VARIANZA, MARKOV, CHEBYCHEV, DISTR. DISCRETE

Note Title 07/11/2016

PROP Sia X funzione misurabile non negativa sullo spazio di misura $(\Omega, \mathcal{E}, \mathbb{P})$. Allora

$$\int_\Omega X(\omega)\,\mathbb{P}(d\omega) = 0 \iff X = 0 \quad \mathbb{P}\text{-q.c.}$$

(NO DIM)

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$(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $X: \Omega \to \mathbb{R}$ v.a. t.c. $\mathbb{E}[X]$ esiste ed è finito

$Var[X] := \mathbb{E}[(X - \mathbb{E}[X])^2]$ si dice VARIANZA di X

PROPRIETÀ

1) $Var[X] \geq 0$, $Var[X] = 0$ SSE $X = \mathbb{E}[X]$ $\mathbb{P}$-q.c.

2)
$$\begin{aligned} Var[X] &= \mathbb{E}[X^2 - 2\mathbb{E}[X]X + (\mathbb{E}[X])^2] = \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\,\mathbb{E}[X] + \mathbb{E}[(\mathbb{E}[X])^2] = \\ &= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2. \end{aligned}$$

3) $Var[\alpha X + \beta] = \mathbb{E}[((\alpha X + \beta) - \mathbb{E}[\alpha X + \beta])^2]$ $\quad \alpha \in \mathbb{R}$ $\beta \in \mathbb{R}$

$$\begin{aligned} &= \mathbb{E}[(\alpha X + \beta - (\alpha\mathbb{E}[X] + \beta))^2] = \\ &= \mathbb{E}[(\alpha X + \cancel{\beta} - \alpha\mathbb{E}[X] - \cancel{\beta})^2] = \\ &= \mathbb{E}[\alpha^2(X - \mathbb{E}[X])^2] = \alpha^2\,\mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \alpha^2 Var[X] \end{aligned}$$

$\sigma(X) := \sqrt{Var[X]}$ si dice SCARTO QUADRATICO MEDIO di X

$\sigma(\alpha X + \beta) = |\alpha|\,\sigma(X)$.

DISUGUAGLIANZA DI MARKOV

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato.
Sia $X: \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$
Sia I un "intervallo" di $\mathbb{R}$ t.c. $X(\Omega) \subseteq I$ e sia
$\varphi: I \to \mathbb{R}$ - strettamente crescente
- non negativa.

Allora $\quad \varphi(t)\mathbb{P}(X > t) \leq \mathbb{E}[\varphi \circ X] \quad \forall t \in I$

DIM $\quad \varphi$ strettamente crescente $\Rightarrow \{X > t\} = \{\varphi \circ X > \varphi(t)\}$

$$\varphi(t)\,\mathbb{P}(X > t) = \varphi(t)\,\mathbb{P}(\varphi \circ X > \varphi(t)) =$$

$$= \varphi(t) \int_\Omega \mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)\,\mathbb{P}(d\omega) = \int_\Omega \varphi(t)\,\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)\,\mathbb{P}(d\omega) \leq (A)$$

$$\varphi(t)\,\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega) \leq (\varphi \circ X)(\omega)\,\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega) \quad \forall \omega \in \Omega$$

$$(A) \leq \int_\Omega (\varphi \circ X)(\omega)\,\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega)\,\mathbb{P}(d\omega) \leq (B)$$

$$(\varphi \circ X)(\omega)\,\mathbb{1}_{\{\varphi \circ X > \varphi(t)\}}(\omega) \leq (\varphi \circ X)(\omega) \quad \forall \omega \in \Omega$$

$$(B) \leq \int_\Omega (\varphi \circ X)(\omega)\,\mathbb{P}(d\omega) = \mathbb{E}[\varphi \circ X].$$

DISUGUAGLIANZA DI CHEBYCHEV

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $X: \Omega \to \mathbb{R}$ una v.a.
con valore atteso $\mathbb{E}[X]$ finito.
Allora
$$\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq \frac{Var[X]}{t^2} \qquad \forall t > 0$$

DIM Applico Markov e $Y := |X - \mathbb{E}[X]|$

$I := [0, +\infty)$

$\varphi : t \in [0, +\infty) \longmapsto t^2 \in \mathbb{R}$

$$t^2 \mathbb{P}(Y > t) \leq \mathbb{E}[\varphi \circ Y] = \mathbb{E}[Y^2]$$

$$t^2 \mathbb{P}(|X - \mathbb{E}[X]| > t) \leq \mathbb{E}[(X - \mathbb{E}[X])^2] = Var[X] \qquad \boxtimes$$

MEDIANA

Sia X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

$t_M := \inf \{ t \in \mathbb{R} : F_X(t) \geq \frac{1}{2} \}$ si dice

MEDIANA di X

$$F_X(t_M) = \lim_{t \to t_M^+} \underbrace{F_X(t)}_{\geq \frac{1}{2} \ \forall t > t_M} \geq \frac{1}{2}$$

$$F_X(t_M^-) := \lim_{t \to t_M^-} \underbrace{F_X(t)}_{< \frac{1}{2}} \leq \frac{1}{2}$$

$$F_X(t_M^-) \leq \frac{1}{2} \leq F_X(t_M)$$

Caratterizzazione variazionale di valore atteso e mediana

Sia X v.a. con $\mathbb{E}[X]$ e $\mathbb{E}[X^2]$ finiti.

