# DISTRIBUZIONI DISCRETE 2

Note Title 09/11/2016

## DISTRIBUZIONE GEOMETRICA DI PARAMETRO $p \in (0,1)$

È la distribuzione $G(p)$ concentrata sugli interi positivi t.c.

$$G(p)(\{k\}) = p(1-p)^{k-1} \qquad k = 1, 2, 3, \ldots$$

$$1 \overset{?}{=} \sum_{k=1}^{\infty} G(p)(\{k\}) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = \qquad j := k-1$$

$$= \sum_{j=0}^{\infty} p(1-p)^{j} = p \sum_{j=0}^{\infty} (1-p)^{j} \qquad p \in (0,1) \Rightarrow (1-p) \in (0,1) \subseteq (-1,1)$$

$$= p \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

Sia $X$ la v.c. $\mathbb{P}_X = G(p)$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\, \mathbb{P}(X=k) = \sum_{k=1}^{\infty} k\, p(1-p)^{k-1} \qquad x := 1-p$$

$$= p \sum_{k=1}^{\infty} k x^{k-1} = p \sum_{k=1}^{\infty} \frac{d}{dx} x^{k} = p \frac{d}{dx} \sum_{k=1}^{\infty} x^{k} =$$

$$= p \frac{d}{dx}\left(\frac{1}{1-x} - 1\right) = p(1-x)^{-2} = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

$$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} k^2\, \mathbb{P}(X=k) = \sum_{k=1}^{\infty} k(k-1+1)\, p(1-p)^{k-1} =$$

$$= \sum_{k=1}^{\infty} k(k-1)\, p(1-p)^{k-1} + \underbrace{\sum_{k=1}^{\infty} k\, p(1-p)^{k-1}}_{\mathbb{E}[X] = \frac{1}{p}} =$$

$$= p(1-p) \sum_{k=1}^{\infty} k(k-1) x^{k-2} + \frac{1}{p} \qquad x := 1-p$$

$$= p(1-p) \sum_{k=1}^{\infty} \frac{d^2}{dx^2} x^{k} + \frac{1}{p}$$

$$= p(1-p)\frac{d^2}{dx^2}\left[(1-x)^{-1}-1\right]+\frac{1}{p} =$$

$$= p(1-p)\, 2\,(1-x)^{-3}\Big|_{x=1-p} + \frac{1}{p} =$$

$$= \frac{2p(1-p)}{p^3} + \frac{1}{p} = \frac{2(1-p)}{p^2} + \frac{1}{p}$$

$$\operatorname{Var}[X] = \frac{2(1-p)}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2-2p+p-1}{p^2} = \frac{1-p}{p^2}$$

$$F_X(k) = \mathbb{P}(X \le k) \qquad k = 1, 2, \ldots$$

$$= \sum_{j=1}^{k} \mathbb{P}(X=j) = \sum_{j=1}^{k} p(1-p)^{j-1} = \sum_{s=0}^{k-1} p(1-p)^s$$

$$= p\,\frac{1-(1-p)^k}{1-(1-p)} = 1-(1-p)^k$$

$$F_X(t) = \begin{cases} 0 & t < 1 \\ 1-(1-p)^k & t \in [k, k+1) \quad k = 1, 2, \ldots \end{cases}$$

$$\mathbb{P}(X \le i+j-1 \mid X \ge j) = \frac{\mathbb{P}(X \le i+j-1,\ X \ge j)}{\mathbb{P}(X \ge j)} = (*)$$

$$\{X \le i+j-1,\ X \ge j\} = \{X \le i+j-1\} \setminus \{X \le j-1\}$$

$$\{X \ge j\} = \Omega \setminus \{X \le j-1\}$$

$$(*) = \frac{\mathbb{P}(X \le i+j-1) - \mathbb{P}(X \le j-1)}{1-\mathbb{P}(X \le j-1)} = \frac{F_X(i+j-1) - F_X(j-1)}{1-F_X(j-1)}$$

$$= \frac{\left(\cancel{1} - (1-p)^{i+j-1}\right) - \left(\cancel{1} - (1-p)^{j-1}\right)}{\cancel{1} - \left(\cancel{1} - (1-p)^{j-1}\right)} =$$

$$= 1 - (1-p)^i$$

$$= F_X(i) = \mathbb{P}(X \le i)$$

$\Rightarrow$ Se $\mathbb{P}_X = \mathcal{G}(p)$ $\quad \mathbb{P}(X \le i+j-1 \mid X \ge j) = \mathbb{P}(X \le i)$
$\forall i,j = 1,2,\_\_$

