\forall A \in \mathcal{B}(\mathbb{R}^N)$ $\{\omega \in \Omega : (X_1(\omega) \dots X_N(\omega)) \in A\}$

è un evento di $\mathcal{E}$, dico che X è una

V.A. VETTORIALE o una V.A. MULTIVARIATA.

PROP. Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $X = (X_1, X_2, \dots, X_N) : \Omega \to \mathbb{R}^N$ una funzione. Allora X è una v.a. multivariata SSE tutte le funzioni $X_i : \Omega \to \mathbb{R}$ $i = 1, 2, \dots, n$ sono v.a. reali.

DIM X v.a. multivariata.

$t \in \mathbb{R}$ $\quad \{X_1 \leq t\} = \{X_1 \leq t, X_2 \in \mathbb{R}, \dots, X_N \in \mathbb{R}\}$
$= \{X \in A\}$

$A := (-\infty, t] \times \mathbb{R}^{N-1} \in \mathcal{B}(\mathbb{R}^N) \Rightarrow \{X_1 \leq t\} \in \mathcal{E}$

$\{X_k \leq t\} = \{X_1 \in \mathbb{R}, \dots X_{k-1} \in \mathbb{R}, X_k \leq t, X_{k+1} \in \mathbb{R} \dots X_N \in \mathbb{R}\}$
$= \{X \in A\}$

$A := \mathbb{R}^{k-1} \times (-\infty, t] \times \mathbb{R}^{N-k} \in \mathcal{B}(\mathbb{R}^N)$

2) $a_1 \dots a_N, b_1 \dots b_N \in \mathbb{R}$ $\quad a_i < b_i \quad \forall i = 1 \dots N$

$$\left\{X \in \prod_{i=1}^{N} (a_i, b_i]\right\} = \bigcap_{i=1}^{N} \underbrace{\{X_i \in (a_i, b_i]\}}_{\in \mathcal{E} \ \forall i = 1 \dots N} \in \mathcal{E}$$

A aperto di $\mathbb{R}^N$ $\quad A = \bigcup_{i=1}^{\infty} R_i$ $\quad R_i$ rettangoli di $\mathbb{R}^N$ disgiunti 2 a 2

$$\{X \in A\} = \left\{X \in \bigcup_{i=1}^{\infty} R_i\right\} = \bigcup_{i=1}^{\infty} \underbrace{\{X \in R_i\}}_{\in \mathcal{E}} \in \mathcal{E}$$

In analogia al caso reale si può dimostrare che

$\mathcal{F} = \{A \subseteq \mathbb{R}^N : X^{-1}(A) \in \mathcal{E}\}$ è una σ-algebra di $\mathbb{R}^N$

$\mathcal{F}$ contiene tutti gli aperti $\Rightarrow \mathcal{F} \supseteq \mathcal{B}(\mathbb{R}^N)$

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato
e sia $X = (X_1 \ldots X_N) : \Omega \to \mathbb{R}^N$ v.a. multivariata.
Per ogni $A \in \mathcal{B}(\mathbb{R}^N)$ $\quad X^{-1}(A) = \{X \in A\} \in \mathcal{E}$
$\Rightarrow$ è ben definita $\mathbb{P}(X \in A)$.
La funzione

$$\mathbb{P}_X : A \in \mathcal{B}(\mathbb{R}^N) \longmapsto \mathbb{P}(X \in A) \in \mathbb{R}$$

si dice
DISTRIBUZIONE DELLA V.A MULTIVARIATA X
DISTRIBUZIONE CONGIUNTA di $X_1 \ldots X_N$.

La funzione $F_X : t = (t_1 \ldots t_N) \in \mathbb{R}^N \longmapsto$

$$\mathbb{P}(X_1 \le t_1, \ldots, X_N \le t_N) \in \mathbb{R}$$

si dice LEGGE CONGIUNTA di $X_1 \ldots X_N$
LEGGE DELLA V.A MULTIVARIATA X

$\mathbb{P}_{X_1}, \mathbb{P}_{X_2} \ldots \mathbb{P}_{X_N}$ si dicono DISTRIBUZIONI MARGINALI di X

$$\mathbb{P}(X_1 \in A) = \mathbb{P}(X_1 \in A, X_2 \in \mathbb{R}, \ldots X_N \in \mathbb{R}) = \mathbb{P}_X(A \times \mathbb{R}^{N-1}) \qquad A \in \mathcal{B}(\mathbb{R})$$

$$\mathbb{P}_{X_1}(A) = \mathbb{P}_X(A \times \mathbb{R}^{N-1})$$

$$\mathbb{P}(X_k \in A) = \mathbb{P}(X_1 \in \mathbb{R}, \ldots, X_{k-1} \in \mathbb{R}, X_k \in A, X_{k+1} \in \mathbb{R} \ldots \ldots X_N \in \mathbb{R})$$

$$= \mathbb{P}(X \in \mathbb{R}^{k-1} \times A \times \mathbb{R}^{N-k})$$

$$\mathbb{P}_{X_k}(A) = \mathbb{P}_X(\mathbb{R}^{k-1} \times A \times \mathbb{R}^{N-k})$$

TEO Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato.
Sia $X = (X_1 \dots X_N) : \Omega \to \mathbb{R}^N$ v.a. multivariata e
sia $\varphi : \mathbb{R}^N \to \mathbb{R}$ funzione di Borel non negativa
Allora $\varphi \circ X : \Omega \to \mathbb{R}$ è una v.a non negativa

$$\int_\Omega (\varphi \circ X)(\omega) \mathbb{P}(d\omega) = \int_\Omega \varphi(X_1(\omega) \dots X_N(\omega)) \mathbb{P}(d\omega) =$$

$$= \int_{\mathbb{R}^N} \varphi(t_1 \dots t_N) \mathbb{P}_X(dt_1 \dots dt_N)$$

Dim Sia $A \in \mathcal{B}(\mathbb{R})$ $\{\varphi \circ X \in A\} = \{X \in \underbrace{\varphi^{-1}(A)}_{\in \mathcal{B}(\mathbb{R}^N)}\} \in \mathcal{E}$

1) Si dimostra per φ funzione semplice non negativa
2) Si considera $\{\varphi_i\}_{i=1}^{\infty}$ $\varphi_i : \mathbb{R}^N \to \mathbb{R}$
semplici non negative
$0 \le \varphi_i(x) \le \varphi_{i+1}(x) \le \varphi(x)$
$\lim_i \varphi_i(x) = \varphi(x) \quad \forall x \in \mathbb{R}^N$

Applico Beppo Levi a $\{\varphi_i\}$ e $\mathbb{P}_X$

$$\int_{\mathbb{R}^N} \varphi_i(t_1 \dots t_N) \mathbb{P}_X(dt_1 \dots dt_N) \to \int_{\mathbb{R}^N} \varphi(t_1 \dots t_N) \mathbb{P}_X(dt_1 \dots dt_N)$$

$\forall i$
$=$

Applico Beppo Levi a $\{\varphi_i \circ X\}$ e alla misura $\mathbb{P}$

$$\int_{\mathbb{R}^N} (\varphi_i \circ X)(\omega) \mathbb{P}(d\omega) \to \int_{\mathbb{R}^N} (\varphi \circ X)(\omega) \mathbb{P}(d\omega)$$