LEGGI CONDIZIONATE

Titolo nota 23/11/2015

$(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

$X : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ t.c. $\mathbb{E}[X]$ esiste finito

Se $\mathcal{G} \subseteq \mathcal{E}$ è una σ-algebra, allora

$\exists Y : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{G}, \mathbb{P}_{\mathcal{G}})$ t.c. $\mathbb{P}_{\mathcal{G}} := \mathbb{P}$ ristretta a $\mathcal{G}$

$$\int_A X(\omega)\, \mathbb{P}(d\omega) = \int_A Y(\omega)\, \mathbb{P}_{\mathcal{G}}(d\omega) \qquad \forall A \in \mathcal{G}$$

Y è $\mathbb{P}_{\mathcal{G}}$-q.c. unica, e si indica $\mathbb{E}[X \mid \mathcal{G}]$

Se $\mathcal{G}$ è la σ-algebra rilevata da una v.a. $Z : \Omega \to \mathbb{R}$

$\mathbb{E}[X \mid \sigma(Z)] =: \mathbb{E}[X \mid Z]$

Se $\mathcal{G}$ è la σ-algebra generata da una partizione finita o numerabile di Ω in eventi

$\mathcal{G} = \sigma\{B_n : n \in \mathcal{J}\}$ $\quad \mathcal{J} = \{1, \ldots, n\}$ o $\mathcal{J} = \mathbb{N}$

allora

$$\mathbb{E}[X \mid \mathcal{G}](\omega) = \sum_{n \in \mathcal{J}} \mathbb{E}_{B_n}[X]\, \mathbb{1}_{B_n}(\omega) \qquad \forall \omega \in \Omega$$

$$\mathbb{E}_{B_n}[X] := \int_\Omega X(\omega)\, \mathbb{P}_{B_n}(d\omega) = \frac{1}{\mathbb{P}(B_n)} \mathbb{E}[X \mathbb{1}_{B_n}]$$

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ESEMPIO

X_1, X_2 v.a. indipendenti $\quad \mathbb{P}_{X_1} = B(n,p) \quad \mathbb{P}_{X_2} = B(m,p)$

$\mathbb{P}_{X_1+X_2} = B(n+m, p)$

$(X_1 + X_2)(\Omega) = \{0, 1, \ldots, n+m\}$

$$\mathbb{P}(X_1 + X_2 = k) = \sum_{j=-\infty}^{+\infty} \mathbb{P}(X_1 = j)\, \mathbb{P}(X_2 = k-j) \qquad \begin{matrix} k-j \geq 0 \\ j \leq k \end{matrix}$$

$$= \sum_{j=0}^{k} \binom{n}{j} p^{\cancel{j}} (1-p)^{n-\cancel{j}} \binom{m}{k-j} p^{k-\cancel{j}} (1-p)^{m-(k-\cancel{j})}$$

$$= p^k (1-p)^{n+m-k} \underbrace{\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}}_{\overset{??}{=} \binom{n+m}{k}}$$

$$\sum_{j=0}^{\infty} \binom{n}{j} x^j = (1+x)^n \qquad \sum_{j=0}^{\infty} \binom{m}{j} x^j = (1+x)^m$$

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} \right) x^k = (1+x)^n (1+x)^m = (1+x)^{n+m}$$

$$= \sum_{k=0}^{\infty} \binom{n+m}{k} x^k$$

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X_1, X_2 independent. $\mathbb{P}_{X_1} = \mathbb{P}_{X_2} = B(p) = B(1,p)$

$Y := X_1 + X_2 \Rightarrow \mathbb{P}_Y = B(2,p)$

$\mathcal{G} := \sigma(Y) = \sigma\{Y=0, Y=1, Y=2\}$

$\mathbb{E}[X_1 | Y](\omega) = ?$

$$\mathbb{E}[X_1 | Y](\omega) = \sum_{j=0}^{2} \mathbb{E}_{\{Y=j\}}[X_1] \mathbb{1}_{\{Y=j\}}(\omega)$$

$$\mathbb{E}_{\{Y=j\}}[X_1] = \frac{1}{\mathbb{P}(Y=j)} \mathbb{E}[X_1 \mathbb{1}_{\{Y=j\}}]$$

