Sia $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ $\quad a_n \to L$

Sia $S_n := \sum_{k=1}^{n} a_k$

$\forall \varepsilon>0 \quad \exists n_0 = n_0(\varepsilon)$ t.c. $\forall n>n_0 \qquad L-\varepsilon \le a_n \le L+\varepsilon$

Per $n>n_0 \qquad S_n = S_{n_0} + \sum_{k=n_0+1}^{n} a_k$

$$\frac{S_{n_0}}{n} + \frac{n-n_0}{n}(L-\varepsilon) \le \frac{S_n}{n} = \frac{S_{n_0}}{n} + \frac{1}{n}\sum_{k=n_0+1}^{n} a_k \le \frac{S_{n_0}}{n} + \frac{n-n_0}{n}(L+\varepsilon)$$

$\forall \varepsilon>0 \quad L-\varepsilon \le \liminf_n \frac{S_n}{n} \le \limsup \frac{S_n}{n} \le L+\varepsilon$

$\Rightarrow \quad \exists \lim \frac{S_n}{n} = L$

— o —

Per ogni $x>0 \qquad \sum_{n>x} \frac{1}{n^2} < \frac{2}{x}$

— o —

Sia $\alpha>1 \quad K_n := \lfloor \alpha^n \rfloor \quad$ Allora $\sum_{n:K_n\ge j} \frac{1}{K_n^2} \le \frac{4\alpha^2}{\alpha^2-1}\frac{1}{j^2} \quad \forall j \ge 1$

DIM 1° PASSO $\quad K_n \ge \frac{1}{2}\alpha^n \quad$ ok

2° PASSO $\quad K_n \ge j \qquad \lfloor \alpha^n \rfloor \ge j$

1° CASO $\quad \alpha^n \in \mathbb{N} \qquad \alpha^n = K_n \qquad \alpha^n = j \qquad e^{n\log\alpha} = e^{\log j}$

$n\log\alpha = \log j \qquad \frac{\log j}{\log\alpha} = n \in \mathbb{N} \qquad$ Pongo $\quad n_0 := \frac{\log j}{\log \alpha}$

$\sum_{n:K_n\ge j} \frac{1}{K_n^2} \le \sum_{n\ge n_0} \frac{4}{\alpha^{2n}} \qquad s = n-n_0 \qquad n = s+n_0$

$$= \sum_{s\ge 0} \frac{4}{\alpha^{2s+2n_0}} = \frac{4}{\alpha^{2n_0}} \sum_{s\ge 0} (\alpha^{-2})^s = \frac{4}{\alpha^{2n_0}} \frac{1}{1-\alpha^{-2}} =$$

$$= \frac{4\alpha^2}{\alpha^2-1}\frac{1}{j^2} \qquad \text{perché} \quad \alpha^{2n_0} = j^2$$

2° caso $\alpha^n \notin \mathbb{N}$ $\quad \alpha^n > k_n \quad k_n \geq j \quad \alpha^n > j \quad e^{n\log\alpha} > e^{\log j}$

$n\log\alpha > \log j \qquad n > \frac{\log j}{\log \alpha} \qquad n_0 := \left\lfloor \frac{\log j}{\log\alpha} \right\rfloor \qquad \text{Ho} \quad n \geq n_0+1$

$$\sum_{n: k_n \geq j} \frac{1}{k_n^2} \leq \sum_{n \geq n_0+1} \frac{4}{\alpha^{2n}} \qquad s = n-(n_0+1) \qquad n = s+n_0+1$$

$$= \sum_{s\geq 0} \frac{4}{\alpha^{2s+2(n_0+1)}} = \frac{4}{\alpha^{2(n_0+1)}}\sum_{s\geq 0}\left(\alpha^{-2}\right)^s = \frac{4\alpha^2}{\alpha^2-1}\frac{1}{\alpha^{2(n_0+1)}}$$

$$n_0 = \left\lfloor \frac{\log j}{\log\alpha}\right\rfloor \qquad n_0 \leq \frac{\log j}{\log \alpha} < n_0+1$$

$$n_0 \log\alpha \leq \log j < (n_0+1)\log\alpha = \log \alpha^{n_0+1}$$

$$j^2 < \alpha^{2(n_0+1)} \qquad \frac{1}{\alpha^{2(n_0+1)}} \leq \frac{1}{j^2}$$

— o —

$\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. i.i.d. e sommabili a valori in $\mathbb{R}$

Sia $E := \mathbb{E}[X_i] \in \mathbb{R}$.

