V.A. INDIPENDENZI

Titolo nota — 13/11/2015

$X_1 \dots X_n$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$X_1 \dots X_n$ si dicono indipendenti se

$\forall A_1 \dots A_n$ boreliani gli insiemi

$E_1 = X_1^{-1}(A_1)$, $E_2 = X_2^{-1}(A_2)$ ... $E_n = X_n^{-1}(A_n)$

sono eventi indipendenti di $(\Omega, \mathcal{E}, \mathbb{P})$

PROP Siano $X_1 \dots X_n$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$.

Allora $X_1 \dots X_n$ sono v.a. indipendenti SSE

$$\mathbb{P}_{X_1 \dots X_n} = \mathbb{P}_{X_1} \times \mathbb{P}_{X_2} \times \dots \times \mathbb{P}_{X_n}$$

DIM. Per semplicità supponiamo $X_i : \Omega \to \mathbb{R}$ $\forall i = 1 \dots n$

Supponiamo che $X_1 \dots X_n$ siano indipendenti.

Siano $A_1 \dots A_n \in \mathcal{B}(\mathbb{R})$

$\mathbb{P}_{X_1 \dots X_n}(A_1 \times A_2 \dots \times A_n) = \mathbb{P}((X_1 \dots X_n) \in A_1 \times A_2 \dots \times A_n) =$

$= \mathbb{P}(X_1 \in A_1, X_2 \in A_2, \dots X_n \in A_n)$

$= \mathbb{P}(X_1^{-1}(A_1) \cap X_2^{-1}(A_2) \cap \dots \cap X_n^{-1}(A_n)) =$

$= \mathbb{P}(X_1^{-1}(A_1)) \mathbb{P}(X_2^{-1}(A_2)) \dots \mathbb{P}(X_n^{-1}(A_n)) =$

$= \mathbb{P}_{X_1}(A_1) \mathbb{P}_{X_2}(A_2) \dots \mathbb{P}_{X_n}(A_n)$ e quindi

$\mathbb{P}_{X_1 \dots X_n} = \mathbb{P}_{X_1} \times \mathbb{P}_{X_2} \times \dots \times \mathbb{P}_{X_n}$.

2) $A_1 \dots A_n \in \mathcal{B}(\mathbb{R})$ $E_i := X_i^{-1}(A_i)$ $i = 1 \dots n$

Fisso $k = 2, \dots, n$ $\{E_{r(1)} \dots E_{r(k)}\} \subseteq \{E_1 \dots E_n\}$

Senza perdere in generalità posso supporre $\{E_{r(1)} \dots E_{r(k)}\} = \{E_1 \dots E_k\}$

Devo far vedere che $\mathbb{P}(E_1 \cap E_2 \dots \cap E_k) = \prod_{i=1}^{k} \mathbb{P}(E_i)$

$$E_1 \cap E_2 \dots \cap E_k = X_1^{-1}(A_1) \cap X_2^{-1}(A_2) \dots \cap X_k^{-1}(A_k) \cap \underbrace{\Omega \cap \Omega \dots \cap \Omega}_{n-k \text{ volte}}$$

$$\Omega = X_j^{-1}(\mathbb{R}) \quad j = k+1 \dots n$$

$$\mathbb{P}\left(\bigcap_{i=1}^{k} E_i\right) = \mathbb{P}\left(X_1^{-1}(A_1) \cap \dots \cap X_k^{-1}(A_k) \cap X_{k+1}^{-1}(\mathbb{R}) \dots \cap X_n^{-1}(\mathbb{R})\right)$$

$$= \mathbb{P}_{X_1 \dots X_n}(A_1 \times A_2 \dots \times A_k \times \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n-k \text{ volte}})$$

$$= \mathbb{P}_{X_1}(A_1)\, \mathbb{P}_{X_2}(A_2) \dots \mathbb{P}_{X_k}(A_k)\, \underbrace{\mathbb{P}_{X_{k+1}}(\mathbb{R})}_{=1} \dots \underbrace{\mathbb{P}_{X_n}(\mathbb{R})}_{=1}$$

$$= \mathbb{P}(E_1)\, \mathbb{P}(E_2) \dots \mathbb{P}(E_k)$$

V.A. DISCRETE

$X, Y : \Omega \to \mathbb{R}$ v.a. discrete indipendenti

$X(\Omega) = \{x_i\}_{i \in I}$ $\qquad (X,Y)(\Omega) \subseteq X(\Omega) \times Y(\Omega)$