**PROPOSIZIONE** Sia $X$ una v.a. distribuita sugli interi positivi t.c. $\mathbb{P}(X=1) = p \in (0,1)$
Allora $\mathbb{P}_X = \mathcal{G}(p)$ SSE $\mathbb{P}(X \le i+j-1 \mid X \ge j) = \mathbb{P}(X \le i)$
$\forall i,j = 1,2,\_\_$

**DIM** Supponiamo $\mathbb{P}(X \le i+j-1 \mid X \ge j) = \mathbb{P}(X \le i)$ $\forall i,j = 1,2,\_\_$
Per $i = 1$ ottengo

$$\mathbb{P}(X \le j \mid X \ge j) = p$$

$$\frac{\mathbb{P}(X \le j, X \ge j)}{\mathbb{P}(X \ge j)} = p$$

$$\mathbb{P}(X=j) = p\,\mathbb{P}(X \ge j)$$
$$\mathbb{P}(X=j+1) = p\,\mathbb{P}(X \ge j+1)$$
$$\mathbb{P}(X=j) - \mathbb{P}(X=j+1) = p\,\underbrace{\{\mathbb{P}(X \ge j) - \mathbb{P}(X \ge j+1)\}}_{=\mathbb{P}(\{X \ge j\} \setminus \{X \ge j+1\}) = \mathbb{P}(X=j)}$$

$$\mathbb{P}(X=j) - \mathbb{P}(X=j+1) = p\,\mathbb{P}(X=j)$$

$$\mathbb{P}(X=j+1) = (1-p)\,\mathbb{P}(X=j) \qquad \forall j = 1,2,\_\_$$

$j=1$ $\quad \mathbb{P}(X=2) = (1-p)\,\mathbb{P}(X=1) = p(1-p)$
$j=2$ $\quad \mathbb{P}(X=3) = (1-p)\,p(1-p) = p(1-p)^2$
Per induzione si dimostra che $\mathbb{P}(X=j) = p(1-p)^{j-1}$

— o —

$\Omega = \{0,1\}^\infty$ = insieme delle successioni a valori in $\{0,1\}$
$\phi$
$\forall n \in \mathbb{N}$ $\quad \forall A \subseteq \{0,1\}^n$ $\quad E_{n,A} := \{\omega \in \{0,1\}^\infty : (\omega_1, \_\_ \omega_n) \in A\}$
CILINDRO su $A$

$E_{n,A}$ , $E_{m,B}$
$k := \max\{n,m\}$ $\qquad \tilde{A} = A \times \{0,1\}^{k-n}$ $\quad E_{n,A} = E_{k,\tilde{A}}$
$\tilde{B} = B \times \{0,1\}^{k-m}$ $\quad E_{m,B} = E_{k,\tilde{B}}$

$\Rightarrow$ Slg posso supporre $n = m$

$E_{n,A} \cup E_{n,B} = E_{n,A\cup B}$ $\Rightarrow$ ho un anello $\mathcal{Q}$

$E_{n,A} \cap E_{n,B} = E_{n,A\cap B}$ su $\Omega$

$E_{n,A} \setminus E_{n,B} = E_{n,A\setminus B}$

Pongo $\mathcal{E} := \sigma(\mathcal{Q})$

Per $E_{n,A} \in \mathcal{Q}$ devo definire $\mathbb{P}(E_{n,A})$

$A \subset \{0,1\}^n$ $(\{0,1\}^n, \mathcal{P}(\Omega), \mathbb{P}_n)$

$\mathbb{P}_n(\{(\omega_1 \_\_ \omega_n)\}) = p^k (1-p)^{n-k}$ $k = \#$ di "1" che ci sono in $\omega_1 \_\_ \omega_n$

