Titolo nota 02/11/2015

FOGLIO 5 - EX 9

$Y = f \circ X \qquad f(s) = a|s-\mu|$

φ funzione di Borel nonnegativa

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} \varphi(f(s))\, \mathbb{P}_X(ds) =$$

$$= \int_{\mathbb{R}} \varphi(a|s-\mu|) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(s-\mu)^2}{2\sigma^2}\right) ds$$

$x = s-\mu$

$$= \int_{\mathbb{R}} \varphi(a|x|) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

$$= 2\int_0^{+\infty} \varphi(ax) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

$y = ax \qquad x = \frac{1}{a} y \qquad dx = \frac{1}{a} dy$

$$= 2\int_0^{+\infty} \varphi(y) \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{a} \exp\left(\frac{-y^2}{2a^2\sigma^2}\right) dy$$

$$= \int_{\mathbb{R}} \varphi(y) g(y)\, dy \qquad g(y) = \begin{cases} 0 & y \le 0 \\ \sqrt{\frac{2}{\pi\sigma^2 a^2}} \exp\left(\frac{-y^2}{2a^2\sigma^2}\right) & y > 0 \end{cases}$$

$$E[Y] = \int_{\mathbb{R}} y g(y)\, dy = \int_0^{+\infty} \sqrt{\frac{2}{\pi\sigma^2 a^2}}\, y \exp\left(\frac{-y^2}{2a^2\sigma^2}\right) dy$$

$$= \sqrt{\frac{2}{\pi\sigma^2 a^2}} (-a^2\sigma^2) \int_0^{+\infty} \frac{-y}{a^2\sigma^2} \exp\left(\frac{-y^2}{2a^2\sigma^2}\right) dy$$

$$= \sqrt{\frac{2}{\pi}} (-a\sigma) \exp\left(\frac{-y^2}{2a^2\sigma^2}\right)\Bigg|_{y=0}^{y\to+\infty} = \sqrt{\frac{2}{\pi}}\, a\sigma$$

Foglio 5 - Ex 10

$X = \mu + \sigma X_0 \qquad \mathbb{P}_{X_0} = N(0,1)$

$\mathbb{P}(|X-\mu| \le 1) = \mathbb{P}(|\mu+\sigma X_0 - \mu| \le 1) = \mathbb{P}(|X_0| \le \frac{1}{\sigma})$

$= \mathbb{P}(-\frac{1}{\sigma} \le X_0 \le \frac{1}{\sigma}) = \Phi(\frac{1}{\sigma}) - \Phi(-\frac{1}{\sigma}) = \circledast$

$\Phi(x) + \Phi(-x) = 1$

$\circledast \ \Phi(\frac{1}{\sigma}) - (1 - \Phi(\frac{1}{\sigma})) = 2\Phi(\frac{1}{\sigma}) - 1 \geq \frac{1}{2}$

$\Phi(\frac{1}{\sigma}) \geq \frac{3}{4} \qquad \frac{1}{\sigma} \geq \Phi^{-1}(\frac{3}{4})$

$\sigma \le \left(\Phi^{-1}(\frac{3}{4})\right)^{-1}$

Foglio 6 - Ex 1

$f(x) > 0 \quad \forall x \in \mathbb{R}$

$\int_{\mathbb{R}} f(x)\,dx = \frac{1}{\pi b} \int_{\mathbb{R}} \frac{1}{1+\left(\frac{x-a}{b}\right)^2}\,dx \qquad y = \frac{x-a}{b}$

$x = a + by$

$dx = b\,dy$

$= \frac{1}{\pi \cancel{b}} \int_{\mathbb{R}} \frac{1}{1+y^2} \cancel{b}\,dy =$

$= \frac{1}{\pi} \operatorname{arctg}(y) \Big|_{y \to -\infty}^{y \to +\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$

$\mathbb{E}[X]$?