Considero $\varphi_2 : s \in \mathbb{R} \longmapsto \int_\Omega |X(\omega) - s|^2 \mathbb{P}(d\omega) \in \mathbb{R}$

$\bar{s} = \mathbb{E}[X]$ è l'unico pto di minimo di φ_2

e il valore minimo $\varphi_2(\mathbb{E}[X])$ è $Var[X]$.

Sia X v.a con $\mathbb{E}[X]$ finito considero

$$\varphi_1 : s \in \mathbb{R} \longmapsto \int_\Omega |X(\omega) - s| \mathbb{P}(d\omega) \in \mathbb{R}$$

Allora $\bar{s} = t_M$ è un pto di minimo.

DISTRIBUZIONI DISCRETE

DISTRIBUZIONE DI DIRAC

Fisso $x_0 \in \mathbb{R}$ e per $A \in \mathcal{B}(\mathbb{R})$ pongo $\delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{cases}$

Sia X v.a. con $\mathbb{P}_X = \delta_{x_0}$

$\mathbb{E}[X] = ?$ $\qquad \mathbb{P}(X = x_0) = \mathbb{P}_X(\{x_0\}) = \delta_{x_0}(\{x_0\}) = 1$

$X = x_0$ $\mathbb{P}$-q.c. $\Rightarrow$ $\mathbb{E}[X] = x_0$

$\mathrm{Var}[X] = 0$

DISTRIBUZIONE DI BERNOULLI DI PARAMETRO $p \in [0,1]$

La distribuzione $B(p)$ concentrata su $\{0,1\}$ t.c.

$B(p)(\{0\}) = 1-p$

$B(p)(\{1\}) = p$

$A \in \mathcal{B}(\mathbb{R})$ $\qquad B(p)(A) =$

$A \supset \{0,1\}$ $\quad B(p)(A) \geq B(p)(\{0\}) + B(p)(\{1\}) = 1-p+p = 1$

$1 \in A \quad 0 \notin A$

$$B(p)(A) = \begin{cases} 1 & 0,1 \in A \\ p & 1 \in A \quad 0 \notin A \\ 1-p & 0 \in A \quad 1 \notin A \\ 0 & 0,1 \notin A \end{cases}$$

$$B(p) = p\,\delta_1 + (1-p)\,\delta_0$$

Sia X v.a. con $\mathbb{P}_X = B(p)$

$\mathbb{P}(X=1) = \mathbb{P}_X(\{1\}) = B(p)(\{1\}) = p$

$\mathbb{P}(X=0) = \mathbb{P}_X(\{0\}) = B(p)(\{0\}) = 1-p$

$\mathbb{E}[X] = 0 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1) = 0 \cdot (1-p) + 1 \cdot p = p$

$\mathbb{E}[X^2] = 0^2\, \mathbb{P}(X=0) + 1^2\, \mathbb{P}(X=1) = 1 \cdot p = p$

$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1-p)$

$\Omega = \{0,1\}$ $\qquad \mathcal{E} = \mathcal{P}(\Omega)$ $\qquad \mathbb{P}(\{0\}) = 1-p$

$\mathbb{P}(\{1\}) = p$

$\omega \in \Omega \qquad X(\omega) = \omega \qquad X: \{0,1\} \to \mathbb{R}$

$\mathbb{P}(X=0) \qquad \{X=0\} = \{0\} \qquad \mathbb{P}(X=0) = 1-p$

$\mathbb{P}(X=1) \qquad \{X=1\} = \{1\} \qquad \mathbb{P}(X=1) = p \qquad \Rightarrow \mathbb{P}_X = B(p)$

DISTRIBUZIONE BINOMIALE DI PARAMETRI n E p

n intero positivo, $p \in [0,1]$

E' la distribuzione $B(n,p)$ concentrata su $\{0,1,2,\dots,n-1,n\}$

T.c.

$$B(n,p)(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \forall k=0,\dots,n$$

$$1 \overset{?}{=} \sum_{k=0}^{n} B(n,p)(\{k\}) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+(1-p))^n = 1$$

