V.A. INDIPENDENZA

Titolo nota 16/11/2015

Distribuzione della somma di v.a. indipendenti

X, Y discrete con $X(\Omega), Y(\Omega) \subset \mathbb{Z}$

X, Y a distribuzione A.C.

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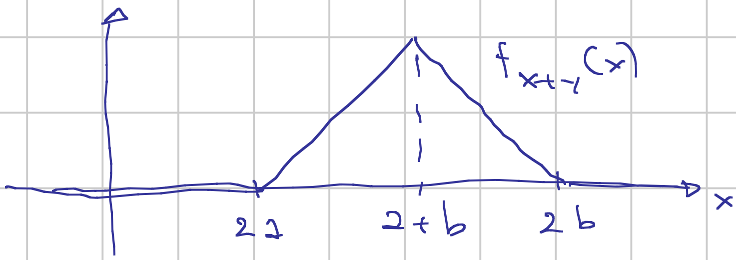
X, Y indipendenti $\mathbb{P}_X = \mathbb{P}_Y = U([a,b])$



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X e Y v.a. indipendenti $\mathbb{P}_X = N(m_1, \sigma_1^2)$

$\mathbb{P}_Y = N(m_2, \sigma_2^2)$

Allora $\mathbb{P}_{X+Y} = N(m_1+m_2, \sigma_1^2+\sigma_2^2)$

$$f(x) = \int_{\mathbb{R}} f_X(y) f_Y(x-y)\,dy = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-(y-m_1)^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(\frac{-(x-y-m_2)^2}{2\sigma_2^2}\right) dy$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{\mathbb{R}} \exp\left(\frac{-(y-m_1)^2}{2\sigma_1^2} - \frac{(x-y-m_2)^2}{2\sigma_2^2}\right) dy$$

Argomento dell'esponenziale

$$\frac{-1}{2\sigma_1^2\sigma_2^2}\left(\sigma_2^2(y-m_1)^2 + \sigma_1^2(y-(x-m_2))^2\right) =$$

$$= \frac{-1}{2\sigma_1^2\sigma_2^2}\left(y^2(\sigma_1^2+\sigma_2^2) - 2y\left(\sigma_2^2 m_1 + \sigma_1^2(x-m_2)\right) + \sigma_2^2 m_1^2 + \sigma_1^2(x-m_2)^2\right)$$

$$= \frac{-1}{2\sigma_1^2\sigma_2^2}\left(\left(y\sqrt{\sigma_1^2+\sigma_2^2}\right)^2 - 2\,y\sqrt{\sigma_1^2+\sigma_2^2}\;\frac{\sigma_2^2 m_1 + \sigma_1^2(x-m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right.$$

$$+\left(\frac{\sigma_2^2 m_1+\sigma_1^2(x-m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2$$

$$-\frac{\cancel{\sigma_2^4 m_1^2+\sigma_1^4(x-m_2)^2}+2\sigma_1^2\sigma_2^2 m_1(x-m_2)}{\sigma_1^2+\sigma_2^2}$$

$$+\frac{(\sigma_1^2+\cancel{\sigma_2^2})\sigma_2^2 m_1^2+(\cancel{\sigma_1^2}+\sigma_2^2)\sigma_1^2(x-m_2)^2}{\sigma_1^2+\sigma_2^2}\Bigg)$$

$$=\frac{-1}{2\sigma_1^2\sigma_2^2}\left(y\sqrt{\sigma_1^2+\sigma_2^2}-\frac{\sigma_2^2 m_1+\sigma_1^2(x-m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2$$

$$-\frac{1}{2\cancel{\sigma_1^2}\cancel{\sigma_2^2}}\left(\frac{\cancel{\sigma_1^2}\cancel{\sigma_2^2}}{\sigma_1^2+\sigma_2^2}\left(-2m_1(x-m_2)+m_1^2+(x-m_2)^2\right)\right)$$

$$=\frac{-1}{2\sigma_1^2\sigma_2^2}(\sigma_1^2+\sigma_2^2)\left(y-\underbrace{\boxed{\frac{\sigma_2^2 m_1+\sigma_1^2(x-m_2)}{\sigma_1^2+\sigma_2^2}}}_{:=a(x)}\right)^2$$