Per $n\in\mathbb{N}$ sia $Y_n := X_n \mathbb{1}_{\{|X_n|\leq n\}}$

Pongo $S_n := \sum_{k=1}^n X_k \quad T_n := \sum_{k=1}^n Y_k$

Se $\frac{T_n}{n} \to E$ $\mathbb{P}$-qc, allora anche $\frac{S_n}{n} \to E$ $\mathbb{P}$-qc.

— o —

$\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. identicamente distribuite, sommabili;

Per $n\in\mathbb{N}$ considero $Y_n := X_n \mathbb{1}_{\{|X_n|\leq n\}}$

Allora
$$\sum_{j=1}^{\infty} \frac{\mathbb{E}[Y_j^2]}{j^2} \leq 4E \qquad \text{dove} \quad E := \mathbb{E}[X_i] \in \mathbb{R}.$$

DIM DEL TEO in ETEMADI

1° PASSO $X_n \geq 0$, i.i.d., sommabili $E := \mathbb{E}[X_n] \in \mathbb{R}$

$Y_n := X_n \mathbb{1}_{\{X_n \leq n\}}$

$T_n := \sum_{k=1}^{n} Y_k$ Mi basta dimostrare che $\frac{T_n}{n} \to E$ $\mathbb{P}$ q.c.

Sia $\alpha > 1$ e sia $k_n := \lfloor \alpha^n \rfloor$

Applico Chebychev $\frac{T_{k_n}}{k_n}$

$$\mathbb{P}\left(\left| \frac{T_{k_n}}{k_n} - \mathbb{E}\left[\frac{T_{k_n}}{k_n}\right] \right| > \delta \right) \leq \frac{Var\left[\frac{T_{k_n}}{k_n}\right]}{\delta^2} \qquad \forall \delta > 0$$

$$= \frac{1}{k_n^2 \delta^2} Var[T_{k_n}]$$

$$= \frac{1}{k_n^2 \delta^2} \sum_{j=1}^{k_n} Var[Y_j] \leq \frac{1}{k_n^2 \delta^2} \sum_{j=1}^{k_n} \mathbb{E}[Y_j^2]$$

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \frac{T_{k_n}}{k_n} - \mathbb{E}\left[\frac{T_{k_n}}{k_n}\right] \right| > \delta \right) \leq \sum_{n=1}^{\infty} \frac{1}{k_n^2 \delta^2} \sum_{j=1}^{k_n} \mathbb{E}[Y_j^2]$$

$$1 \leq j \leq k_n < +\infty$$

$$= \sum_{j=1}^{\infty} \sum_{n: k_n \geq j} \frac{1}{k_n^2 \delta^2} \mathbb{E}[Y_j^2] = \frac{1}{\delta^2} \sum_{j=1}^{\infty} \mathbb{E}[Y_j^2] \sum_{n: k_n \geq j} \frac{1}{k_n^2} \quad \leq \frac{4\alpha^2}{\alpha^2 - 1} \frac{1}{j^2}$$

$$\leq \frac{1}{\delta^2} \frac{4\alpha^2}{\alpha^2 - 1} \sum_{j=1}^{\infty} \frac{\mathbb{E}[Y_j^2]}{j^2} \leq \frac{1}{\delta^2} \frac{16\alpha^2}{\alpha^2 - 1} E \qquad \text{dove } E := \mathbb{E}[X_n]$$

$$A_{n,\delta} := \left\{ \left| \frac{T_{k_n}}{k_n} - \mathbb{E}\left[\frac{T_{k_n}}{k_n}\right] \right| > \delta \right\}$$

$\sum_{n=1}^{\infty} \mathbb{P}(A_{n,\delta}) < +\infty \Rightarrow$ per il lemma di Borel-Cantelli.