$\mathbb{E}[X^+] = \int_0^{+\infty} x f(x)\,dx = \frac{1}{\pi b} \int_0^{+\infty} \frac{x}{1+\left(\frac{x-a}{b}\right)^2}\,dx$

$y = \frac{x-a}{b} \qquad x = a + by \qquad dx = b\,dy$

$= \frac{1}{\pi \cancel{b}} \int_{-\frac{a}{b}}^{+\infty} \frac{a+by}{1+y^2} \cancel{b}\,dy = \frac{1}{\pi} a \operatorname{arctg}(y) + \frac{b}{2\pi} \log(1+y^2) \Big|_{y = -\frac{a}{b}}^{y \to +\infty}$

$= +\infty$

$\mathbb{E}[X^-] = \int_{-\infty}^{0} -x f(x)\,dx = \frac{1}{\pi b} \int_{-\infty}^{0} \frac{-x}{1+\left(\frac{x-a}{b}\right)^2}\,dx$

$y = \frac{x-a}{b} \qquad x = a + by$

$$= \frac{1}{\pi \cancel{b}} \int_{-\infty}^{-\frac{a}{b}} \cancel{b}\, \frac{-a-by}{1+y^2}\, dy =$$

$$= -\frac{a}{\pi} \arctan(y) - \frac{b}{2\pi} \log(1+y^2) \Big|_{y \to -\infty}^{y=-\frac{a}{b}} = +\infty$$

$$\mathbb{E}[X^+] = \mathbb{E}[X^-] = +\infty \qquad \nexists\ \mathbb{E}[X]$$

FOGLIO 6 - EX 2

$$f(x) > 0 \quad \forall x \in \mathbb{R}$$

$$\int_{\mathbb{R}} f(x)\,dx = 2\int_0^{+\infty} \frac{\exp(-x)}{2}\,dx = -\exp(-x)\Big|_{x=0}^{x\to+\infty} = -(-1) = 1$$

$$\mathbb{E}[|X|] = \int_{\mathbb{R}} |x|\, \frac{1}{2} \exp(-|x|)\,dx = 2\int_0^{+\infty} \frac{1}{2}\, x \exp(-x)\,dx < +\infty$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x\, \frac{1}{2} \exp(-|x|)\,dx = 0$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] = \int_{\mathbb{R}} x^2\, \frac{1}{2} \exp(-|x|)\,dx = \cancel{2}\int_0^{+\infty} \frac{\overbrace{x^2}^{f}\ \overbrace{\exp(-x)}^{g'}}{\cancel{2}}\,dx$$

$$= x^2(-\exp(-x))\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} \underbrace{2x}_{f}\, \underbrace{\exp(-x)}_{g'}\,dx =$$

$$= 2x(-\exp(-x))\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} 2\exp(-x)\,dx =$$

$$= -2\exp(-x)\Big|_{x=0}^{x\to+\infty} = 2$$

$$Y := \frac{X - \mathbb{E}[X]}{\mathrm{Var}[X]} = \frac{X}{2} = \frac{1}{2}\cdot X = aX+b$$

$$\mathbb{P}_Y = g(y)\,dy \qquad g(y) = \frac{1}{\frac{1}{2}}\, f\left(\frac{y-0}{\frac{1}{2}}\right) = 2f(2y) =$$

$$= 2\,\frac{1}{2}\exp(-|2y|) = \exp(-2|y|)$$

FOGLIO 6 - EX 3

$$\mathbb{E}[X^b] = \int_0^{+\infty} x^b \lambda e^{-\lambda x}\, dx \qquad y = \lambda x \quad x = \frac{1}{\lambda} y \quad dx = \frac{1}{\lambda} dy$$

$$= \int_0^{+\infty} \frac{1}{\lambda^b} y^b \cancel{\lambda} e^{-y} \frac{1}{\cancel{\lambda}} dy = =$$

$$= \frac{1}{\lambda^b} \int_0^{+\infty} y^{(b+1)-1} e^{-y} dy = \frac{\Gamma(b+1)}{\lambda^b}$$

$$\varphi(t) = \begin{cases} 0 & t < 0 \\ t^b & t \geq 0 \end{cases} \implies X^b = \varphi \circ X$$

$\forall$ funzione di Borel nonnegative

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{X^b}(dt) = \int_{\mathbb{R}} \psi(\varphi(s))\, \mathbb{P}_X(ds) = \int_{\mathbb{R}} \psi(\varphi(s)) f(s) ds =$$

$$= \int_0^{+\infty} \psi(s^b) \lambda e^{-\lambda s} ds \qquad x = s^b \qquad s = x^{\frac{1}{b}}$$

$$= \int_0^{+\infty} \psi(x) \underbrace{\lambda \exp(-\lambda x^{\frac{1}{b}}) \frac{1}{b} x^{\frac{1}{b}-1}}\, dx =$$