$$-\frac{1}{2(\sigma_1^2+\sigma_2^2)}\left(x-m_1-m_2\right)^2$$

$$f(x)=\frac{1}{2\pi\sigma_1\sigma_2}\exp\left(\frac{-(x-m_1-m_2)^2}{\sigma_1^2+\sigma_2^2}\right)\cdot\int_{\mathbb{R}}\exp\left(\frac{-(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2}\left(y-a(x)\right)^2\right)dy$$

$$t=y-a(x) \qquad dt=dy \qquad y\to-\infty \quad t\to-\infty$$

$$y\to+\infty \quad t\to+\infty$$

$$f(x)=\frac{1}{2\pi\sigma_1\sigma_2}\exp\left(\frac{-(x-m_1-m_2)^2}{\sigma_1^2+\sigma_2^2}\right)\int_{\mathbb{R}}\exp\left(\frac{-(\sigma_1^2+\sigma_2^2)t^2}{2\sigma_1^2\sigma_2^2}\right)dt$$

$$u=\frac{t\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2}\,\sigma_1\sigma_2} \qquad dt=\frac{\sigma_1\sigma_2\sqrt{2}}{\sqrt{\sigma_1^2+\sigma_2^2}}\,du$$

$$f(x)=\frac{1}{2\pi\cancel{\sigma_1}\cancel{\sigma_2}}\exp\left(\frac{-(x-(m_1+m_2))^2}{\sigma_1^2+\sigma_2^2}\right)\frac{\cancel{\sigma_1}\cancel{\sigma_2}\sqrt{2}}{\sqrt{\sigma_1^2+\sigma_2^2}}\overbrace{\int_{\mathbb{R}}\exp(-u^2)\,du}^{=\sqrt{\pi}}$$

$$=\frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\exp\left(\frac{-(x-(m_1+m_2))^2}{\sigma_1^2+\sigma_2^2}\right) \qquad \mathbb{P}_{X+Y}=\mathcal{N}(m_1+m_2,\sigma_1^2+\sigma_2^2)$$

X, Y indipendenti

$\mathbb{P}_X = \text{Poisson}(\lambda) \qquad \mathbb{P}_Y = \text{Poisson}(\mu)$

$\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad \mathbb{P}(Y=k) = e^{-\mu}\frac{\mu^k}{k!} \qquad k=0,1,2,\ldots\ldots$

$X(\Omega) = Y(\Omega) \subseteq \mathbb{Z}$

$\forall k \in \mathbb{Z} \qquad \mathbb{P}(X+Y=k) = \sum_{j=-\infty}^{+\infty} \mathbb{P}(X=j)\mathbb{P}(Y=k-j)$

$= \sum_{j=0}^{+\infty} \mathbb{P}(X=j)\mathbb{P}(Y=k-j) \qquad k-j \geq 0 \qquad j \leq k$

$= \sum_{j=0}^{k} \mathbb{P}(X=j)\mathbb{P}(Y=k-j) = \sum_{j=0}^{k} e^{-\lambda}\frac{\lambda^j}{j!} e^{-\mu}\frac{\mu^{k-j}}{(k-j)!}\frac{k!}{k!}$

$= \frac{1}{k!} e^{-\lambda-\mu} \underbrace{\sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j}}_{=(\lambda+\mu)^k} = \frac{1}{k!} e^{-(\lambda+\mu)} (\lambda+\mu)^k$

$\mathbb{P}_{X+Y} = \text{Poisson}(\lambda+\mu)$

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X, Y v.a. indipendenti $\qquad \mathbb{P}_X = \exp(\lambda) \qquad \mathbb{P}_Y = \exp(\mu)$

$Z := \min\{X, Y\}$

$Z = \varphi \circ (X,Y) \qquad \varphi : (x,y) \in \mathbb{R}^2 \longmapsto \min\{x,y\} \in \mathbb{R}$