$$\mathbb{P}(\limsup_{n \to \infty} A_{n,\delta}) = 0$$

cioè

$$\frac{T_{k_n}}{k_n} - \mathbb{E}\left[\frac{T_{k_n}}{k_n}\right] \longrightarrow 0 \quad \mathbb{P}\text{-qc}$$

$$\mathbb{E}[Y_n] = \mathbb{E}[X_n \mathbb{1}_{\{X_n \le n\}}] = \mathbb{E}[X_1 \mathbb{1}_{\{X_1 \le n\}}]$$

$$\varphi_n := X_1 \mathbb{1}_{\{X_1 \le n\}} \qquad 0 \le \varphi_n \le \varphi_{n+1} \qquad \varphi_n(\omega) \to X_1(\omega) \ \forall \omega \in \Omega$$

Per Beppo Levi:

$$\mathbb{E}[Y_n] \to \mathbb{E}[X_1] = E$$

$$\mathbb{E}\left[\frac{T_{k_n}}{k_n}\right] = \frac{1}{k_n} \sum_{j=1}^{k_n} \mathbb{E}[Y_j] \xrightarrow{\text{per Cesàro}} E$$

$$\frac{T_{k_n}}{k_n} - E = \underbrace{\frac{T_{k_n}}{k_n} - \mathbb{E}\left[\frac{T_{k_n}}{k_n}\right]}_{\to 0 \ \mathbb{P}-q.c.} + \underbrace{\mathbb{E}\left[\frac{T_{k_n}}{k_n}\right] - E}_{\to 0 \text{ per Cesàro}} \to 0 \quad \mathbb{P}-q.c.$$

Cioè $\frac{T_{k_n}}{k_n} \to E \quad \mathbb{P}-q.c.$

Per ogni $j \in \mathbb{N}$ $\quad \exists!\, n = n(j)$ $\quad \underbrace{k_n < j \le k_{n+1}}$

e $j \to \infty$ sse $k_n \to \infty$

$$\underbrace{\frac{T_{k_n}}{k_n}}_{\downarrow \ E \ \mathbb{P} q.c.} \frac{k_n}{j} = \frac{T_{k_n}}{j} \le \frac{T_j}{j} \le \frac{T_{k_{n+1}}}{j} = \underbrace{\frac{T_{k_{n+1}}}{k_{n+1}}}_{\downarrow \ E \ \mathbb{P}-q.c.} \frac{k_{n+1}}{j}$$

$$\alpha^n - 1 < k_n \le \alpha^n < k_n + 1 \le j \le k_{n+1} \le \alpha^{n+1}$$

$$\frac{k_n}{j} \ge \frac{k_n}{\alpha^{n+1}} \ge \frac{\alpha^n - 1}{\alpha^{n+1}} \qquad \frac{k_{n+1}}{j} \le \frac{\alpha^{n+1}}{\alpha^n} = \alpha$$

$$\frac{\alpha^n - 1}{\alpha^{n+1}} \frac{T_{k_n}}{k_n} \le \frac{T_j}{j} \le \frac{T_{k_{n+1}}}{k_{n+1}} \alpha$$

quando $j \to \infty$

$$\frac{E}{\alpha} \le \liminf \frac{T_j}{j} \le \limsup \frac{T_j}{j} \le \alpha E \qquad \mathbb{P}-q.c.$$

$\forall \alpha > 1 \quad \exists N_\alpha \in \mathcal{E} \quad \mathbb{P}(N_\alpha) = 0$ t.c.

$\forall \omega \in \Omega \setminus N_\alpha \qquad \frac{E}{\alpha} \le \liminf_j \frac{T_j(\omega)}{j} \le \limsup_j \frac{T_j(\omega)}{j} \le \alpha E$

$\alpha_k = 1 + \frac{1}{k}$

$\forall \omega \in \Omega \setminus N_{\alpha_k} \qquad \frac{E}{\alpha_k} \le \liminf_j \frac{T_j(\omega)}{j} \le \limsup_j \frac{T_j(\omega)}{j} \le \alpha_k E$

Pongo $N_0 = \bigcup_{k=1}^{\infty} N_{\alpha_k} \qquad \mathbb{P}(N_0) = 0$

Se $\omega \in \Omega \setminus N_0$

$$\frac{E}{\alpha_k} \le \liminf_j \frac{T_j(\omega)}{j} \le \limsup_j \frac{T_j(\omega)}{j} \le E\alpha_k \qquad \forall k \in \mathbb{N}$$