Posto $a = \frac{1}{b}$ $\quad \mathbb{P}_{X^b} = g(x) dx$

$$g(x) = \begin{cases} 0 & x \leq 0 \\ \lambda a x^{a-1} \exp(-\lambda x^a) & x > 0 \end{cases}$$

$$\mathbb{E}[X^b] = \frac{\Gamma(b+1)}{\lambda^b} \qquad Y = X^b = X^{\frac{1}{a}}$$

$$\mathbb{E}[Y] = \frac{\Gamma(1+\frac{1}{a})}{\lambda^{1/a}}$$

FOGLIO 6 - EX 4

$X_i = 1$ se all' i-esima estrazione estraggo bianca

$X_i = 0$ rossa

$S_{n-1} :=$ # bianche estratte nelle prime n-1 estrazioni

$$\mathbb{P}(X_n = 1)$$

$$\{X_n = 1\} = \bigcup_{j=0}^{b-1} \{X_n = 1, S_{n-1} = j\}$$

$$\mathbb{P}(X_n = 1) = \sum_{j=0}^{b-1} \mathbb{P}(X_n = 1 \mid S_{n-1} = j)\,\mathbb{P}(S_{n-1} = j)$$

$$= \sum_{j=0}^{b-1} \frac{b-j}{b+r-(n-1)} \frac{\binom{b}{j}\binom{r}{(n-1)-j}}{\binom{b+r}{n-1}} = \frac{b}{b+r}$$

V.A. VETTORIALI – DISTRIBUZIONE E INTEGRAZIONE

$X : \Omega \to \mathbb{R}^N$ v.a. vettoriale su $(\Omega, \mathcal{E}, \mathbb{P})$

$X = (X_1 \dots X_N)$ X è v.a. sse tutte le X_i sono v.a.

$\mathbb{P}_X(A) = \mathbb{P}(X \in A)$ $A \in \mathcal{B}(\mathbb{R}^N)$

$F_X(t_1 \dots t_N) = \mathbb{P}(X_1 \le t_1, X_2 \le t_2 \dots, X_n \le t_n)$

$X_1 \dots X_N : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$\varphi : \mathbb{R}^N \to \mathbb{R}$ funzione di Borel non negativa

$Y = \varphi \circ (X_1 \dots X_N)$ è una v.a.

$$\int_\Omega Y(\omega)\,\mathbb{P}(d\omega) = \int_\Omega \varphi(X_1(\omega) \dots X_N(\omega))\,\mathbb{P}(d\omega)$$

$$= \int_{\mathbb{R}^N} \varphi(t_1 \dots t_N)\,\mathbb{P}_{(X_1 \dots X_N)}(dt_1 \dots dt_N)$$

DIM 1) φ semplice non negativa $\varphi(t_1 \dots t_N) = \sum_{i=1}^{k} c_i \mathbb{1}_{E_i}(t_1 \dots t_N)$

Si fa vedere che per φ la Tesi è vera

2) φ Borel-misurabile nonnegativa

$\exists \{f_j\}$ successione di funzioni semplici che approssimano φ dal basso

Per le f_j la Tesi è vera

Applico Beppo Levi a $\{f_j\}$ e $\mathbb{P}_{X_1 \dots X_n}$

e a $f_j \circ (X_1 \dots X_N)$ e $\mathbb{P}$

FORMULA DI COMPOSIZIONE

Sia $X : \Omega \to \mathbb{R}^N$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

sia $f : \mathbb{R}^N \to \mathbb{R}^k$ di Borel

sia $\varphi : \mathbb{R}^k \to \mathbb{R}$ di Borel nonnegativa

Allora $f \circ X : \Omega \to \mathbb{R}^k$ è una v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

e

$$\int_{\mathbb{R}^k} \varphi(t_1 \dots t_k)\,\mathbb{P}_{f \circ X}(dt_1 \dots dt_k) = \int_{\mathbb{R}^N} \varphi(f(s_1 \dots s_N))\,\mathbb{P}_X(ds_1 \dots s_N)$$