Pongo $\mathbb{P}(E_{n,A}) := \mathbb{P}_n(A)$

$E_{n,A}, E_{n,B} \in \mathcal{Q}$ $E_{n,A} \cap E_{n,B} = \emptyset$ $\Rightarrow$ $A \cap B = \emptyset$

$\mathbb{P}(E_{n,A} \cup E_{n,B}) = \mathbb{P}(E_{n,A\cup B}) = \mathbb{P}_n(A \cup B) = \mathbb{P}_n(A) + \mathbb{P}_n(B)$

$= \mathbb{P}(E_{n,A}) + \mathbb{P}(E_{n,B})$

$X : \omega \in \{0,1\}^\infty \longrightarrow \overline{\mathbb{R}}$

$$X(\omega) := \begin{cases} \min\{i : \omega_i = 1\} & \text{se } \exists i \text{ t.c. } \omega_i = 1 \\ +\infty & \text{se } \omega_i = 0 \ \forall i \end{cases}$$

$X(\Omega) = \mathbb{N} \cup \{+\infty\}$

$\hookrightarrow$ interi positivi

$\{X = +\infty\}$

$\{X = k\}$ $k = 1, 2 \_\_\_$

$\{X = k\} = \{\omega \in \{0,1\}^\infty : \omega_1 = \dots = \omega_{k-1} = 0, \omega_k = 1\}$

$= E_{k,A}$ $A \subseteq \{0,1\}^k$ $A = \{(0, \_\_ 0, 1)\}$

$\{X = +\infty\} = \{(0, \_\_\_ 0 \_\_\_)\}$

$= \bigcap_{k=1} E_{k,B_k}$ $B_k = \{\underbrace{(0, \_\_ 0)}_{k\text{-volte}}\}$

$\mathbb{P}(E_{k,B_k}) = \mathbb{P}_k(B_k) = (1-p)^k$

$\mathbb{P}(X = +\infty) = \lim_{k\to\infty} \mathbb{P}_k(B_k) = \lim_{k\to\infty} (1-p)^k = 0$

$$\mathbb{P}(X=k) = \mathbb{P}(E_{k,A}) = \mathbb{P}_k(A) = (1-p)^{k-1}p$$

$$\Rightarrow \mathbb{P}_X = G(p)$$

—— o ——

$X$ = # di prova a cui ottengo il 1° successo
$Y$ = # di insuccessi che ottengo prima del 1° successo

$$\Rightarrow Y = X-1$$

$$E[Y] = E[X-1] = E[X] - E[1] = E[X] - 1 =$$
$$= \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$Var[Y] = Var[X-1] = Var[X] = \frac{1-p}{p^2}$$

$Y(\Omega) = \mathbb{N}_0 \cup \{+\infty\}$ $\qquad \mathbb{P}(Y=+\infty) = \mathbb{P}(X=+\infty) = 0$

$k \in \mathbb{N}_0$

$$\mathbb{P}(Y=k) = \mathbb{P}(X-1=k) = \mathbb{P}(X=k+1) = p(1-p)^k \quad k=0,1,\ldots$$

Chiamo DISTRIBUZIONE GEOMETRICA MODIFICATA $G'(p)$, la distribuzione concentrata sugli interi non negativi t.c.

$$G'(p)(\{k\}) = (1-p)^k \quad \forall k=0,1,\ldots$$

—— o ——

DISTRIBUZIONE IPERGEOMETRICA DI PARAMETRI $n,b,r \in \mathbb{N}$

E' la distribuzione $H(b,r,n)$ concentrata su $\{0,1,\ldots,n\}$ t.c.

$$H(b,r,n)(\{k\}) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}}$$

E' vero che $\sum_{k=0}^{n} H(b,r,n)(\{k\}) = 1$ ?