$Y(\Omega) = \{y_j\}_{j \in J}$ $\qquad p_i := \mathbb{P}(X = x_i)$

$\qquad q_j := \mathbb{P}(Y = y_j)$

$\forall (x_i, y_j) \in X(\Omega) \times Y(\Omega)$

$$p_{ij} := \mathbb{P}(X = x_i, Y = y_j) = \mathbb{P}(X = x_i)\,\mathbb{P}(Y = y_j) = p_i q_j$$

X, Y v.a. indipendenti con $\mathbb{P}_{X,Y}$ A.C.

$$\exists f : \mathbb{R}^2 \to \overline{\mathbb{R}}_+ \quad : \quad \mathbb{P}_{X,Y}(A) = \int_A f(x,y)\,dx\,dy \quad \forall A \in \mathcal{B}(\mathbb{R}^2)$$

Abbiamo visto che anche $\mathbb{P}_X$ e $\mathbb{P}_Y$ sono A.C.

e
$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy \qquad f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

$\forall \varphi : \mathbb{R}^2 \to \mathbb{R}_+$ di Borel

$$\int_{\mathbb{R}^2} \varphi(x,y)\, \mathbb{P}_{X,Y}(dx\,dy) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y)\, \mathbb{P}_Y(dy) \right) \mathbb{P}_X(dx)$$

$$\int_{\mathbb{R}^2} \varphi(x,y)\, f(x,y)\,dx\,dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y)\, f_Y(y)\,dy \right) f_X(x)\,dx$$

$$= \int_{\mathbb{R}^2} \varphi(x,y) f_X(x) f_Y(y)\, dx\, dy$$

$$\Rightarrow f(x,y) = f_X(x) f_Y(y)$$

X, Y v.a. indipendenti t.c. $\mathbb{P}_X$ e $\mathbb{P}_Y$ sono distribuzioni A.C.

$\varphi : \mathbb{R}^2 \to \mathbb{R}_+$ f. Borel

$$\int_{\mathbb{R}^2} \varphi(x,y)\, \mathbb{P}_{X,Y}(dx\, dy) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y)\, \mathbb{P}_Y(dy) \right) \mathbb{P}_X(dx)$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(x,y) f_Y(y)\, dy \right) f_X(x)\, dx$$

$$= \int_{\mathbb{R}^2} \varphi(x,y) f_X(x) f_Y(y)\, dx\, dy$$

$\Rightarrow \mathbb{P}_{X,Y}$ è A.C. con densità $f(x,y) = f_X(x) f_Y(y)$

— o —

$\mathrm{Var}[X+Y]$

$\mathrm{Var}[\alpha X + \beta] = \alpha^2 \mathrm{Var}[X] \quad \forall \alpha, \beta \in \mathbb{R}$

$$\mathrm{Var}[X+Y] = \mathbb{E}\left[\left((X+Y) - \mathbb{E}[X+Y]\right)^2\right] =$$

$$= \mathbb{E}\left[\left((X - \mathbb{E}[X]) + (Y - \mathbb{E}[Y])\right)^2\right]$$

$$= \mathbb{E}\left[(X - \mathbb{E}[X])^2 + (Y - \mathbb{E}[Y])^2 + 2(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$

$$= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y)$$

$X_1 \dots X_n$

$$\mathrm{Var}[X_1 + \dots + X_n] = \sum_{i=1}^n \mathrm{Var}[X_i] + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^n \mathrm{Var}[X_i] + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

ESERCIZIO

Tiro un dado equilibrato ed una moneta equilibrata.
Calcolare la probabilità che la 1ª Testa ed il 1° 6 arrivino allo stesso lancio;
calcolare la probabilità che la 1ª Testa arrivi prima del 1° 6

— o —

T = v.a. che mi dice a quale lancio ottengo la 1ª Testa.