$\boxed{j=0}$ $\quad (X_1 \mathbb{1}_{\{Y=0\}})(\omega) = 0 \quad \forall \omega \in \Omega \Rightarrow \mathbb{E}[X_1 \mathbb{1}_{\{Y=0\}}] = 0$

$j=1$ $\quad (X_1 \mathbb{1}_{\{Y=1\}})(\omega) \neq 0$ sse $Y(\omega) = 1$, $X_1(\omega) \neq 0$

sse $\begin{cases} X_1(\omega) + X_2(\omega) = 1 \\ X_1(\omega) \neq 0 \end{cases}$ sse $\begin{cases} X_1(\omega) = 1 \\ X_2(\omega) = 0 \end{cases}$

$$\mathbb{E}[X_1 \mathbb{1}_{\{Y=1\}}] = 1 \cdot \mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0) = p(1-p)$$

$$\mathbb{P}(Y=1) = \binom{2}{1} p(1-p) = 2p(1-p)$$

$$\mathbb{E}_{\{Y=1\}}[X_1] = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

$j=2 \qquad \mathbb{E}[X_1 \mathbb{1}_{\{Y=2\}}] = ?$

$(X_1 \mathbb{1}_{\{Y=2\}})(\omega) \neq 0 \qquad \begin{cases} Y(\omega)=2 \\ X_1(\omega) \neq 0 \end{cases} \qquad \begin{cases} X_1(\omega)+X_2(\omega)=2 \\ X_1(\omega)=1 \end{cases}$

$\begin{cases} X_1(\omega)=1 \\ X_2(\omega)=1 \end{cases}$

$$\mathbb{E}[X_1 \mathbb{1}_{\{Y=2\}}] = 1 \cdot \mathbb{P}(X_1=1,\ X_2=1) = \mathbb{P}(X_1=1)\mathbb{P}(X_2=1)$$
$$= p \cdot p = p^2$$

$\mathbb{P}(Y=2) = \binom{2}{0} p^2 (1-p)^0 = p^2 \qquad \mathbb{E}_{\{Y=2\}}[X_1] = \dfrac{p^2}{p^2} = 1$

$$\mathbb{E}[X_1 | Y](\omega) = \frac{1}{2} \mathbb{1}_{\{Y=1\}}(\omega) + \mathbb{1}_{\{Y=2\}}(\omega) \qquad \forall \omega \in \Omega$$

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$(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

$B \in \mathcal{E}$ fissato

$X : \Omega \to \mathbb{R}$ v.a. discreta $\qquad X(\Omega) = \{x_i\}_{i \in J}$

$$\varphi_B(x) = \begin{cases} \mathbb{P}(B \mid X=x) & x \in \{x_i\}_{i\in J} \quad \mathbb{P}(X=x)>0 \\ 0 & \text{altrimenti} \end{cases}$$

$\forall A \in \mathcal{B}(\mathbb{R}) \qquad \int_A \varphi_B(x)\, \mathbb{P}_X(dx) = \mathbb{P}(\{X\in A\} \cap B)$

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$(X,Y) : \Omega \to \mathbb{R}^2$ con distribuzione congiunta A.C.

$\mathbb{P}_{X,Y} = f(x,y)\, dx\, dy$

$D \in \mathcal{B}(\mathbb{R})$ fissato $\qquad B := Y^{-1}(D)$

$\forall A \in \mathcal{B}(\mathbb{R}) \qquad \mathbb{P}(\{X\in A\}\cap B) = \int_A \varphi_B(x) f_X(x)\, dx = \int_A \varphi_B(x)\, \mathbb{P}_X(dx)$

dove $\qquad \varphi_B(x) = \int_D \dfrac{f(x,y)}{f_X(x)}\, dy \qquad \mathbb{P}_{X,Y}-$q.c.

$$\frac{f(x,y)}{f_X(x)} = h(y|x)$$

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Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $(\Omega', \mathcal{E}')$ spazio misurabile
Sia $X : \Omega \to \Omega'$ $(\mathcal{E}, \mathcal{E}')$-misurabile cioè
$\forall A \in \mathcal{E}' \Rightarrow X^{-1}(A) \in \mathcal{E}$
Definisco $\mathbb{P}_X : A \in \mathcal{E}' \mapsto \mathbb{P}(X \in A) \in [0,1]$
Si dimostra che $(\Omega', \mathcal{E}', \mathbb{P}_X)$ è uno spazio probabilizzato

