Osserviamo che $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$

Chiamiamo distribuzione di Poisson di parametro $\lambda > 0$ la distribuzione concentrata sugli interi non negativi t.c.

$$P(\lambda)(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!} \quad \forall k \in \mathbb{N}_0$$

Se X è una v.a. t.c. $\mathbb{P}_X = P(\lambda) \Rightarrow$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \, e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} =$$

$\boxed{k-1=j}$

$$= e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!} = e^{-\lambda} \lambda e^{\lambda} = \lambda$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^k}{(k-1)!} \qquad j = k-1$$

$$= e^{-\lambda} \sum_{j=0}^{\infty} \frac{(j+1) \lambda^{j+1}}{j!} =$$

$$= \lambda e^{-\lambda} \left(\sum_{j=0}^{\infty} \frac{j \lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \right)$$

$$= \lambda \left(\lambda + e^{-\lambda} \cdot e^{\lambda} \right) = \lambda(\lambda + 1)$$

$$\Rightarrow \mathrm{Var}[X] = \lambda(\lambda+1) - (\lambda)^2 = \lambda$$

3 DISTRIBUZIONE IPERGEOMETRICA di parametri $b, r, n \in \mathbb{N}$

Dati $b, r, n \in \mathbb{N}$ t.c. $b + r \leq n$ ~~si dice~~ è la distribuzione concentrata su $\{0, 1, \ldots n\}$ t.c.

$$H(b, r, n)(\{k\}) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}}$$

N.B. Bisogna verificare che

$$\sum_{k=0}^{n} \binom{b}{k}\binom{r}{n-k} = \binom{b+r}{n}$$

Sappiamo che $\sum_{k=0}^{\infty} \binom{b}{k} z^k = (1+z)^b$

$$\sum_{k=0}^{\infty} \binom{r}{k} z^k = (1+z)^r$$

$$\Rightarrow (1+z)^b (1+z)^r = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \binom{b}{k}\binom{r}{n-k} \right) z^n$$

$$= (1+z)^{b+r} = \sum_{n=0}^{\infty} \binom{b+r}{n} z^n$$

$\Rightarrow$ i coefficienti corrispondenti devono essere uguali:

$$\sum_{k=0}^{n} \binom{b}{k}\binom{r}{n-k} = \binom{b+r}{n}$$

$$\Rightarrow \sum_{k=0}^{n} H(b,r,n)(\{k\}) = \frac{\binom{b+r}{n}}{\binom{b+r}{n}} = 1$$

~~ES~~ Se X è una v.a. con $P_X = H(b,r,n)$ $\Rightarrow$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}} = \frac{1}{\binom{b+r}{n}} \sum_{k=0}^{n} k \binom{b}{k}\binom{r}{n-k}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k\binom{b}{k}\binom{r}{n-k}\right) z^n = \left(\sum_{n=0}^{\infty} n\binom{b}{n} z^n\right)\left(\sum_{n=0}^{\infty}\binom{r}{n} z^n\right)$$

$$= \left(z\sum_{n=1}^{\infty} n\binom{b}{n} z^{n-1}\right)\left(\underbrace{\sum_{n=0}^{\infty}\binom{r}{n} z^n}\right) =$$

$$= z\frac{d}{dz}(1+z)^b \, (1+z)^r = bz(1+z)^{b-1+r} =$$

$$= bz\sum_{j=0}^{\infty}\binom{b+r-1}{j} z^j \qquad j+1=n \qquad j=n-1$$

$$= \sum_{n=1}^{\infty} b\binom{b+r-1}{n-1} z^n$$

$$\Rightarrow \sum_{k=0}^{n} k\binom{b}{k}\binom{r}{n-k} = b\binom{b+r-1}{n-1}$$

$$\Rightarrow \mathbb{E}[X] = \frac{b(b+r-1)!}{(n-1)!\,\cancel{(b+r-n)!}} \cdot \frac{n!\,\cancel{(b+r-n)!}}{(b+r)!}$$

$$= \frac{bn}{b+r}$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2 \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{b+r}{n}} = \qquad k^2 = k(k-1)+k$$

$$= \frac{1}{\binom{b+r}{n}}\left\{\sum_{k=2}^{n} k(k-1)\binom{b}{k}\binom{r}{n-k} + \sum_{k=1}^{n} k\binom{b}{k}\binom{r}{n-k}\right\}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=2}^{n} k(k-1)\binom{b}{k}\binom{r}{n-k}\right) z^n =$$

$$= \left(\sum_{n=0}^{\infty} n(n-1)\binom{b}{n} z^n\right)\left(\sum_{n=0}^{\infty}\binom{r}{n} z^n\right)$$

$$= z^2\left(\frac{d^2}{dz^2}\sum_{n=0}^{\infty}\binom{b}{n} z^n\right)\sum_{n=0}^{\infty}\binom{r}{n} z^n$$