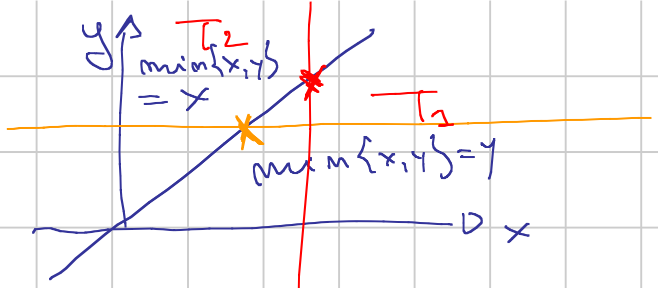
$f : \mathbb{R} \to \mathbb{R}_+$ funzione di Borel

$$\int_{\mathbb{R}} f(t)\,\mathbb{P}_Z(dt) = \int_{\mathbb{R}} f(t)\,\mathbb{P}_{\varphi\circ(X,Y)}(dt) = \int_{\mathbb{R}^2} f(\varphi(x,y))\,\mathbb{P}_{X,Y}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} f(\min\{x,y\})\,\mathbb{P}_X(dx)\,\mathbb{P}_Y(dy) = \int_{\mathbb{R}^2} f(\min\{x,y\}) f_X(x) f_Y(y)\,dx\,dy$$

$f_X(x) = \lambda e^{-\lambda x}\mathbb{1}_{(0,+\infty)}(x) \qquad f_Y(y) = \mu e^{-\mu y}\mathbb{1}_{(0,+\infty)}(y)$

$$= \int_{(0,+\infty)^2} f(\min\{x,y\})\,\lambda e^{-\lambda x}\mu e^{-\mu y}\,dx\,dy$$



$$= \int_{T_1} f(y)\,\lambda e^{-\lambda x}\mu e^{-\mu y}\,dx\,dy + \int_{T_2} f(x)\,\lambda e^{-\lambda x}\mu e^{-\mu y}\,dx\,dy$$

$$= \int_0^{+\infty} dy \left(\int_y^{+\infty} f(y)\,\lambda e^{-\lambda x}\mu e^{-\mu y}\,dx \right) + \int_0^{+\infty} dx \left(\int_x^{+\infty} f(x)\,\lambda e^{-\lambda x}\mu e^{-\mu y}\,dy \right)$$

$$= \int_0^{+\infty} f(y)\,\mu e^{-\mu y} \left(\int_y^{+\infty} \lambda e^{-\lambda x}\,dx \right) dy + \int_0^{+\infty} f(x)\,\lambda e^{-\lambda x} \left(\int_x^{+\infty} \mu e^{-\mu y}\,dy \right) dx$$

$$= \int_0^{+\infty} f(y)\,\mu e^{-\mu y} \left(-e^{-\lambda x} \right)_{x=y}^{x\to+\infty} dy + \int_0^{+\infty} f(x)\,\lambda e^{-\lambda x} \left(-e^{-\mu y} \right)_{y=x}^{y\to+\infty} dx$$

$$= \int_0^{+\infty} f(y)\,\mu e^{-\mu y} e^{-\lambda y}\,dy + \int_0^{+\infty} f(x)\,\lambda e^{-\lambda x} e^{-\mu x}\,dx$$

$$= \int_0^{+\infty} f(t)\, e^{-(\mu+\lambda)t} (\mu+\lambda)\,dt = \int_{\mathbb{R}} f(t) \boxed{(\mu+\lambda) e^{-(\mu+\lambda)t}\, \mathbb{1}_{(0,+\infty)}(t)}\, dt$$

$$\mathbb{P}_{\min\{X,Y\}} = \exp(\lambda+\mu)$$

X, Y indipendenti $\quad \mathbb{P}_X = \Gamma(\alpha,\lambda) \quad$ N.B. $\Gamma(1,\lambda) = \exp(\lambda)$

