LEGGE FORTE DEI GRANDI NUMERI

Titolo nota 30/11/2015

$\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ e X v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

1) CONVERGENZA IN MEDIA QUADRATICA

Dico che $X_n \longrightarrow X$ in media quadratica ($X_n \to X$ in $\mathcal{L}^2$)

se $\mathbb{E}[(X_n - X)^2] = \int_\Omega |X_n - X|^2(\omega)\,\mathbb{P}(d\omega) \to 0$

2) CONVERGENZA IN PROBABILITÀ

Dico che $X_n \to X$ in probabilità $\mathbb{P}$ se

$\forall \delta > 0 \qquad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \delta) = 0$

3) CONVERGENZA $\mathbb{P}$-QUASI CERTA

Dico che $X_n \to X$ $\mathbb{P}$-q.c. se $\exists N \subset \Omega$ $\mathbb{P}(N) = 0$ t.c.

$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall \omega \in \Omega \setminus N$

$\delta > 0$ $n \in \mathbb{N}$ $\quad A_{n,\delta} = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \delta\}$

$X_n \to X$ in probabilità SSE $\lim_{n\to\infty} \mathbb{P}(A_{n,\delta}) = 0 \quad \forall \delta > 0$

LEMMA $\quad X_n \to X$ $\mathbb{P}$-q.c. SSE

$\mathbb{P}(\limsup_n A_{n,\delta}) = 0 \quad \forall \delta > 0$

DIM $\quad E_\delta = \{\omega \in \Omega : \limsup_{n\to\infty} |X_n(\omega) - X(\omega)| > \delta\} \quad \delta \geq 0$

$X_n \to X$ $\mathbb{P}$-q.c. SSE $\mathbb{P}(E_0) = 0$

$\delta > 0 \qquad E_\delta \subseteq E_0 \qquad E_0 = \bigcup_{\delta > 0} E_\delta$

$\mathbb{P}(E_0) = 0$ SSE $\mathbb{P}(E_\delta) = 0$

Dimostriamo che $E_\delta = \limsup_n A_{n,\delta}$

1) Sia $\omega \in E_\delta$:

$\limsup_n |X_n(\omega) - X(\omega)| > \delta$

$\forall k \in \mathbb{N} \quad \exists n > k \quad |X_n(\omega) - X(\omega)| > \delta$

$\forall k \in \mathbb{N} \quad \exists n \geq k \quad$ t.c. $\omega \in A_{n,\delta}$

$\forall k \in \mathbb{N} \quad \omega \in \bigcup_{n=k}^{\infty} A_{n,\delta} \quad \Rightarrow \quad \omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_{n,\delta} = \limsup A_{n,\delta}$

2) $\omega \in \limsup_n A_{n,\delta} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_{n,\delta}$

$\forall k \in \mathbb{N} \quad \omega \in \bigcup_{n=k}^{\infty} A_{n,\delta}$

$\forall k \in \mathbb{N} \quad \exists n \geq k \quad \omega \in A_{n,\delta}$

$\forall k \in \mathbb{N} \quad \exists n \geq k \; : |X_n(\omega) - X(\omega)| > \delta$

$\Rightarrow \limsup |X_n(\omega) - X(\omega)| > \delta$

Allora $X_n \to X$ $\mathbb{P}$-q.c. sse $\mathbb{P}(E_\delta) = 0$

cioè sse $\mathbb{P}(\limsup_n A_{n,\delta}) = 0$

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LEMMA Sia $\{Y_n\}_{n \in \mathbb{N}}$ successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

Supponiamo
- $\mathbb{E}[Y_n] = E \in \mathbb{R} \quad \forall n \in \mathbb{N}$
- $\sum_{n=1}^{\infty} Var[Y_n] < +\infty$

Allora Y_n converge $\mathbb{P}$-q.c. alla v.a. costante $Y \equiv E$

DIM $A_{n,\delta} = \{\omega \in \Omega : |Y_n(\omega) - E| > \delta\}$

Scrivo la disuguaglianza di Chebychev.

$$\mathbb{P}(A_{n,\delta}) \leq \frac{Var[Y_n]}{\delta^2} \qquad \forall \delta > 0$$

$$\sum_{n=1}^{\infty} \mathbb{P}(A_{n,\delta}) \leq \frac{1}{\delta^2} \sum_{n=1}^{\infty} Var[Y_n] < +\infty \quad \text{per ipotesi}$$

Applico il lemma di Borel-Cantelli

$\mathbb{P}(\limsup_n A_{n,\delta}) = 0$ cioè $Y_n \to E$ $\mathbb{P}$-q.c.

