DISTRIBUZIONI A.C.

Titolo nota 27/10/2015

Per $\alpha \in \mathbb{R}$, $\alpha > 0$ considero $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx$

L'integrale è convergente e l'integrando è positivo.

Considero $\Gamma(\alpha+1) = \int_0^{+\infty} x^{\alpha} e^{-x}\, dx$ = per parti

$$= -x^{\alpha} e^{-x} \Big|_{x=0}^{x \to +\infty} + \int_0^{+\infty} \alpha x^{\alpha-1} e^{-x}\, dx = \alpha \Gamma(\alpha)$$

$\Rightarrow \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \forall \alpha > 0$

Inoltre $\Gamma(1) = \int_0^{+\infty} e^{-x}\, dx = -e^{-x} \Big|_{x=0}^{x \to +\infty} = 1$

$\Rightarrow \Gamma(2) = 1 \cdot \Gamma(1) = 1$, $\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2$,

$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 = 3!$

Per induzione $\Gamma(n) = (n-1)!$

$\Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)! = n!$ o.k.

DISTRIBUZIONI GAMMA DI PARAMETRI $\alpha, \lambda > 0$

È la distribuzione A.C. associata alla densità

$$f(t) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Facciamo vedere che si tratta di una densità di probabilità.

$$\int_{\mathbb{R}} f(t)\,dt = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t}\, dt$$

Pongo $s = \lambda t$, $t = \dfrac{s}{\lambda}$ $\qquad = \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} \displaystyle\int_0^{+\infty} \frac{s^{\alpha-1}}{\lambda^{\alpha-\cancel{1}}} e^{-s} \frac{1}{\cancel{\lambda}}\, ds$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{1}{\lambda^{\alpha}} \int_0^{+\infty} s^{\alpha-1} e^{-s}\, ds = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{1}{\lambda^{\alpha}} \Gamma(\alpha) = 1$$

Sia X una v.a. t.c. $\mathbb{P}_X = \Gamma(\alpha, \lambda) \Rightarrow X \geq 0$ $\mathbb{P}$-q.c. $\Rightarrow$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^{+\infty} x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx =$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x}\,dx \qquad s=\lambda x \quad , \quad x=\frac{s}{\lambda}$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \frac{s^\alpha}{\cancel{\lambda^\alpha}} e^{-s} \frac{1}{\lambda}\,ds = \frac{1}{\lambda\Gamma(\alpha)}\Gamma(\alpha+1) = \frac{1}{\lambda\Gamma(\alpha)}\alpha\Gamma(\alpha) = \frac{\alpha}{\lambda}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^{+\infty} x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} e^{-\lambda x}\,dx$$

$$s=\lambda x \qquad x=\frac{s}{\lambda} \qquad dx = \frac{1}{\lambda}\,ds \qquad \Rightarrow$$

$$\mathbb{E}[X^2] = \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} \frac{s^{\alpha+1}}{\lambda^{\cancel{\alpha}+1}} e^{-s} \frac{1}{\lambda}\,ds = \frac{1}{\lambda^2\Gamma(\alpha)}\Gamma(\alpha+2) = \frac{(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\lambda^2\cancel{\Gamma(\alpha)}}$$

$$\Rightarrow \mathrm{Var}[X] = \frac{\alpha(\alpha+1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}$$

DISTRIBUZIONE BETA DI PARAMETRI $q,r>0$

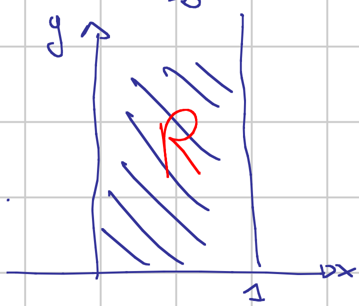
$\beta(q,r)$ è la distribuzione A.C. associata alla densità

$$f(x) = \begin{cases} \dfrac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} x^{q-1}(1-x)^{r-1} & x\in(0,1) \\ 0 & \text{altrimenti} \end{cases}$$

Calcolo $\int_{\mathbb{R}} f(x)\,dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^{q-1}(1-x)^{r-1}\,dx = 1$??