$T(\Omega) = 1, 2, 3, \dots \qquad \mathbb{P}(T=k) = p(1-p)^{k-1} = \frac{1}{2}\left(1-\frac{1}{2}\right)^{k-1} = \left(\frac{1}{2}\right)^k$

S = v.a. che mi dice a quale lancio ottengo il 1° 6

$S(\Omega) = 1, 2, 3, \dots \qquad \mathbb{P}(S=k) = p(1-p)^{k-1} = \frac{1}{6}\left(1-\frac{1}{6}\right)^{k-1} = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}$

$\mathbb{P}(T=S) \qquad \{T=S\} = \bigcup_{k=1}^{\infty}\{T=k, S=k\}$

$$\mathbb{P}(T=S) = \sum_{k=1}^{\infty}\mathbb{P}(T=k, S=k) = \sum_{k=1}^{\infty}\mathbb{P}(T=k)\mathbb{P}(S=k) =$$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} = \frac{1}{12}\sum_{k=1}^{\infty}\left(\frac{1}{2}\cdot\frac{5}{6}\right)^{k-1} \qquad j = k-1$$

$$= \frac{1}{12}\sum_{j=0}^{\infty}\left(\frac{5}{12}\right)^j = \frac{1}{12}\,\frac{1}{1-\frac{5}{12}} = \frac{1}{12}\,\frac{1}{7/12} = \frac{1}{7}$$

— o —

$\mathbb{P}(T<S) \qquad \{T<S\} = \bigcup_{k=1}^{\infty}\{T=k, S>k\}$

$$\mathbb{P}(T<S) = \mathbb{P}\left(\bigcup_{k=1}^{\infty}\{T=k, S>k\}\right) = \sum_{k=1}^{\infty}\mathbb{P}(T=k, S>k) =$$

$$= \sum_{k=1}^{\infty}\mathbb{P}(T=k)\,\mathbb{P}(S>k) = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \sum_{j=k+1}^{\infty}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{j-1} = \bigstar$$

$\{S>k\} = \bigcup_{j=k+1}^{\infty}\{S=j\} \qquad \mathbb{P}(S>k) = \sum_{j=k+1}^{\infty}\mathbb{P}(S=j)$

$$(*)\quad \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \sum_{\ell=0}^{\infty}\left(\frac{5}{6}\right)^{\ell+k}$$

$$\ell = j-(k+1)$$
$$j = \ell+k+1$$

$$= \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\left(\frac{5}{6}\right)^k \sum_{\ell=0}^{\infty}\left(\frac{5}{6}\right)^{\ell}$$

$$= \cancel{\frac{1}{6}}\sum_{k=1}^{\infty}\left(\frac{5}{12}\right)^k \frac{1}{\cancel{1-5/6}} = \frac{1}{1-\frac{5}{12}} - 1 = \frac{12}{7}-1 = \frac{5}{7}$$

SOMMA DI v.A. INDIPENDENTI

1° CASO X,Y discrete con $X(\Omega), Y(\Omega) \subseteq \mathbb{Z}$

Sia $\{p_k\}_{k\in\mathbb{Z}}$ la densità di X : $p_k = \mathbb{P}(X=k)$

Sia $\{q_k\}_{k\in\mathbb{Z}}$ la densità di Y : $q_k = \mathbb{P}(Y=k)$

$$\{X+Y=k\} = \bigcup_{j\in\mathbb{Z}}\{X=j, Y=k-j\}$$

$$\mathbb{P}(X+Y=k) = \sum_{j\in\mathbb{Z}}\mathbb{P}(X=j, Y=k-j) = \sum_{j\in\mathbb{Z}}\mathbb{P}(X=j)\mathbb{P}(Y=k-j)$$

$$\mathbb{P}(X+Y=k) = \sum_{j\in\mathbb{Z}} p_j q_{k-j}$$

2° CASO X,Y v.a. indipendenti entrambe con distribuzione A.C.