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LEGGE FORTE DEI GRANDI NUMERI

(TEOREMA DI RAJCHMANN)

Sia $\{X_n\}_{n \in \mathbb{N}}$ una successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

Supponiamo che: 1) le X_n siano a 2 a 2 scorrelate

2) $\mathbb{E}[X_n] = E \in \mathbb{R} \quad \forall n \in \mathbb{N}$

3) $\exists\, C^2 > 0 : \ Var[X_n] \le C^2 \quad \forall n \in \mathbb{N}$

Allora $\frac{S_n}{n} := \frac{1}{n}\sum_{k=1}^{n} X_k$ converge a E in probabilità, in media quadratica e $\mathbb{P}$-q.c.

Dim Introduco la successione $\{Y_n\}$ definita da

$Y_n = X_n - E$

Le Y_n sono a 2 a 2 scorrelate

$\mathbb{E}[Y_n] = 0 \quad \forall n \in \mathbb{N}$

$Var[Y_n] = Var[X_n] \le C^2 \quad \forall n \in \mathbb{N}$

Ovviamente $\frac{1}{n}\sum_{k=1}^{n} X_k \to E$ $\mathbb{P}$-q.c. sse $\frac{1}{n}\sum_{k=1}^{n} Y_k \to 0$ $\mathbb{P}$-q.c.

$S_0(\omega) \equiv 0 \qquad S_n = \sum_{k=1}^{n} Y_k$

$j, i \in \mathbb{N}_0,\ j \ge i \qquad Var[S_j - S_i] = Var\left[\sum_{k=i+1}^{j} Y_k\right] =$

$$= \sum_{k=i+1}^{j} Var[Y_k] \le \sum_{k=i+1}^{j} C^2 = C^2(j-i)$$

Scelgo $i = 0 \quad j = n^2$

$Var[S_{n^2}] \le C^2 n^2 \qquad Var\left[\frac{S_{n^2}}{n^2}\right] \le \frac{1}{n^4} C^2 n^2 = \frac{C^2}{n^2}$

$$\sum_{n=1}^{\infty} Var\left[\frac{S_{n^2}}{n^2}\right] \le \sum_{n=1}^{\infty} \frac{C^2}{n^2} < +\infty \qquad \mathbb{E}\left[\frac{S_{n^2}}{n^2}\right] = \frac{1}{n^2}\mathbb{E}[S_{n^2}] = 0$$

Allora $\frac{S_{n^2}}{n^2}$ converge $\mathbb{P}$-q.c. alla v.a. identicamente nulla.

Sia $p \in \mathbb{N}$ $\quad \exists!\, n = n(p)$ t.c. $\quad n^2 \le p < (n+1)^2$

$$\frac{n^2}{(n+1)^2}\frac{S_{n^2}(\omega)}{n^2} < \frac{S_{n^2}(\omega)}{p} = \frac{S_{n^2}(\omega)}{n^2}\frac{n^2}{p} < \frac{S_{n^2}(\omega)}{n^2}\cdot\frac{n^2}{n^2} \qquad \text{se } S_{n^2}(\omega) \ge 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{S_{n^2(p)}(\omega)}{p} = 0$$. Analogo se $S_{n^2}(\omega) < 0$