TEO Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato, sia $(\Omega', \mathcal{E}')$ uno spazio misurabile e sia $X : \Omega \to \Omega'$ una funzione $(\mathcal{E}, \mathcal{E}')$-misurabile.
Sia $B \in \mathcal{E}$.
Allora $\exists\, \varphi_B : \Omega' \to \mathbb{R}$ v.a. su $(\Omega', \mathcal{E}')$ t.c.

$$\mathbb{P}(\{X \in A\} \cap B) = \int_A \varphi_B(\omega')\, \mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

DIM $\mu_B : A \in \mathcal{E}' \mapsto \mathbb{P}(\{X \in A\} \cap B) \in [0,1]$
Allora $(\Omega', \mathcal{E}', \mu_B)$ è uno spazio di misura con μ_B misura finita
e $\mu_B \ll \mathbb{P}_X$
Sia $A \in \mathcal{E}'$ t.c. $\mathbb{P}_X(A) = 0$ sse $\mathbb{P}(X \in A) = 0 \Rightarrow \mathbb{P}(\{X \in A\} \cap B) = 0$
Per il Teorema di Radon-Nikodym
$\exists\, \varphi_B : \Omega' \to \mathbb{R}$ $\quad \varphi_B \geq 0$ $\quad \varphi_B \in \mathcal{L}^1(\mathbb{P}_X)$

$$\mathbb{P}(\{X \in A\} \cap B) = \mu_B(A) = \int_A \varphi_B(\omega')\, \mathbb{P}_X(d\omega') \quad \forall A \in \mathcal{E}'$$

e se ψ è un'altra funzione che gode di queste proprietà, allora
$\psi = \varphi_B$ $\quad \mathbb{P}_X$-q.c.

$\varphi_B(\omega')$ si indica $\boxed{\mathbb{P}(B | X = \omega')}$

TEO Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $(\Omega', \mathcal{E}')$ spazio misurabile.

Sia $X : \Omega \to \Omega'$ $(\mathcal{E}, \mathcal{E}')$-misurabile

sia $Y : \Omega \to \mathbb{R}$ v.a. con $\mathbb{E}[Y]$ finito

Allora $\exists\, \varphi_Y : \Omega' \to \mathbb{R}$ $\mathcal{E}'$-misurabile t.c.

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \int_A \varphi_Y(\omega')\,\mathbb{P}_X(d\omega') \qquad \forall A \in \mathcal{E}'$$

Se ψ è un'altra funzione che gode delle stesse proprietà, allora $\psi = \varphi_Y$ $\mathbb{P}_X$-q.c.

DIM 1° CASO $Y \geq 0$

Considero $\mu : A \in \mathcal{E}' \longmapsto \int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) \in \mathbb{R}$

$(\Omega', \mathcal{E}', \mu)$ è uno spazio di misura

$\mu(\Omega') = \int_\Omega Y(\omega)\,\mathbb{P}(d\omega) = \mathbb{E}[Y] < +\infty \Rightarrow \mu$ misura finita

$\mu \ll \mathbb{P}_X$

Sia $A \in \mathcal{E}'$ $\mathbb{P}_X(A) = 0$ sse $\mathbb{P}(X^{-1}(A)) = 0 \Rightarrow \mu(A) = 0$

Per Radon-Nikodým

$\exists\, \varphi : \Omega' \to \mathbb{R}$ $\varphi \geq 0$ $\varphi \in L^1(\mathbb{P}_X)$ t.c.

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \mu(A) = \int_A \varphi(\omega')\,\mathbb{P}_X(d\omega') \qquad \forall A \in \mathcal{E}'$$

φ è $\mathbb{P}_X$-q.c. unica.