Considero $\Gamma(q+r)\int_0^1 x^{q-1}(1-x)^{r-1}\,dx = \int_0^{+\infty} y^{(q+r)-1}e^{-y}\,dy \int_0^1 x^{q-1}(1-x)^{r-1}\,dx$

$$= \iint_R e^{-y}\, y^{(q+r)-1} x^{q-1} (1-x)^{r-1}\,dx\,dy$$



$$= \iint_R e^{-y} (xy)^{q-1} y^r (1-x)^{r-1}\,dx\,dy$$

Cambio variabile $\begin{cases} s = xy \\ t = (1-x)y \end{cases}$

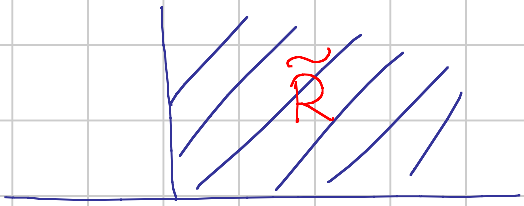
$$\frac{1-x}{x} = \frac{t}{s} \qquad \frac{1}{x} - 1 = \frac{t}{s} \qquad \frac{1}{x} = \frac{t+s}{s} \qquad x = \frac{s}{t+s}$$

$$y = \frac{s}{x} = t+s \qquad \begin{cases} x = \dfrac{s}{t+s} \\ y = t+s \end{cases} \implies 1-x = \frac{t}{t+s}$$

$$J = \begin{pmatrix} \dfrac{t+\cancel{s}-\cancel{s}}{(t+s)^2} & \dfrac{-s}{(t+s)^2} \\ 1 & 1 \end{pmatrix} \qquad |\det J| = \frac{|t+s|}{(t+s)^2} = |t+s|^{-1}$$

$$\begin{matrix} 0<x<1 \\ y>0 \end{matrix} \qquad \begin{cases} 0 < \dfrac{s}{t+s} < 1 \\ t+s>0 \end{cases} \qquad \begin{cases} t+s>0 \\ 0<s<t+s \end{cases}$$

$$\begin{cases} t+s>0 \\ s>0 \, , \, t>0 \end{cases}$$



$$\Gamma(q+r) \int_0^1 x^{q-1} (1-x)^{r-1} dx = \iint_{(0,+\infty)^2} e^{-t-s} s^{q-1} t^r \frac{\cancel{t+s}}{t} \frac{1}{\cancel{t+s}} \, ds dt$$

$$= \left(\int_0^{+\infty} e^{-s} s^{q-1} ds \right) \left(\int_0^{+\infty} e^{-t} t^{r-1} \right) dt = \Gamma(q)\Gamma(r)$$

$$\implies \int_{\mathbb{R}} f(x)\,dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^{q-1}(1-x)^{r-1} dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)} = 1$$

Se X è una v.a. t.c. $\mathbb{P}_X = \mathcal{B}(q,r)$, allora $X \in (0,1)$ $\mathbb{P}$-q.c.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x \cdot x^{q-1} (1-x)^{r-1} dx =$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^{(q+1)-1} (1-x)^{r-1} = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{\Gamma(q+1)\Gamma(r)}{\Gamma(q+1+r)} =$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{q\Gamma(q)\Gamma(r)}{(q+r)\Gamma(q+r)} = \frac{q}{q+r}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^2 x^{q-1} (1-x)^{r-1} dx =$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \int_0^1 x^{(q+2)-1} (1-x)^{r-1} dx = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{\Gamma(q+2)\Gamma(r)}{\Gamma(q+2+r)} =$$

$$= \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)} \frac{(q+1)q\Gamma(q)\Gamma(r)}{(q+1+r)(q+r)\Gamma(q+r)} = \frac{q(q+1)}{(q+r)(q+r+1)}$$

$$Var[X] = \frac{q(q+1)}{(q+r)(q+r+1)} - \left(\frac{q}{q+r}\right)^2 = \frac{q\left((q+1)(q+r) - q(q+r+1)\right)}{(q+r)^2(q+r+1)}$$

$$= \frac{q(q+r-q)}{(q+r)^2(q+r+1)} = \frac{qr}{(q+r)^2(q+r+1)}$$

Foglio 4 - Ex 5

La v.a. X è distribuita uniformemente sull'intervallo $[0,4]$

Sia $Y = \max\{3X-2, X^2\}$

Calcolare densità e valore atteso di Y

Sia $f(t) = \max\{3t-2, t^2\} \Rightarrow Y = f \circ X$

Sia φ di Borel non negativa

$$\int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} \varphi(t)\, \mathbb{P}_{f\circ X}(dt) = \int_{\mathbb{R}} \varphi(f(s))\, \mathbb{P}_X(ds) =$$