Devo verificare che $\sum_{k=0}^{n} \binom{b}{k}\binom{r}{n-k} = \binom{b+r}{n}$

$$(1+z)^b = \sum_{k=0}^{b} \binom{b}{k} z^k = \sum_{k=0}^{\infty} \binom{b}{k} z^k$$

$$(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} z^k$$

$$(1+z)^b (1+z)^r = \left(\sum_{k=0}^{\infty} \overset{a_k}{\binom{b}{k} z^k}\right)\left(\sum_{k=0}^{\infty} \overset{b_k}{\binom{r}{k} z^k}\right) =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{b}{k}\binom{r}{n-k}\right) z^n$$

$$(1+z)^{b+r} = \sum_{n=0}^{\infty} \binom{b+r}{n} z^n$$

$$\Rightarrow \binom{b+r}{n} = \sum_{k=0}^{n} \binom{b}{k}\binom{r}{n-k}$$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}} = \frac{1}{\binom{b+r}{n}} \sum_{k=0}^{n} k \binom{b}{k}\binom{r}{n-k}$$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k \binom{b}{k}\binom{r}{n-k}\right) z^n = \left(\sum_{k=0}^{\infty} k \binom{b}{k} z^k\right)\left(\sum_{k=0}^{\infty} \binom{r}{k} z^k\right) =$$

$$= z \left(\sum_{k=1}^{b} \binom{b}{k} k z^{k-1}\right)\left(\sum_{k=0}^{r} \binom{r}{k} z^k\right) =$$

$$= z \frac{d}{dz} \underbrace{\left(\sum_{k=1}^{b} \binom{b}{k} z^k\right)}_{(1+z)^b} (1+z)^r = z b (1+z)^{b-1+r}$$

$$= b z \sum_{j=0}^{b+r-1} \binom{b+r-1}{j} z^j \qquad n := j+1$$

$$= \sum_{n=1}^{b+r} b \binom{b+r-1}{n-1} z^n$$

$$\sum_{k=0}^{n} k \binom{b}{k}\binom{r}{n-k} = b\binom{b+r-1}{n-1}$$

$$\mathbb{E}[X] = \frac{b\binom{b+r-1}{n-1}}{\binom{b+r}{n}} = \frac{b\,(b+r-1)!}{(n-1)!\,\cancel{(b+r-n)!}} \cdot \frac{n!\,\cancel{(b+r-n)!}}{(b+r)!}$$

$$= b\,\frac{n}{b+r} = \frac{nb}{b+r} = np \qquad p := \frac{b}{b+r}$$

$$\mathrm{Var}[X] = np(1-p)\,\frac{1-\frac{n}{b+r}}{1-\frac{1}{b+r}}$$

— o —

FOGLIO 3 EX 1

$$Y = X^2 = \varphi \circ X \qquad \varphi(t) = t^2$$

$$\{Y \le t\} = \{X^2 \le t\} = \begin{cases} \emptyset & t<0 \\ \{X=0\} & t=0 \\ \{-\sqrt{t} \le X \le \sqrt{t}\} & t>0 \end{cases}$$

$$F_Y(t) = 0 \qquad t<0$$

$$t>0 \qquad F_Y(t) = \mathbb{P}(-\sqrt{t} \le X \le \sqrt{t})$$

$$\{-\sqrt{t} \le X \le \sqrt{t}\} = \{X \le \sqrt{t}\} \setminus \{X < -\sqrt{t}\}$$

$$F_Y(t) = \mathbb{P}(X \le \sqrt{t}) - \mathbb{P}(X < -\sqrt{t}) =$$

$$= F_X(\sqrt{t}) - \mathbb{P}(\{X \le -\sqrt{t}\} \setminus \{X = -\sqrt{t}\}) =$$

$$= F_X(\sqrt{t}) - \mathbb{P}(X \le -\sqrt{t}) + \mathbb{P}(X = -\sqrt{t})$$

$$= F_X(\sqrt{t}) - F_X(-\sqrt{t}) + \mathbb{P}(X = -\sqrt{t})$$

— o —

$X$ v.a. $\qquad X : \Omega \to \mathbb{R} \qquad \varphi : \mathbb{R} \to \mathbb{R}$ di Borel

$\psi : \mathbb{R} \to \mathbb{R}$ di Borel nonnegativa

$$\Rightarrow \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi \circ X}(dt) = \int_{\mathbb{R}} (\psi \circ \varphi)(s)\, \mathbb{P}_X(ds)$$