Sia X v.a. t.c. $\mathbb{P}_X = B(n,p)$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k\,\mathbb{P}(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= (1-p)^n \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{p}{1-p}\right)^{k-1+1} =$$

$$= (1-p)^n \frac{p}{1-p} \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} = \qquad x := \frac{p}{1-p}$$

$$= (1-p)^{n-1} p \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx} x^k =$$

$$= p(1-p)^{n-1} \frac{d}{dx} \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k}_{\sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} = (1+x)^n} = p(1-p)^{n-1} \frac{d}{dx} (1+x)^n =$$

$$= p(1-p)^{n-1} n (1+x)^{n-1} \Big|_{x=\frac{p}{1-p}} \qquad 1+x = 1+\frac{p}{1-p} = \frac{1}{1-p}$$

$$= p\,\cancel{(1-p)^{n-1}}\, n\, \cancel{(1-p)^{-(n-1)}} = np \qquad \Rightarrow \mathbb{E}[X] = np$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2\, \mathbb{P}(X=k) = \sum_{k=0}^{n} k(k-1+1)\binom{n}{k} p^k (1-p)^{n-k} =$$

$$= \sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k (1-p)^{n-k} + \underbrace{\sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k}}_{\mathbb{E}[X] = np}$$

$$= (1-p)^n \sum_{k=0}^{n} k(k-1)\binom{n}{k}\left(\frac{p}{1-p}\right)^{k-2+2} + np$$

$$= (1-p)^n\, p^2 (1-p)^{-2} \sum_{k=0}^{n} \binom{n}{k} \underbrace{k(k-1)\, x^{k-2}}_{\frac{d^2}{dx^2} x^k} + np, \quad x := \frac{p}{1-p}$$

$$= (1-p)^{n-2}\, p^2 \frac{d^2}{dx^2} \sum_{k=0}^{n} \binom{n}{k} x^k + np$$

$$= (1-p)^{n-2}\, p^2 \frac{d^2}{dx^2} (1+x)^n + np =$$

$$= (1-p)^{n-2}\, p^2\, n(n-1)(1+x)^{n-2}\Big|_{x = \frac{p}{1-p}} + np$$

$$= \cancel{(1-p)^{n-2}}\, p^2\, n(n-1)\, \cancel{(1-p)^{-(n-2)}} + np = np\big((n-1)p + 1\big)$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = np(\cancel{np} - p + 1) - \cancel{n^2p^2} =$$

$$= np(1-p)$$

$\Omega = \{0,1\}^n \qquad \mathcal{E} = \mathcal{P}(\Omega)$

$\omega = (\omega_1 \_\_ \omega_n) \in \Omega \qquad \mathbb{P}(\{\omega\}) = p^k (1-p)^{n-k}$ dove

k = # di componenti di ω che valgono 1

$n-k$ = # di componenti di ω che valgono 0

$X(\omega) = \sum_{i=1}^{n} \omega_i \qquad X : \Omega \to \mathbb{R}$

$X(\Omega) = \{0, 1, \_\_ n\}$

Fissato $k \in \{0, 1, \_\_ n\}$ $\quad \mathbb{P}(X=k)$

$\{X=k\} = \{\omega \in \Omega : \omega$ ha k componenti uguali ad 1
e $n-k$ componenti uguali a $0\}$

$\Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} = B(n,p)(\{k\})$

$B(n, p_n)$ $\qquad p_n \in [0,1]$ $\qquad n \in \mathbb{N}$

$\exists \lim_{n\to\infty} np_n = \lambda \in (0,+\infty)$ $\qquad (\Rightarrow p_n \to 0)$

Fissato $k \in \mathbb{N}$ mi chiedo se $\exists \lim_{n\to\infty} B(n,p_n)(\{k\})$

$k > n$ $\qquad B(n,p_n)(\{k\}) = 0$

$n \geq k$ $\qquad B(n,p_n)(\{k\}) = \binom{n}{k} p_n^k (1-p_n)^{n-k} =$

$$= \frac{n!}{k!(n-k)!} p_n^k (1-p_n)^{n-k} =$$

$$= \frac{1}{k!} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k} n^k p_n^k (1-p_n)^{n-k} =$$

$$= \frac{1}{k!} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k} (np_n)^k \underbrace{\left[(1-p_n)^{\frac{1}{p_n}}\right]}_{\to e^{-1}}^{\underbrace{p_n(n-k)}_{\to\lambda}}$$

$$\to \frac{1}{k!} \lambda^k e^{-\lambda}$$

$$P(\lambda)(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad \forall k = 0,1,2,\dots$$

definisce una distribuzione concentrata sugli interi non negativi?

$$1 \overset{?}{=} \sum_{k=0}^{\infty} P(\lambda)(\{k\}) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Chiamo $P(\lambda)$ DISTRIBUZIONE di POISSON di PARAMETRO λ

Se X v.a. t.c. $P_X = P(\lambda)$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\,\mathbb{P}(X=k) = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \qquad k-1=j$$

$$= e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!} = e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{-\lambda} \lambda e^{\lambda} = \lambda$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2\,\mathbb{P}(X=k) = \sum_{k=0}^{\infty} k(k-1+1)\, e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=0}^{\infty} k(k-1)\, e^{-\lambda} \frac{\lambda^k}{k!} + \underbrace{\sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!}}_{=\mathbb{E}[X] = \lambda}$$

$$= \sum_{k=2}^{\infty} \frac{e^{-\lambda} \lambda^k}{(k-2)!} + \lambda = \qquad j = k-2$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\lambda}\, \lambda^j \cdot \lambda^2}{j!} + \lambda = e^{-\lambda} \lambda^2 \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \lambda$$

$$= e^{-\lambda} \lambda^2 e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mathrm{Var}[X] = \lambda^2 + \lambda - (\lambda)^2 = \lambda$$

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