Considero $\sum_{p=1}^{\infty} Var\left[\frac{S_p}{p} - \frac{S_{n^2(p)}}{p}\right] =$

$$= \sum_{p=1}^{\infty} \frac{1}{p^2} Var\left[S_p - S_{n^2(p)}\right] \le \sum_{p=1}^{\infty} \frac{1}{p^2} c^2 \left(p - n^2(p)\right)$$

$$n^2 \le p < (n+1)^2 \qquad \Rightarrow n \le \sqrt{p}$$

$$p - n^2 \le (n+1)^2 - n^2 = 2n+1 \le 2\sqrt{p} + 1$$

$$\le \sum_{p=1}^{\infty} \frac{1}{p^2} c^2 \left(1 + 2\sqrt{p}\right) = c^2 \sum_{p=1}^{\infty} \frac{1+2\sqrt{p}}{p^2} < +\infty$$

Per il lemma $\frac{S_p}{p} - \frac{S_{n^2(p)}}{p} \to 0$ per $p \to \infty$ $\mathbb{P}$-q.c.

Sapevo già che $\frac{S_{n^2(p)}}{p} \to 0$ $\mathbb{P}$-q.c. e dunque $\frac{S_p}{p} \to 0$ $\mathbb{P}$-q.c.

LEGGE FORTE DEI GRANDI NUMERI
(TEOREMA DI ETEMADI)

Sia $\{X_n\}_{n\in\mathbb{N}}$ una successione di v.a. identicamente distribuite e a 2 a 2 indipendenti e sia $E[X_i] = E \in \mathbb{R}$. Allora, posto $S_n := \sum_{k=1}^{n} X_k$ si ha che $\frac{S_n}{n} \to E$ $\mathbb{P}$-q.c.

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LEMMA Se $\{a_n\}_{n\in\mathbb{N}}$ è una successione in $\mathbb{R}$ t.c. a_n converge a $L \in \mathbb{R}$ per $n \to \infty$, allora $\frac{1}{n}\sum_{k=1}^{n} a_k \to L$

LEMMA * Per ogni $x > 0$ $\sum_{n > x} \frac{1}{n^2} \le \frac{2}{x}$

LEMMA Sia $\alpha > 1$ e per $n \in \mathbb{N}$ pongo $k_n := \lfloor \alpha^n \rfloor$.
Allora $\forall j \in \mathbb{N}$ $\sum_{n : k_n \ge j} \frac{1}{k_n^2} \le \frac{4\alpha^2}{\alpha^2 - 1} \frac{1}{j^2}$

DIM 1° PASSO $\boxed{k_n \ge \frac{1}{2}\alpha^n}$

a) $\alpha^n \le 2 \Rightarrow \frac{\alpha^n}{2} \le 1$ $\quad \lfloor k_n \rfloor = \lfloor \alpha^n \rfloor \ge 1 \ge \frac{1}{2}\alpha^n$

b) $\alpha^n > 2$ $\quad k_n = \lfloor \alpha^n \rfloor \geq \alpha^n - 1 > \alpha^n - \frac{\alpha^n}{2} = \frac{\alpha^n}{2}$

2° PASSO, 1° CASO : $\alpha^n = k_n$

$k_n = j \iff \alpha^n = j \iff e^{n\log\alpha} = e^{\log j} \iff n = \frac{\log j}{\log\alpha} \Rightarrow \frac{\log j}{\log \alpha} = n_0 \in \mathbb{N}_0$

$$\sum_{n: k_n \geq j} \frac{1}{k_n^2} \leq \sum_{n\geq n_0} \frac{4}{\alpha^{2n}} = \frac{4}{\alpha^{2n_0}} \sum_{s\geq 0} \frac{1}{\alpha^{2s}} = \frac{4}{\alpha^{2n_0}} \frac{1}{1-\alpha^{-2}} = \frac{4\alpha^2}{\alpha^2-1}\frac{1}{j^2}$$

2° PASSO, 2° CASO $\alpha^n > k_n$

$k_n \geq j \Rightarrow \alpha^n > j \quad e^{n\log\alpha} > e^{\log j} \quad n\log\alpha > \log j \quad n > \frac{\log j}{\log\alpha} \Rightarrow$