$\qquad\qquad\qquad\quad \mathbb{P}_Y = \Gamma(\beta,\lambda)$

$$\mathbb{P}_X = f(x)\,dx \qquad f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x>0 \\ 0 & x \le 0 \end{cases}$$

$$\mathbb{P}_Y = g(x)\,dx \qquad g(x) = \begin{cases} \dfrac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x} & x>0 \\ 0 & x \le 0 \end{cases}$$

$$\mathbb{P}_{X+Y} = h(x)\,dx$$

$$h(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy = \int_0^x f(y)\, g(x-y)\, dy \qquad \begin{cases} x-y \geq 0 & y \leq x \\ y \geq 0 \end{cases} \qquad 0 \leq y \leq x$$

$$= \int_0^x \frac{\lambda^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} \cancel{e^{-\lambda y}} \frac{\lambda^{\beta}}{\Gamma(\beta)} (x-y)^{\beta-1} e^{-\lambda(x-\cancel{y})}\, dy$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} \int_0^x y^{\alpha-1} (x-y)^{\beta-1}\, dy \qquad y = tx \qquad t = \frac{y}{x}$$

$$dy = x\, dt$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} \int_0^1 t^{\alpha-1} x^{\alpha-\cancel{1}} (x(1-t))^{\beta-1} \cancel{x}\, dt$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda x} x^{\alpha+\beta-1} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1}\, dt$$

$$= \frac{\lambda^{\alpha+\beta}}{\cancel{\Gamma(\alpha)\Gamma(\beta)}} e^{-\lambda x} x^{\alpha+\beta-1} \frac{\cancel{\Gamma(\alpha)\Gamma(\beta)}}{\Gamma(\alpha+\beta)} = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} e^{-\lambda x} x^{\alpha+\beta-1}$$

$$h(x) = \begin{cases} \dfrac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} e^{-\lambda x} x^{\alpha+\beta-1} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\mathbb{P}_{X+Y} = \Gamma(\alpha+\beta, \lambda)$$

$$\mathbb{P}_X = \mathbb{P}_Y = \exp(\lambda) = \Gamma(1, \lambda) \qquad \mathbb{P}_{X+Y} = \Gamma(2, \lambda)$$

$X_1 \ldots X_n$ indipendenti $\quad \mathbb{P}_{X_i} = \exp(\lambda) \quad \forall i = 1 \ldots n$

$$\mathbb{P}_{X_1 + \cdots + X_n} = \Gamma(n, \lambda)$$

X, Y v.a. scalari t.c. $\mathbb{P}_{X,Y}$ è la distribuzione uniforme su $Q=[0,1]^2$

Calcolare la distribuzione della v.a. $Z=XY$

$$f(x,y) = \begin{cases} 1 & (x,y)\in[0,1]^2 \\ 0 & \text{altrimenti} \end{cases} = \mathbb{1}_{(0,1)^2}(x,y)$$

$$= \mathbb{1}_{(0,1)}(x)\,\mathbb{1}_{(0,1)}(y)$$

$$Z = \varphi\circ(X,Y) \qquad \varphi:(x,y)\in\mathbb{R}^2 \mapsto xy\in\mathbb{R}$$

$$\int_{\mathbb{R}} \psi(t)\,\mathbb{P}_Z(dt) = \int_{\mathbb{R}} \psi(t)\,\mathbb{P}_{\varphi\circ(X,Y)}(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\,\mathbb{P}_{X,Y}(dx\,dy)$$

$$= \int_{\mathbb{R}^2} \psi(xy)\,\mathbb{1}_{(0,1)^2}(x,y)\,dx\,dy = \int_{(0,1)^2} \psi(xy)\,dx\,dy =$$

$$= \int_0^1 dx \left(\int_0^1 \psi(xy)\,dy\right) \qquad t=xy \qquad y=\frac{t}{x} \qquad dy=\frac{1}{x}\,dt$$

$$y=0 \quad t=0$$

$$y=1 \quad t=x$$

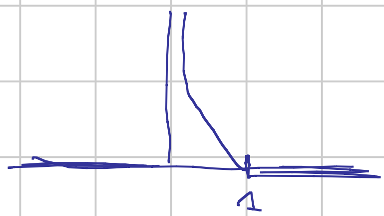
$$= \int_0^1 dx \int_0^x \psi(t)\,\frac{1}{x}\,dt$$

$$= \int_0^1 dt \int_t^1 \psi(t)\,\frac{1}{x}\,dt =$$

$$= \int_0^1 \psi(t)\left(\int_t^1 \frac{1}{x}\,dx\right)dt = \int_0^1 \psi(t)\,\log(x)\Big|_{x=t}^{x=1}\,dt =$$

$$= \int_0^1 \psi(t)\,(-\log t)\,dt = \int_{\mathbb{R}} \psi(t)\,g(t)\,dt$$

$$g(t) = \begin{cases} -\log(t) & t\in(0,1) \\ 0 & \text{altrimenti} \end{cases}$$



SPERANZE CONDIZIONATE

Sia $(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato e sia $B \in \mathcal{E}$ t.c. $\mathbb{P}(B) > 0$

Definisco $\mathbb{P}_B : A \in \mathcal{E} \mapsto \mathbb{P}(A|B) \in \mathbb{R}$

Allora $(\Omega, \mathcal{E}, \mathbb{P}_B)$ è ancora uno spazio probabilizzato.