Passando a limite per $k \to +\infty$ ottengo

$$\exists \lim_j \frac{T_j(\omega)}{j} = E \qquad \forall \omega \in \Omega \setminus N_0$$

con $\mathbb{P}(N_0) = 0$

— o —

CASO GENERALE: Rimuovo l'ipotesi $X_n \geq 0$

$X_n = X_n^+ - X_n^-$

$\{X_n^+\}$ è una successione di v.a. i.i.d., sono sommabili

idem per $\{X_n^-\}$

$S_n^+ := \sum_{k=1}^{n} X_k^+ \qquad$ so che $\frac{S_n^+}{n} \to \mathbb{E}[X_1^+]$ $\mathbb{P}$-q.c. $\star$

$S_n^- := \sum_{k=1}^{n} X_k^- \qquad$ so che $\frac{S_n^-}{n} \to \mathbb{E}[X_1^-]$ $\mathbb{P}$.q.c. $\star$

$S_n^+ - S_n^- = S_n := \sum_{k=1}^{n} X_k$

$\frac{S_n}{n} \to \mathbb{E}[X_1^+] - \mathbb{E}[X_1^-] = \mathbb{E}[X_1] = E \quad \mathbb{P}$-q.c.

ESERCIZIO

X, Y i.i.d $\mathbb{P}_X = \mathbb{P}_Y = N(0,1)$ Calcolare $\mathbb{P}_{X^2+Y^2}$

$\mathbb{P}_X = \mathbb{P}_Y = f(x)\,dx \qquad f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \qquad x \in \mathbb{R}$

$$\mathbb{P}_{X^2} = \mathbb{P}_{Y^2} = g(x)\,dx \qquad g(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2\sqrt{x}}\left(f(\sqrt{x}) + f(-\sqrt{x})\right) & x > 0 \end{cases}$$

$x > 0$

$$g(x) = \frac{1}{2\sqrt{x}} 2f(\sqrt{x}) = \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x}{2}\right) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{1}{2}x}$$

$$= \frac{1}{\sqrt{2\pi}} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$\Gamma(\alpha, \lambda)$

$$g(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\frac{1}{\sqrt{2\pi}} = \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} \qquad \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}\,\Gamma\left(\frac{1}{2}\right)} \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$Z = X^2 + Y^2 \qquad \mathbb{P}_Z = h(t)\,dt$

$$h(t) = \int_{\mathbb{R}} g(x) g(t-x)\,dx \qquad \begin{cases} x > 0 \\ t - x > 0 \end{cases} \qquad 0 < x < t$$

1) $t \leq 0 \qquad h(t) = 0$

2) $t > 0$

$$h(t) = \int_0^t \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-\frac{\cancel{x}}{2}} \cdot \frac{1}{\sqrt{2\pi}} (t-x)^{-1/2} e^{-\frac{(t-\cancel{x})}{2}}\,dx$$

$$= \frac{1}{2\pi} e^{-t/2} \int_0^t x^{-1/2} (t-x)^{-1/2}\,dx \qquad x = ts \quad dx = t\,ds \quad s = \frac{x}{t}$$

$$= \frac{1}{2\pi} e^{-t/2} \int_0^1 \cancel{t^{-1/2}} s^{-1/2} \cancel{t^{-1/2}} (1-s)^{-1/2} \cancel{t}\,ds$$

$$= \frac{1}{2\pi} e^{-t/2} \int_0^1 s^{\frac{1}{2}-1} (1-s)^{\frac{1}{2}-1}\,ds$$