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} \psi(s^2)\, \mathbb{P}_X(ds) = \int_{\mathbb{R}} \psi(s^2) f(s)\, ds = (*)$$

se scrivo $\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} \psi(t) g(t)\, dt$

$A \in \mathcal{B}(\mathbb{R})$ $\quad \psi(t) = \mathbb{1}_A(t) \quad \Rightarrow \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Y(dt) = \mathbb{P}_Y(A)$

$\Rightarrow$ otterrei

$$\mathbb{P}_Y(A) = \int_{\mathbb{R}} \mathbb{1}_A(t) g(t)\, dt = \int_A g(t)\, dt$$

$$= \int_0^{+\infty} \psi(s^2) f(s)\, ds + \int_{-\infty}^{0} \psi(s^2) f(s)\, ds$$

$t = s^2$ $\qquad\qquad\qquad t = s^2$

$s = \sqrt{t}$ $\qquad\qquad\qquad s = -\sqrt{t}$

$ds = \frac{1}{2\sqrt{t}}\, dt$ $\qquad\qquad ds = \frac{-1}{2\sqrt{t}}\, dt$

$$= \int_0^{+\infty} \psi(t) f(\sqrt{t}) \frac{1}{2\sqrt{t}}\, dt + \int_{+\infty}^{0} \psi(t) f(-\sqrt{t}) \frac{-1}{2\sqrt{t}}\, dt$$

$$= \int_0^{+\infty} \psi(t) \frac{f(\sqrt{t})}{2\sqrt{t}}\, dt + \int_0^{+\infty} \psi(t) \frac{f(-\sqrt{t})}{2\sqrt{t}}\, dt =$$

$$= \int_0^{+\infty} \psi(t) \frac{1}{2\sqrt{t}} \left( f(\sqrt{t}) + f(-\sqrt{t}) \right) dt \overset{?}{=} \int_{\mathbb{R}} \psi(t) g(t)\, dt$$

$$g(t) = \begin{cases} \frac{1}{2\sqrt{t}} \left( f(\sqrt{t}) + f(-\sqrt{t}) \right), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

---

Foglio 3 - Ex 2

$X$ v.a con $\quad \mathbb{P}_X = f(x)\, dx \qquad a, b \in \mathbb{R} \quad a \neq 0$

$Y := aX + b$

$Y = \varphi \circ X \qquad \varphi(t) = at + b \qquad \psi$ di Borel non negativa $\Rightarrow$

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} (\psi \circ \varphi)(s)\, \mathbb{P}_X(ds) =$$

$$= \int_{\mathbb{R}} \psi(as + b) f(s)\, ds$$

$$t = as + b \qquad s = \frac{t-b}{a} \qquad ds = \frac{1}{a}\,dt$$

$a > 0$

$$\int_{-\infty}^{+\infty} \varphi(t)\, f\left(\frac{t-b}{a}\right) \frac{1}{a}\, dt$$

$$\mathbb{P}_Y = g(t)\,dt \qquad g(t) = \frac{1}{a} f\left(\frac{t-b}{a}\right)$$

$a < 0$

$$\int_{+\infty}^{-\infty} \varphi(t)\, f\left(\frac{t-b}{a}\right) \frac{1}{a}\, dt$$

$$= \int_{-\infty}^{+\infty} \varphi(t)\, f\left(\frac{t-b}{a}\right) \frac{1}{-a}\, dt$$

$$g(t) = f\left(\frac{t-b}{a}\right) \frac{1}{-a}$$

$$\Rightarrow \mathbb{P}_Y = g(t)\,dt \qquad g(t) = \frac{1}{|a|} f\left(\frac{t-b}{a}\right)$$

FOGLIO 3 - EX 5

$T = \#$ Teste $\qquad \mathbb{P}_T = B(n,p)$

$C = \#$ croci $\qquad \mathbb{P}_C = B(n, 1-p)$

$$X = T - (n - T) = 2T - n \qquad X + n = 2T$$

$$k \in \{-n, -n+2, \dots, n-2, n\}$$

$$\mathbb{P}(X = k) = \mathbb{P}\left(T = \frac{k+n}{2}\right) = B(n,p)\left(\left\{\frac{k+n}{2}\right\}\right)$$