2° CASO Y di segno qualsiasi con $\mathbb{E}[Y]$ esiste finito

Considero le v.a. Y^+ e Y^-

$\exists\, \varphi^+$ e $\varphi^- \geq 0$, $L^1(\mathbb{P}_X)$ t.c.

$$\int_{X^{-1}(A)} Y^+(\omega)\,\mathbb{P}(d\omega) = \int_A \varphi^+(\omega')\,\mathbb{P}_X(d\omega') \qquad \forall A \in \mathcal{E}'$$

$$\int_{X^{-1}(A)} Y^-(\omega)\,\mathbb{P}(d\omega) = \int_A \varphi^-(\omega')\,\mathbb{P}_X(d\omega') \qquad \forall A\in\mathcal{E}'$$

Sottraggo membro a membro

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \int_A \underbrace{(\varphi^+-\varphi^-)}_{\varphi_Y}(\omega')\,\mathbb{P}_X(d\omega') \quad \forall A\in\mathcal{E}'$$

Supponiamo che esistano 2 funzioni che soddisfano l'uguaglianza

$$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) = \int_A \varphi_1(\omega')\,\mathbb{P}_X(d\omega') = \int_A \varphi_2(\omega')\,\mathbb{P}_X(d\omega') \quad \forall A\in\mathcal{E}'$$

$A=\{\varphi_1>\varphi_2\}\in\mathcal{E}'$ $\qquad \int_{\{\varphi_1>\varphi_2\}} (\varphi_1-\varphi_2)(\omega')\,\mathbb{P}_X(d\omega')=0$

$\mathbb{P}_X(A)=0$ e analogamente $\mathbb{P}_X(\varphi_1<\varphi_2)=0$

$\Rightarrow \varphi_1=\varphi_2$ $\mathbb{P}_X$-q.c.

La funzione $\varphi_Y(\omega')$ si indica $\mathbb{E}[Y\,|\,X=\omega']$

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CASO PARTICOLARE: X DISCRETA

$X:\Omega\to\mathbb{R}$ discreta $\qquad \Omega'=X(\Omega)=\{x_i\}_{i\in J}$

$Y:\Omega\to\mathbb{R}$ v.a.

$\int_{X^{-1}(A)} Y(\omega)\,\mathbb{P}(d\omega) \qquad X^{-1}(A)=\bigcup_{x_i\in A}\{X=x_i\}$

$$=\sum_{x_i\in A}\int_{\{X=x_i\}} Y(\omega)\,\mathbb{P}(d\omega) = \sum_{\substack{x_i\in A\\ \mathbb{P}(X=x_i)>0}} \mathbb{P}(X=x_i)\underbrace{\frac{1}{\mathbb{P}(X=x_i)}\int_{\{X=x_i\}} Y(\omega)\,\mathbb{P}(d\omega)}_{\varphi(x_i)}$$

$$=\sum_{\substack{x_i\in A\\ \mathbb{P}(X=x_i)>0}} \varphi(x_i)\,\mathbb{P}(X=x_i) = \int_A \varphi(t)\,\mathbb{P}_X(dt)$$

$$\varphi(t)=\begin{cases} \dfrac{1}{\mathbb{P}(X=t)}\displaystyle\int_{\{X=t\}} Y(\omega)\,\mathbb{P}(d\omega) & t\in X(\Omega)\text{ e } \mathbb{P}(X=t)>0 \\ 0 & \text{altrimenti}\end{cases}$$

$$\varphi_Y(\omega') = \mathbb{E}[Y \mid X = \omega']$$

$$\mathbb{E}[Y \mid B] = \frac{1}{\mathbb{P}(X=t)} \int_\Omega Y(\omega) \mathbb{1}_{\{X=t\}}(\omega) \mathbb{P}(d\omega)$$

dove $B := \{X = t\}$

CASO PARTICOLARE X, Y v.a. con $\mathbb{P}_{X,Y}$ A.C.