Se ho $X : \Omega \to \mathbb{R}$ v.a. calcolare $\int_\Omega X(\omega)\,\mathbb{P}_B(d\omega)$

1) Supponiamo X v.a. semplice

$X = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$ $\qquad 0 \le c_1 < c_2 < \dots < c_n$

$\{E_i\}_{i=1}^n$ sono una partizione di Ω

$$\int_\Omega X(\omega)\,\mathbb{P}_B(d\omega) = \sum_{i=1}^n c_i \mathbb{P}_B(E_i) = \sum_{i=1}^n c_i \frac{\mathbb{P}(E_i \cap B)}{\mathbb{P}(B)} =$$

$$= \frac{1}{\mathbb{P}(B)} \sum_{i=1}^n c_i \mathbb{P}(E_i \cap B) \qquad \mathbb{P}(E_i) = \int_\Omega \mathbb{1}_{E_i \cap B}(\omega)\,\mathbb{P}(d\omega)$$

$$= \frac{1}{\mathbb{P}(B)} \sum_{i=1}^n c_i \int_\Omega \mathbb{1}_{E_i \cap B}(\omega)\,\mathbb{P}(d\omega) = \frac{1}{\mathbb{P}(B)} \int_\Omega \underbrace{\sum_{i=1}^n c_i \mathbb{1}_{E_i \cap B}(\omega)}_{X\mathbb{1}_B}\,\mathbb{P}(d\omega)$$

$$= \frac{1}{\mathbb{P}(B)} \mathbb{E}[X \mathbb{1}_B]$$

$\int_\Omega X(\omega)\,\mathbb{P}_B(d\omega) = \mathbb{E}_B[X]$ speranza condizionata di X dato l'evento B

$$\mathbb{E}_B[X] = \frac{1}{\mathbb{P}(B)} \mathbb{E}[X \mathbb{1}_B]$$

2) X v.a. non negativa

$\exists \{\varphi_k\}_{k=1}^\infty$ successione monotone crescente di v.a semplici

t.c. $\lim_{k \to \infty} \varphi_k(\omega) = X(\omega) \quad \forall \omega \in \Omega$

$$\mathbb{E}_B[\varphi_k] = \int_\Omega \varphi_k(\omega)\,\mathbb{P}_B(d\omega) \longrightarrow \int_\Omega X(\omega)\,\mathbb{P}_B(d\omega)$$

$$\mathbb{E}[\varphi_k \mathbb{1}_B] = \int_\Omega \varphi_k(\omega) \mathbb{1}_B(\omega)\,\mathbb{P}(d\omega)$$

$$\varphi_k(\omega) = (\varphi_k \cdot \mathbb{1}_B)(\omega) \longrightarrow (X \mathbb{1}_B)(\omega) \quad \forall \omega \in \Omega$$

$$\int_\Omega (\varphi_k \mathbb{1}_B)(\omega) \longmapsto \int_\Omega (X \mathbb{1}_B)(\omega) \mathbb{P}(d\omega) = \mathbb{E}[X \mathbb{1}_B]$$

$$\mathbb{E}_B[X] \longleftarrow \mathbb{E}_B[\varphi_k] = \frac{1}{\mathbb{P}(B)} \mathbb{E}[\varphi_k \mathbb{1}_B] \longrightarrow \frac{1}{\mathbb{P}(B)} \mathbb{E}[X \mathbb{1}_B]$$

2) Se X ha segno variabile e integrabile si considerano X^+ e $X^+ \mathbb{1}_B$, X^- e $X^- \mathbb{1}_B$

e si ritrova la stessa formula.