$$= \int_{\mathbb{R}} \varphi(\max\{3s-2, s^2\})\, \frac{1}{4} \mathbb{1}_{(0,4)}(s)\, ds =$$

$$= \int_0^4 \frac{1}{4} \varphi(\max\{3s-2, s^2\})\, ds$$

$3s-2 \geq s^2 \qquad s^2 - 3s + 2 \leq 0 \qquad (s-2)(s-1) \leq 0$

$s \in (1,2) \qquad f(s) = 3s-2$

$s \leq 1,\ s \geq 2 \qquad f(s) = s^2$

$$= \frac{1}{4}\int_0^1 \varphi(s^2)\, ds + \frac{1}{4}\int_1^2 \varphi(3s-2)\, ds + \frac{1}{4}\int_2^4 \varphi(s^2)\, ds$$

$u = s^2 \qquad s = \sqrt{u}$

$\rightarrow u = 3s-2 \qquad s = \frac{u+2}{3}$

$s=1\ u=1$

$s=2\ u=4$

$$= \frac{1}{4}\int_0^1 \varphi(u) \frac{1}{2\sqrt{u}}\, du + \frac{1}{4}\int_1^4 \varphi(u) \frac{1}{3}\, du + \frac{1}{4}\int_4^{16} \varphi(u) \frac{1}{2\sqrt{u}}\, du$$

$$= \int_{\mathbb{R}} \varphi(u)\, g(u)\, du \quad \text{dove}$$

$$g(u) = \begin{cases} \frac{1}{8\sqrt{u}} & u \in (0,1) \cup (4,16) \\ \frac{1}{12} & u \in (1,4) \\ 0 & \text{altrimenti} \end{cases}$$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} u\, g(u)\, du = \int_0^1 8\sqrt{u}\, du + \int_4^{16} 8\sqrt{u}\, du + \int_1^4 \frac{4}{12}\, du$$

$$= 8\cdot\frac{2}{3}u^{3/2}\Big|_{u=0}^{u=1} + 8\cdot\frac{2}{3}u^{3/2}\Big|_{u=4}^{u=16} + \frac{u^2}{24}\Big|_{u=1}^{u=4}$$

$$= \frac{16}{3} + \frac{16}{3}\left(64-8\right) + \frac{16-1}{24} = \frac{16}{3}\cdot 57 + \frac{15}{24}$$

Ripetere l'esercizio con $Z = \max\{X, 1\}$

$f(t) = \max\{t, 1\}$

ψ di Borel nonnegative

$$\int_{\mathbb{R}} \psi(t)\,\mathbb{P}_Z(dt) = \int_{\mathbb{R}} \psi(f(s))\,\mathbb{P}_X(dx) = \int_0^4 \psi(\max(s,1))\,\frac{1}{4}\,ds$$

$$= \frac{1}{4}\int_0^1 \psi(1)\,ds + \frac{1}{4}\int_1^4 \psi(s)\,ds = \frac{1}{4}\psi(1) + \frac{1}{4}\int_1^4 \psi(s)\,ds$$

$$= \int_{\mathbb{R}} \psi(s)\,\frac{1}{4}\,\delta_1(ds) + \int_{\mathbb{R}} \psi(s)\,g(s)\,ds \qquad g(s) = \frac{1}{4}\mathbb{1}_{(1,4)}(s)$$

$$\Rightarrow \quad \mathbb{P}_Z = \frac{1}{4}\delta_1(\cdot) + \frac{1}{4}\mathbb{1}_{(1,4)}\,ds$$

$$= \frac{1}{4}\delta_1(\cdot) + \frac{3}{4}\,U((1,4))$$

FOGLIO 5 - EX 1

Sicuramente $X(\Omega) \subseteq \{-n, -n+1, \_\_, 0, \_\_, n-1, n\}$

Sia Y la v.a. che conta il numero di teste

Sappiamo che $\mathbb{P}_Y = B(n,p)$

Se $X = k$ e $Y = j$ $\Rightarrow$ # croci $= n-j$ $\Rightarrow k = j-(n-j)$

$k = 2j-n \qquad j = \frac{k+n}{2}$

Ma j è un intero $\Rightarrow$ $k \equiv n \bmod (2)$

Cioè $X(\Omega) = \{-n, -n+2, \_\_\_, n-2, n\}$

Poiché $X = k$ sse $Y = \frac{k+n}{2}$ abbiamo

$$\mathbb{P}(X=k) = \mathbb{P}\left(Y = \frac{k+n}{2}\right) = B(n,p)\left(\left\{\frac{k+n}{2}\right\}\right) = \binom{n}{\frac{n+k}{2}} p^{\frac{k+n}{2}} (1-p)^{n-\frac{n+k}{2}}$$