$A \in \mathcal{B}(\mathbb{R})$ $\qquad \Omega = \Omega' \quad X, Y : \Omega \to \mathbb{R}$

$$\mathbb{P}_{X,Y} = f(x,y)\,dx\,dy$$

$$\int_{X^{-1}(A)} Y(\omega) \mathbb{P}(d\omega) =$$

$$= \int_{(X,Y)^{-1}(A \times \mathbb{R})} Y(\omega) \mathbb{P}(d\omega) = \int_{A \times \mathbb{R}} y \, \mathbb{P}_{X,Y}(dx\,dy) = \int_{A \times \mathbb{R}} y f(x,y)\,dx\,dy =$$

$$= \int_{A \times \mathbb{R}} y \underbrace{\frac{f(x,y)}{f_X(x)}}_{h(y|x)} f_X(x)\,dx\,dy = \int_A \left(\int_\mathbb{R} y\, h(y|x)\,dy \right) f_X(x)\,dx$$

$$= \int_A \underbrace{\left(\int_\mathbb{R} y\, h(y|x)\,dy \right)}_{\varphi(x)} \mathbb{P}_X(dx) = \int_A \varphi(x) \mathbb{P}_X(dx)$$

$$\Rightarrow \mathbb{E}[Y \mid X = x] = \int_\mathbb{R} y\, h(y|x)\,dy$$

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FOGLIO 7 - EX 2

$$F_Y(t) = \mathbb{P}(Y \le t) \qquad \{Y \le t\} = \{Y \le t, X = 0\} \cup \{Y \le t, X = 1\}$$

$$= \mathbb{P}(Y \le t, X = 0) + \mathbb{P}(Y \le t, X = 1) =$$

$$= \mathbb{P}(Y \le t \mid X = 0)\, \mathbb{P}(X = 0) + \mathbb{P}(Y \le t \mid X = 1)\, \mathbb{P}(X = 1)$$

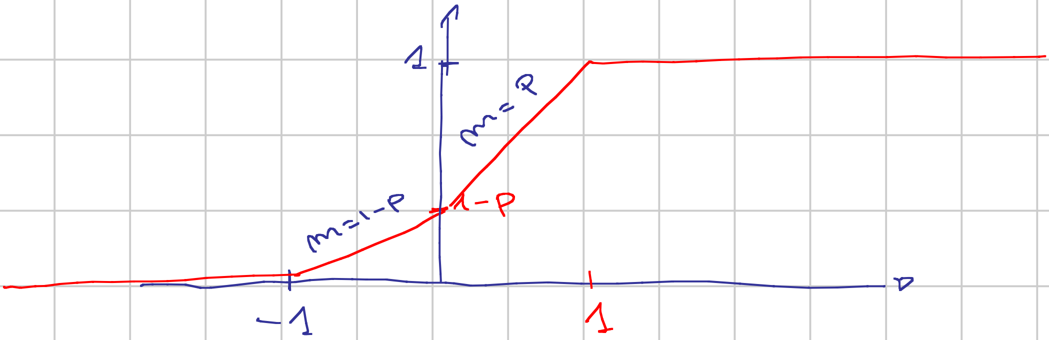
$$= (1-p)\, \mathbb{P}(Y \le t \mid X = 0) + p\, \mathbb{P}(Y \le t \mid X = 1)$$

$t < -1 \qquad F_Y(t) = 0$

$t \in [-1, 0) \qquad F_Y(t) = (1-p)(t+1) + 0 = (1-p)(t+1)$

$t \in [0, 1) \qquad F_Y(t) = (1-p)\,1 + pt = pt + 1 - p$

$t \geq 1 \qquad F_Y(t) = (1-p)\,1 + p \cdot 1 = 1 - p + p = 1$



$$f_Y(t) = \begin{cases} 0 & t<-1,\ t>1 \\ 1-p & t\in(-1,0) \\ p & t\in(0,1) \end{cases} \Rightarrow F_Y(t) = \int_{-\infty}^{t} f_Y(x)\,dx$$

$$\mathbb{P}_Y = (1-p)\,\mathcal{U}((-1,0)) + p\,\mathcal{U}((0,1))$$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} t f_Y(t)\,dt = \int_{-1}^{0} t(1-p)\,dt + \int_{0}^{1} tp\,dt = \frac{1-p}{2}t^2\Big|_{t=-1}^{t=0} + \frac{p}{2}t^2\Big|_{t=0}^{t=1} =$$

$$= \frac{-(1-p)}{2} + \frac{p}{2} = \frac{2p-1}{2}$$

$$\mathbb{E}[Y^2] = \int_{-1}^{0} t^2(1-p)\,dt + \int_{0}^{1} t^2 p\,dt \ \ldots\ldots = \frac{1}{3}$$

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

Calcolare $\mathbb{E}[Y \mid X=t]$