DIM $A \in \mathcal{B}(\mathbb{R}^k)$ $\{\varphi \circ X \in A\} = \{X \in \varphi^{-1}(A)\} \in \mathcal{E}$

perché φ di Borel $\Rightarrow \varphi^{-1}(A) \in \mathcal{B}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^k} \psi(t_1 \dots t_k)\, \mathbb{P}_{\varphi\circ X}(dt_1 \dots dt_k) = \int_\Omega \psi(\varphi(X))\, \mathbb{P}(d\omega) \quad \textcircled{=}$$

$$\int_{\mathbb{R}^N} \psi(\varphi(s_1 \dots s_N))\, \mathbb{P}_X(ds_1 \dots ds_N) = \int_\Omega (\psi\circ\varphi)(X(\omega))\, \mathbb{P}(d\omega)$$

CASO PARTICOLARE

$X, Y : \Omega \to \mathbb{R}^N$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$\varphi : (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto x+y \in \mathbb{R}^N$

$\psi : \mathbb{R}^N \to \mathbb{R}$ di Borel nonnegative

Applico la formula di composizione con la v.a $Z = (X,Y)$

$$\int_{\mathbb{R}^N} \psi(t_1 \dots t_N)\, \mathbb{P}_{\varphi\circ Z}(dt_1 \dots dt_N) = \int_{\mathbb{R}^{2N}} \psi(\varphi(x,y))\, \mathbb{P}_Z(dx_1 \dots dx_N dy_1 \dots dy_N)$$

$$\int_{\mathbb{R}^N} \psi(t_1 \dots t_N)\, \mathbb{P}_{X+Y}(dt_1 \dots dt_N) = \int_{\mathbb{R}^{2N}} \psi(x_1+y_1, \dots x_N+y_N)\, \mathbb{P}_{X,Y}(dx_1 \dots dx_N dy_1 \dots dy_N)$$

$A \in \mathcal{B}(\mathbb{R}^N)$ $\psi = \mathbb{1}_A$

$$\mathbb{P}_{X+Y}(A) = \int_{\mathbb{R}^{2N}} \mathbb{1}_A(x_1+y_1 \dots x_N+y_N)\, \mathbb{P}_{X,Y}(dx_1 \dots dx_N\, dy_1 \dots dy_N)$$

$$= \mathbb{P}_{X,Y}(\{(x,y) \in \mathbb{R}^{2N} : x+y \in A\})$$

$A = \prod_{i=1}^N (-\infty, t_i]$ $\mathbb{P}_{X+Y}(A) = F_{X+Y}(t_1, t_2 \dots t_N)$

$$F_{X+Y}(t_1 \dots t_N) = \mathbb{P}_{X,Y}(\{(x,y) \in \mathbb{R}^{2N} : x_i + y_i \le t_i \;\; \forall i = 1 \dots N\})$$

— o —

Sia $X : \Omega \to \mathbb{R}^N$ una v.a su $(\Omega, \mathcal{E}, \mathbb{P})$

Dico che la distribuzione di X, $\mathbb{P}_X$ È ASSOLUTAMENTE CONTINUA RISPETTO ALLA MISURA DI LEBESGUE su $\mathbb{R}^N$

se $\exists\, f : \mathbb{R}^N \to [0, +\infty]$ $\mathcal{L}^N$- misurabile detta DENSITÀ

t.c. $\mathbb{P}_X(A) = \int_{\mathbb{R}^N} \mathbb{1}_A(x) f(x)\,dx = \int_A f(x)\,dx \quad \forall A \in \mathcal{B}(\mathbb{R}^N)$

PROPRIETÀ $\mathbb{P}_X$ è A.C. rispetto alla misura di Lebesgue in $\mathbb{R}^N$

SSE $$\int_{\mathbb{R}^N} \varphi(x)\,\mathbb{P}_X(dx) = \int_{\mathbb{R}^N} \varphi(x) f(x)\,dx$$

$\forall \varphi$ funzione di Borel non negativa.

Siano $X, Y : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ e supponiamo che $(X,Y) : \Omega \to \mathbb{R}^2$ abbia distribuzione A.C.