$n \geq 1 + n_0$ dove $n_0 := \left\lfloor \frac{\log j}{\log \alpha}\right\rfloor$

$$\Rightarrow \sum_{n:k_n\geq j} \frac{1}{k_n^2} \leq \sum_{n=n_0+1}^{+\infty} \frac{4}{\alpha^{2n}} = \frac{4}{\alpha^{2(n_0+1)}} \sum_{s\geq0}\frac{1}{\alpha^{2s}} = \frac{4}{\alpha^{2(n_0+1)}}\frac{1}{1-\alpha^{-2}} =$$

$$= \frac{4\alpha^2}{\alpha^2-1}\frac{1}{\alpha^{2(n_0+1)}} < \frac{4\alpha^2}{\alpha^2-1}\frac{1}{j^2}$$

Ma $n_0 = \left\lfloor \frac{\log j}{\log\alpha}\right\rfloor \Rightarrow n_0+1 \geq \frac{\log j}{\log \alpha} \quad (1+n_0)\log\alpha \geq \log j$

$\Rightarrow \alpha^{1+n_0} > j \Rightarrow \frac{1}{\alpha^{2(n_0+1)}} < \frac{1}{j^2}$ da cui la tesi

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LEMMA Sia $\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. identicamente distribuite, a valori reali e sommabili ($E[X_i] = E \in \mathbb{R}$ $\forall i \in\mathbb{N}$)

Per $n\in\mathbb{N}$ pongo $Y_n(\omega) = \begin{cases} X_n(\omega) & \text{se } |X_n(\omega)|\leq n \\ 0 & \text{altrimenti}\end{cases}$

cioè $Y_n(\omega) = X_n(\omega)\mathbb{1}_{\{|X_n|\leq n\}}(\omega)$

Sia $S_n(\omega) := \frac{1}{n}\sum_{k=1}^n X_k(\omega)$ e sia $T_n(\omega) = \frac{1}{n}\sum_{k=1}^n Y_k(\omega)$

Se $\frac{T_n}{n}$ converge ad E $\mathbb{P}$-qc, allora anche $\frac{S_n}{n}$ converge a E $\mathbb{P}$-qc.

DIM Per $n\in\mathbb{N}$ considero $A_n := \{\omega\in\Omega : X_n(\omega)\neq Y_n(\omega)\}$

cioè $A_n = \{\omega\in\Omega : |X_n(\omega)| > n\}$

Considero $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n|>n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_1|>n)$

$$= \sum_{n=1}^{\infty} \int_{n-1}^{n} \mathbb{P}(|X_1|>n)\,dt \leq$$

$t \in (n-1, n)$ $\quad |X_1|>n \Rightarrow |X_1|>t$

$$\mathbb{P}(|X_1|>n) \leq \mathbb{P}(|X_1|>t)$$

$$\sum_{n=1}^{\infty} \int_{n-1}^{n} \mathbb{P}(|X_1|>t)\,dt = \int_{0}^{+\infty} \mathbb{P}(|X_1|>t)\,dt = \mathbb{E}[|X_1|]$$

finito per ipotesi

$\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ è una serie convergente

Per il Lemma di Borel-Cantelli $\mathbb{P}(\limsup_n A_n) = 0$

Dunque $\mathbb{P}((\limsup A_n)^c) = 1$

Sia $\omega \in (\limsup A_n)^c$ $\quad \omega \notin \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$\exists \bar{k}$ t.c. $\omega \notin \bigcup_{n=\bar{k}}^{\infty} A_n$ cioè $\exists \bar{k}$ t.c. $\omega \in A_n^c \ \forall n \geq \bar{k}$

$\exists \bar{k}$ t.c. $X_n(\omega) = Y_n(\omega) \quad \forall n \geq \bar{k}$

$$n \geq \bar{k} \qquad S_n(\omega) - T_n(\omega) = \sum_{k=1}^{n} X_k(\omega) - \sum_{k=1}^{n} Y_k(\omega)$$

$$= \sum_{k=1}^{n} (X_k - Y_k)(\omega) = S_{\bar{k}}(\omega) - T_{\bar{k}}(\omega)$$

Considero, per $n \geq \bar{k}$

$$\frac{S_n(\omega)}{n} - \frac{T_n(\omega)}{n} = \frac{1}{n}\left(S_{\bar{k}}(\omega) - T_{\bar{k}}(\omega)\right) \xrightarrow{n\to\infty} 0$$

Quindi, se $\frac{T_n}{n}(\omega) \to \mathbb{E}$, anche $\frac{S_n(\omega)}{n} \to \mathbb{E}$ $\mathbb{P}$-q.c.