$$= \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}} (1-p)^{\frac{n-k}{2}} \qquad \forall k = -n, -n+2, \dots n-2, n$$

FOGLIO 5 - Ex 2

X_1 = # estratto nella 1ª urna, X_2 = # estratto nella 2ª urna

$\{X = k\} = \{X_1 = k, X_2 = k\} \cup \{X_1 = k, X_2 < k\} \cup \{X_1 < k, X_2 = k\}$

$$\mathbb{P}(X = k) = \mathbb{P}(X_1 = k, X_2 = k) + \mathbb{P}(X_1 = k)\mathbb{P}(X_2 < k) + \mathbb{P}(X_1 < k)\mathbb{P}(X_2 = k)$$

$$= \frac{1}{n}\cdot\frac{1}{n} + \frac{1}{n}\cdot\frac{k-1}{n} + \frac{k-1}{n}\cdot\frac{1}{n} = \frac{2k-1}{n^2} \qquad k = 1, 2, \dots, n$$

$$\mathbb{E}[X] = \sum_{k=1}^{n} k\,\mathbb{P}(X = k) = \sum_{k=1}^{n} \frac{k(2k-1)}{n^2} = \frac{2}{n^2}\sum_{k=1}^{n} k^2 - \frac{1}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{2}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{1}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)}{2} = \frac{(n+1)\left(4n+2-3\right)}{6n}$$

$$= \frac{(n+1)(4n-1)}{6}$$

FOGLIO 5 - Ex 3

$$p = \frac{\binom{6}{3}\binom{6}{3}}{\binom{12}{6}} = \frac{6!}{3!}\;\frac{6!}{3!}\;\frac{6!\,6!}{12!} = \left(\frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3}\cdot\cancel{2}}\right)^2 \frac{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{12}\cdot 11\cdot \underset{2}{\cancel{10}}\cdot \underset{3}{\cancel{9}}\cdot \underset{2}{\cancel{8}}\cdot 7}$$

$$= \frac{100}{11\cdot 3\cdot 7} = \frac{100}{231}$$

$$\mathbb{P}(X = n) = p\,(1-p)^{n-1} = \frac{100}{231}\left(\frac{131}{231}\right)^{n-1}$$

FOGLIO 5 - Ex 4

X = # Teste $\qquad \mathbb{P}_X = B(n, p)$

$\mathbb{P}(X \text{ sia pari}) = ?$

$$\mathbb{P}(X = 2j) = \binom{n}{2j} p^{2j}(1-p)^{n-2j} \qquad 0 \le 2j \le n \qquad 0 \le j \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$\mathbb{P}(X \text{ sia pari}) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} p^{2j}(1-p)^{n-2j}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(-x+y)^n = \sum_{k=0}^{n} \binom{n}{k}(-x)^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k}(-1)^k x^k y^{n-k}$$

$$(x+y)^n + (-x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\left(1 + (-1)^k\right) =$$

$$= 2\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} x^{2j} y^{n-2j}$$

$$\mathbb{P}(X \text{ è pari}) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \underbrace{p}_{=:x}{}^{2j} \underbrace{(1-p)}_{=:y}{}^{n-2j} = \frac{1}{2}\left\{(p+(1-p))^n + (-p+1-p)^n\right\}$$

$$= \frac{1}{2}\left(1+(1-2p)^n\right)$$

Foglio 5 - Ex 5

$$\mathbb{P}(X \text{ è pari}) = \sum_{j=0}^{\infty} \mathbb{P}(X=2j) = \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^{2j}}{2j!} = e^{-\lambda} \cosh(\lambda) =$$

$$= e^{-\lambda} \frac{e^{\lambda} + e^{-\lambda}}{2} = \frac{1}{2} + \frac{1}{2} e^{-2\lambda}$$