$Z = (X, Y) : \Omega \to \mathbb{R}^2$

$\rho : (x,y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R} \quad \Rightarrow \quad X = \rho \circ Z$

$\varphi : \mathbb{R} \to [0, +\infty]$ funzione di Borel non negativa

$$\int_{\mathbb{R}} \varphi(x)\,\mathbb{P}_X(dx) = \int_{\mathbb{R}} \varphi(x)\,\mathbb{P}_{\rho\circ Z}(dx) = \int_{\mathbb{R}^2} \varphi(\rho(x,y))\,\mathbb{P}_Z(dx\,dy)$$

$$= \int_{\mathbb{R}^2} \varphi(x) f(x,y)\,dy = \int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} \varphi(x) f(x,y)\,dy \right) =$$

Teo di Fubini

$$= \int_{\mathbb{R}} \varphi(x) \left(\int_{\mathbb{R}} f(x,y)\,dy \right) dx$$

$\mathbb{P}_X$ è A.C. con densità $g(x) = \int_{\mathbb{R}} f(x,y)\,dy$

Analogamente $\mathbb{P}_Y$ è A.C. con densità $h(y) = \int_{\mathbb{R}} f(x,y)\,dx$

— o —

$$F_{X,Y}(s,t) = \mathbb{P}_{X,Y}((-\infty, s] \times (-\infty, t]) = \int_{-\infty}^{s} dx \int_{-\infty}^{t} f(x,y)\,dy$$

$$\frac{\partial F_{X,Y}(s,t)}{\partial s} = \int_{-\infty}^{t} f(s,y)\,dy \qquad \frac{\partial^2 F_{X,Y}}{\partial s\,\partial t}(s,t) = f(s,t)$$

$X, Y : \Omega \to \mathbb{R}$ v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$

$XY : \Omega \to \mathbb{R}$ è ancora una v.a.

Ts) $\mathbb{E}[|XY|] \leq \left(\mathbb{E}[X^2] \mathbb{E}[Y^2] \right)^{1/2}$

1) Se $\mathbb{E}[X^2]$ o $\mathbb{E}[Y^2] = +\infty$ non c'è niente da dimostrare

2) Se $\mathbb{E}[X^2] = 0$ o $\mathbb{E}[Y^2] = 0$

Supponiamo $\mathbb{E}[X^2] = 0 \Rightarrow X^2 = 0$ $\mathbb{P}$-q.c. $\Rightarrow X = 0$ $\mathbb{P}$-q.c.

$\Rightarrow XY = 0$ $\mathbb{P}$-q.c. $\Rightarrow \mathbb{E}[|XY|] = 0$

3) Suppongo $\mathbb{E}[X^2]$ e $\mathbb{E}[Y^2]$ siano finiti e positivi

$$\alpha(\omega) = \frac{|X(\omega)|}{\sqrt{\mathbb{E}[X^2]}} \qquad \beta(\omega) = \frac{|Y(\omega)|}{\sqrt{\mathbb{E}[Y^2]}}$$

$$2\alpha(\omega)\beta(\omega) \leq \alpha^2(\omega) + \beta^2(\omega) \quad \forall \omega \in \Omega$$

$$2 \int_\Omega \alpha(\omega)\beta(\omega)\,\mathbb{P}(d\omega) \leq \int_\Omega \left(\alpha^2(\omega) + \beta^2(\omega) \right) \mathbb{P}(d\omega)$$

$$\frac{2}{\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}} \int_\Omega |XY|(\omega)\,\mathbb{P}(d\omega) \leq \frac{1}{\mathbb{E}[X^2]} \int_\Omega |X(\omega)|^2\,\mathbb{P}(d\omega) + \frac{1}{\mathbb{E}[Y^2]} \int_\Omega |Y(\omega)|^2\,\mathbb{P}(d\omega)$$

$$\frac{2}{\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}} \mathbb{E}[|XY|] \leq 1 + 1$$

$$\mathbb{E}[|XY|] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

$k \in \mathbb{N}$ $\mathcal{L}^k(\Omega, \mathbb{P}) = \{ X : \Omega \to \mathbb{R}$ v.a. t.c. $\mathbb{E}[|X|^k] < +\infty \}$

1) $X, Y \in \mathcal{L}^2(\Omega, \mathbb{P}) \Rightarrow \mathbb{E}[XY]$ esiste ed è finito

2) $\forall k \in \mathbb{N}$ $\mathcal{L}^k(\Omega, \mathbb{P})$ è uno spazio vettoriale (NO DIM)

3) Sia $\alpha : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{R}$

$\alpha : (X, Y) \mapsto \mathbb{E}[XY]$

Allora α è una forma bilineare simmetrica semidefinita positiva, su $\mathcal{L}^2$

$\alpha(X,X)=0 \quad \mathbb{E}[X^2]=0 \quad \Rightarrow \quad X=0 \quad \mathbb{P}$-q.c