<u>LEMMA</u> Sia $\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. identicamente distribuite e sommabili ($\mathbb{E}[X_i] = E \in \mathbb{R} \ \forall i \in \mathbb{N}$) su $(\Omega, \mathcal{E}, \mathbb{P})$

Sia $Y_n(\omega) := X_n(\omega)\, \mathbb{1}_{\{|X_n| \leq n\}}(\omega)$

Allora

$$\sum_{j=1}^{\infty} \frac{\mathbb{E}[Y_j^2]}{j^2} \leq 4\, \mathbb{E}[|X_1|]$$

DIM $\quad \mathbb{E}[Y_j^2] = \int_0^{+\infty} \mathbb{P}(Y_j^2 > t)\,dt = \int_0^{+\infty} \mathbb{P}(|Y_j| > \sqrt{t})\,dt$

$s = \sqrt{t} \quad t = s^2$

$$= \int_0^{+\infty} 2s\,\mathbb{P}(|Y_j| > s)\,ds =$$

$$= \int_0^{j} 2s\,\mathbb{P}(|Y_j| > s)\,ds \le \int_0^{j} 2s\,\mathbb{P}(|X_j| > s)\,ds =$$

$$= \int_0^{j} 2s\,\mathbb{P}(|X_1| > s)\,ds = \int_0^{+\infty} 2s\,\mathbb{1}_{[0,j]}(s)\,\mathbb{P}(|X_1| > s)\,ds$$

$$\sum_{j=1}^{+\infty} \frac{\mathbb{E}[Y_j]^2}{j^2} = \int_0^{+\infty} 2s \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \mathbb{1}_{[0,j]}(s) \right) \mathbb{P}(|X_1| > s)\,ds$$

$$= \int_0^{+\infty} 2s \sum_{j \geq s} \frac{1}{j^2} \mathbb{P}(|X_1| > s)\,ds \le \int_0^{+\infty} 2\cancel{s}\, \frac{2}{\cancel{s}} \mathbb{P}(|X_1| > s)\,ds$$

$$= 4 \int_0^{+\infty} \mathbb{P}(|X_1| > s)\,ds = 4\,\mathbb{E}[|X_1|]$$

DIM DEL LEMMA $\star$

$x > 0 \quad \sum_{n > x} \frac{1}{n^2}$

$$\frac{1}{n^2} = \int_{n-1}^{n} \frac{1}{n^2}\,dt \qquad n-1 < t < n \qquad \frac{1}{t} > \frac{1}{n} \qquad \frac{1}{n^2} < \frac{1}{t^2}$$

$$\sum_{n > x} \frac{1}{n^2} = \sum_{n=1+\lfloor x \rfloor}^{+\infty} \frac{1}{n^2} = \frac{1}{(1+\lfloor x \rfloor)^2} + \sum_{n=2+\lfloor x \rfloor}^{+\infty} \frac{1}{n^2} \le$$

$$\le \frac{1}{(1+\lfloor x \rfloor)^2} + \sum_{n=2+\lfloor x \rfloor}^{+\infty} \int_{n-1}^{n} \frac{1}{t^2}\,dt = \frac{1}{(1+\lfloor x \rfloor)^2} + \int_{1+\lfloor x \rfloor}^{+\infty} \frac{1}{t^2}\,dt$$

$$= \frac{1}{(1+\lfloor x \rfloor)^2} + \left. \frac{-1}{t} \right|_{t=1+\lfloor x \rfloor}^{t \to +\infty} = \frac{1}{(1+\lfloor x \rfloor)^2} + \frac{1}{1+\lfloor x \rfloor} \le \frac{2}{1+\lfloor x \rfloor} < \frac{2}{x}$$

$$(1+\lfloor x \rfloor)^2 \geq 1 + \lfloor x \rfloor \qquad \frac{1}{(1+\lfloor x \rfloor)^2} \le \frac{1}{1+\lfloor x \rfloor}$$