$$= z^2\left(\frac{d^2}{dz^2}(1+z)^b\right)(1+z)^r =$$

$$= z^2\, b(b-1)\,(1+z)^{b-2+r} =$$

$$= b(b-1)\sum_{j=0}^{\infty}\binom{b-2+r}{j} z^{j+2}$$

$j+2=n$

$j=n-2$

$$= \sum_{n=2}^{\infty} b(b-1)\binom{b-2+r}{n-2} z^n$$

$$\Rightarrow \sum_{k=2}^{n} k(k-1)\binom{b}{k}\binom{r}{n-k} = b(b-1)\binom{b-2+r}{n-2}$$

$$\Rightarrow \mathbb{E}[X^2] = \frac{1}{\binom{b+r}{n}}\left\{b(b-1)\binom{b+r-2}{n-2} + b\binom{b+r-1}{n-1}\right\}$$

$$= \frac{n!\, b\,\cancel{(b+r-n)!}}{(b+r)!}\left\{\frac{(b-1)(b+r-2)!}{(n-2)!\,\cancel{(b+r-n)!}} + \frac{(b+r-1)!}{(n-1)!\,\cancel{(b+r-n)!}}\right\}$$

$$= \frac{b(b-1)\,n(n-1)}{(b+r)(b+r-1)} + \frac{bn}{b+r} =$$

$$= \frac{bn}{(b+r)(b+r-1)}\left(\overbrace{(b-1)(n-1) + b+r-1}^{bn-\cancel{b}-n+\cancel{1}+\cancel{b}+r-\cancel{1}}\right)$$

$$= \frac{bn}{(b+r)(b+r-1)}\left(n(b-1)+r\right)$$

$$\mathrm{Var}[X] = \frac{bn}{(b+r)(b+r-1)}\Big(n(b-1)+r \Big) - \left(\frac{bn}{b+r}\right)^2 =$$

$$= \frac{bn}{(b+r)^2(b+r-1)}\Big\{ (b+r)(bn-n+r) - bn(b+r-1) \Big\}$$

$$= \frac{bn}{(b+r)^2(b+r-1)}\Big\{ \cancel{b^2n} - \cancel{bn} + br + \cancel{brn} - nr + r^2 - \cancel{b^2n} - \cancel{brn} + \cancel{bn} \Big\}$$

$$= \frac{brn\,(b+r-n)}{(b+r)^2(b+r-1)} =$$

$$= n\,\frac{b}{b+r}\,\frac{r}{b+r}\left(\frac{b+r-n}{b+r-1}\right)$$

$$= n\,\frac{b}{b+r}\left(1-\frac{b}{b+r}\right)\frac{b+r-1-(n-1)}{b+r-1}$$

$$= n\,\frac{b}{b+r}\left(1-\frac{b}{b+r}\right)\left(1-\frac{n-1}{b+r-1}\right)$$

N.B. Se consideriamo le estrazioni da un'urna:

Urna che contiene b palline bianche

r palline rosse

Estraggo n palline

Sia X la v.a. che conta il # di palline bianche estratte $\Rightarrow$ $\mathbb{P}_X = H(b,r,n)$

Ω = sottoinsiemi di $\{1, \_\_, b, b+1, \_\_, b+r\}$ aventi cardinalità n

$\mathcal{E} = \mathcal{P}(\Omega)$

$\mathbb{P}(\{\omega\}) = \frac{1}{\binom{b+r}{n}}$ $\forall \omega$ ~~sottoinsieme di~~ elemento di Ω

$X(\omega)$ = # di elementi di $\{\omega$ ~~contenuti~~ che sono $\leq b$

5 DISTRIBUZIONE BINOMIALE NEGATIVA

di parametri $-n$ e p

Per $n \in \mathbb{N}_0$ e $p \in (0,1)$ considero la distribuzione concentrata sugli interi nonnegativi con densità

$$B(-n,p)(\{k\}) = \binom{n+k-1}{k} p^n (1-p)^k \quad k = 0,1,2 \dots$$

1) Facciamo vedere che è una probabilità

$$\sum_{k=0}^{\infty} B(-n,p)(\{k\}) = p^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} (1-p)^k$$

Considero $\binom{n+k-1}{k}(-1)^k = \frac{(n+k-1)!}{k!(n-1)!}(-1)^k =$

$$= \frac{(n+k-1)(n+k-2) \dots (n+1)\, n}{k!} (-1)^k$$

$$= \frac{-n(-n-1) \dots (-n-k+2)(-n-k+1)}{k!}$$

$$= \binom{-n}{k}$$

$$\Rightarrow \sum_{k=0}^{\infty} \binom{n+k-1}{k} (-1)^k z^k = \sum_{k=0}^{\infty} \binom{-n}{k} z^k = (1+z)^{-n}$$

$$\forall z \in (-1,1)$$

$$\Rightarrow \sum_{k=0}^{\infty} B(-n,p)(\{k\}) = p^n \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k (1-p)^k$$

$$= p^n \sum_{k=0}^{\infty} \binom{-n}{k} (p-1)^k = p^n (1+p-1)^{-n} = 1 \text{ ok}$$