Sia $f(x)$ la densità di X : $\mathbb{P}_X = f(x)dx$

Sia $g(x)$ la densità di Y : $\mathbb{P}_Y = g(x)dx$

Sia $\psi: \mathbb{R}\to\mathbb{R}_+$ di Borel

Sia $\varphi: (x,y)\in\mathbb{R}^2 \mapsto x+y\in\mathbb{R} \Rightarrow X+Y = \varphi\circ(X,Y)$

$$\int_{\mathbb{R}}\psi(t)\,\mathbb{P}_{X+Y}(dt) = \int_{\mathbb{R}}\psi(t)\,\mathbb{P}_{\varphi\circ(X,Y)}(dt) =$$

$$= \int_{\mathbb{R}^2}\psi(\varphi(x,y))\,\mathbb{P}_{(X,Y)}(dx\,dy) = \int_{\mathbb{R}^2}\psi(x+y)f(x)g(y)\,dx\,dy$$

$$s = x \qquad x = s$$
$$t = x+y \qquad y = t-s \qquad J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \int_{\mathbb{R}^2} \varphi(t) f(s) g(t-s)\, ds\,dt =$$

$$= \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} f(s) g(t-s)\, ds \right)}\varphi(t)\, dt$$

$X+Y$ ha distribuzione A.C. con densità $h(t) = \int_{\mathbb{R}} f(s) g(t-s)\, ds$

— o —

X, Y v.a. scalari con densità congiunta

$$f(x,y) = \begin{cases} c(x-y)^2 & (x,y) \in (0,1)^2 \\ 0 & \text{altrimenti} \end{cases}$$

Determinare c e calcolare la densità congiunta di X^2 e X^2Y^2

$f \geq 0 \qquad c \geq 0$

$$\int_{\mathbb{R}^2} f(x,y)\, dx\,dy = 1$$

$$= \int_{(0,1)^2} c(x-y)^2\, dx\,dy = c \int_0^1 dy \int_0^1 (x-y)^2\, dx$$

$$= c \int_0^1 \frac{1}{3}(x-y)^3 \Big|_{x=0}^{x=1} = \frac{c}{3} \int_0^1 \left[(1-y)^3 - (-y)^3 \right] dy =$$

$$= \frac{c}{3} \int_0^1 \left[y^3 - (y-1)^3 \right] dy = \frac{c}{3} \frac{1}{4} \left[y^4 - (y-1)^4 \right] \Big|_{y=0}^{y=1} =$$

$$= \frac{c}{12} \left[1+1 \right] = \frac{c}{6} = 1 \quad \text{SSE} \quad \boxed{c=6}$$

$$(X^2, X^2Y^2) \qquad \varphi_0(X,Y)$$

$$\varphi : (x,y) \in \mathbb{R}^2 \mapsto (x^2, x^2y^2) \in \mathbb{R}^2$$

$\psi : \mathbb{R}^2 \to \mathbb{R}_+$ di Borel

$$\int_{\mathbb{R}^2} \psi(s,t)\, \mathbb{P}_{X^2, X^2Y^2}(ds\,dt) = \int_{\mathbb{R}^2} \psi(s,t)\, \mathbb{P}_{\varphi_0(X,Y)}(ds\,dt) =$$

$$= \int_{\mathbb{R}^2} \psi(\varphi(x,y))\, \mathbb{P}_{X,Y}(dx\,dy) = \int_{\mathbb{R}^2} \psi(x^2, x^2y^2)\, f(x,y)\, dx\,dy =$$