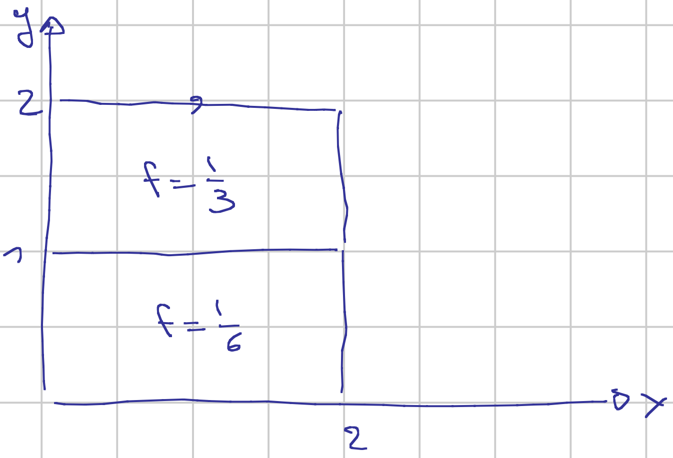
$$= \frac{1}{2\pi} \frac{\sqrt{\pi}\sqrt{\pi}}{1} e^{-t/2} = \frac{1}{2} e^{-t/2}$$

$$h(t) = \begin{cases} 0 & t \le 0 \\ \frac{1}{2} e^{-t/2} & t > 0 \end{cases} \qquad \mathbb{P}_{X^2+Y^2} = \exp\left(\frac{1}{2}\right)$$

$$h(t) = \begin{cases} 0 & t \le 0 \\ C e^{-t/2} & t > 0 \end{cases}$$

— o —

FOGLIO 7 EX 10



$Z := X + Y$

$$h_Z(t) = \int_{\mathbb{R}} f(x, t-x)\, dx = \int_0^2 f(x, t-x)\, dx$$

$Z = p \circ (X, Y) \qquad p(x,y) = x + y$

$\varphi : \mathbb{R} \to \mathbb{R}$ di Borel nonnegative

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_Z(dt) = \int_{\mathbb{R}^2} (\varphi \circ p)(x,y)\, \mathbb{P}_{X,Y}(dx\,dy) =$$

$$= \int_{\mathbb{R}^2} \varphi(x+y)\, f(x,y)\, dx\, dy = \int_{\mathbb{R}} dx \int_{\mathbb{R}} \varphi(x+y) f(x,y)\, dy$$

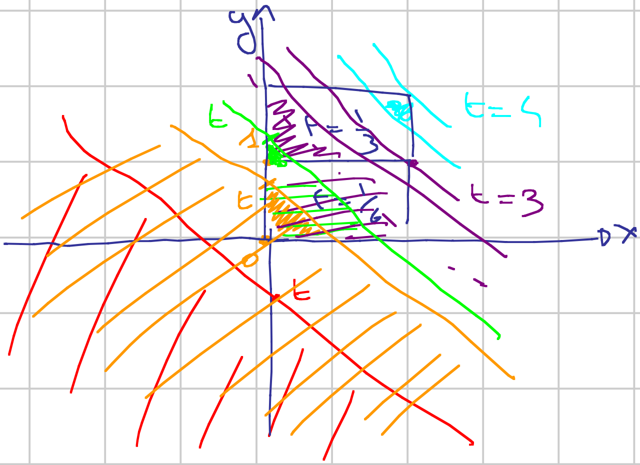
$x + y = s \qquad y = s - x \qquad dy = ds$

$$\int_{\mathbb{R}} dx \int_{\mathbb{R}} \varphi(s)\, f(x, s-x)\, ds = \int_{\mathbb{R}} \varphi(s) \left(\int_{\mathbb{R}} f(x, s-x)\, dx \right) ds$$

$$\mathbb{P}_Z(dt) = h(t)\, dt \qquad h(t) = \int_{\mathbb{R}} f(x, t-x)\, dx$$

$$h_Z(t) = \frac{d}{dt} F_Z(t)$$

$$F_Z(t) = \mathbb{P}(Z \le t) = \mathbb{P}(X+Y \le t) = \int_{\{(x,y)\in\mathbb{R}^2:\ x+y\le t\}} f(x,y)\,dx\,dy$$



$x+y \le t \qquad y \le t-x$

$t \le 0 \Rightarrow F_Z(t) = 0$

$0 < t \le 1 \qquad F_Z(t) = \frac{1}{6}\cdot\frac{1}{2}t^2 = \frac{1}{12}t^2$

$1 < t \le 2 \qquad F_Z(t) = \frac{1}{3}\frac{1}{2}(t-1)^2 + \frac{1}{6}\cdot\frac{1}{2}\cdot 1\,(t+t-1)$

$2 \le t < 3$