$$= \int_{(0,1)^2} \psi(x^2, x^2y^2)\, 6(x-y)^2\, dx\,dy$$

$s = x^2 \qquad x \in (0,1) \qquad x = s^{1/2}$

$t = x^2y^2 \qquad y \in (0,1) \qquad t = sy^2 \qquad y^2 = ts^{-1}$

$x = s^{1/2} \qquad 0 < s < 1$

$y = t^{1/2} s^{-1/2} \qquad t = x^2y^2 = sy^2 < s$

$0 < t < s < 1$

$$J = \begin{pmatrix} \frac{1}{2} s^{-1/2} & 0 \\ \cancel{x} & \frac{1}{2} t^{-1/2} s^{-1/2} \end{pmatrix} \qquad \det J = \frac{1}{4} t^{-1/2} s^{-1}$$

$$= \int_T \psi(s,t)\, \overset{3}{\cancel{6}}\, (s^{1/2} - t^{1/2} s^{-1/2})^2\, \frac{1}{\underset{2}{\cancel{4}}}\, t^{-1/2} s^{-1}\, ds\,dt$$

$$\mathbb{P}_{X^2, X^2Y^2} = g(s,t)\, ds\,dt$$

$$g(s,t) = \begin{cases} \frac{3}{2} t^{-1/2} s^{-1} (s^{1/2} - t^{1/2} s^{-1/2})^2 & 0 < t < s < 1 \\ 0 & \text{altrimenti.} \end{cases}$$

X, Y v.a. indipendenti $\mathbb{P}_X = \mathbb{P}_Y = U((a,b))$

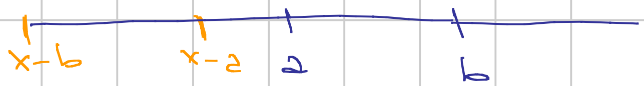
$$\mathbb{P}_X = \mathbb{P}_Y = \frac{1}{b-a}\mathbb{1}_{(a,b)}(x)\,dx \qquad \frac{1}{b-a}\mathbb{1}_{(a,b)}(x) = f(x)$$

$$\mathbb{P}_{X+Y} = h(x)\,dx \qquad h(x) = \int_{\mathbb{R}} f(y)f(x-y)\,dy =$$

$$= \int_{\mathbb{R}} \frac{1}{(b-a)^2}\mathbb{1}_{(a,b)}(y)\,\mathbb{1}_{(a,b)}(x-y)\,dy$$

$$= \frac{1}{(b-a)^2}\int_a^b \mathbb{1}_{(a,b)}(x-y)\,dy \qquad x-y=u \qquad y=x-u$$

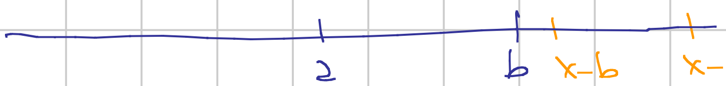
$$= \frac{1}{(b-a)^2}\int_{x-a}^{x-b} \mathbb{1}_{(a,b)}(u)\,(-1)\,du = \frac{1}{(b-a)^2}\int_{x-b}^{x-a}\mathbb{1}_{(a,b)}(u)\,du$$



$x-a<a \Rightarrow h(x)=0$

$a<x-a<b \qquad \frac{1}{(b-a)^2}\int_a^{x-a} 1\,du = \frac{x-2a}{(b-a)^2}$

$x-b<b<x-a \qquad \frac{1}{(b-a)^2}\int_{x-b}^{b} 1\,du = \frac{2b-x}{(b-a)^2}$



$x-b>b \qquad h(x)=0$

$$h(x) = \begin{cases} 0 & x<2a \vee x>2b \\ \dfrac{x-2a}{(b-a)^2} & 2a<x<a+b \\ \dfrac{2b-x}{(b-a)^2} & a+b<x<2b \end{cases}$$