Titolo nota — 16/10/2015

Esercizi - Semaforo - Distribuzioni discrete

Ex 4 - Foglio 4

$\mathbb{P}_X = f(x)dx \qquad f(x) = \frac{1}{2}\mathbb{1}_{(0,2)}(x)$

$Y = \sqrt{X} \qquad \varphi(t) = \begin{cases} 0 & t \le 0 \\ \sqrt{t} & t > 0 \end{cases}$

$Y = \varphi \circ X$

Sia ψ funzione di Borel nonnegativa

$$\int_{\mathbb{R}} \psi(t)\mathbb{P}_Y(dt) = \int_{\mathbb{R}} \psi(t)\mathbb{P}_{\varphi\circ X}(dt) = \int_{\mathbb{R}} \psi(\varphi(s))\mathbb{P}_X(ds)$$

$$= \int_{\mathbb{R}} \psi(\varphi(s)) f(s)\, ds =$$

$$= \int_0^2 \psi(\varphi(s)) \frac{1}{2} ds = \frac{1}{2}\int_0^2 \psi(\sqrt{s})\, ds \qquad u = \sqrt{s}, \quad s = u^2$$

$$= \frac{1}{2}\int_0^{\sqrt{2}} \psi(u)\, 2u\, du = \int_0^{\sqrt{2}} \psi(u)\frac{2u}{2} du =$$

$$= \int_{\mathbb{R}} \psi(u)\frac{2u}{2}\mathbb{1}_{(0,\sqrt{2})}(u)\, du$$

$\mathbb{P}_{\sqrt{X}}$ è A.C. con densità $f_Y(u) = \frac{2u}{2}\mathbb{1}_{(0,\sqrt{2})}(u)$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} u f_Y(u)\, du = \int_0^{\sqrt{2}} u \frac{2u}{2} du = \frac{2}{2}\frac{u^3}{3}\Big|_{u=0}^{u=\sqrt{2}} = \frac{2}{3}\sqrt{2}$$

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} u^2 f_Y(u)\, du = \int_0^{\sqrt{2}} u^2\frac{2u}{2} du = \frac{2}{2}\frac{u^4}{4}\Big|_{u=0}^{u=\sqrt{2}} =$$

$$= \frac{1}{2}2$$

$$\Rightarrow \mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{2}{2} - \frac{4}{9}2 = \frac{2}{18}(9-8) = \frac{2}{18}$$

$$F_Y(t) = \int_{-\infty}^{t} f_Y(u)\,du$$

$$F_Y(t) = 0 \qquad t < 0$$
$$F_Y(t) = 1 \qquad t \geq \sqrt{2}$$

$$F_Y(t) = \int_0^t \frac{2u}{2}\,du \qquad t \in [0, \sqrt{2})$$

$$= \frac{u^2}{2}\Big|_{u=0}^{u=t} = \frac{t^2}{2}$$

$$\begin{cases} \frac{t^2}{2} = \frac{1}{2} \\ t \in (0, \sqrt{2}) \end{cases} \qquad \begin{cases} t \in (0, \sqrt{2}) \\ t^2 = \frac{2}{2} \end{cases} \Rightarrow t = \sqrt{\frac{2}{2}}$$

$$Z = X^2 \qquad f_Z(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{2\sqrt{t}}\left(f_X(\sqrt{t}) + f_X(-\sqrt{t})\right) & t > 0 \end{cases}$$

$$f_X(t) = \frac{1}{2}\mathbb{1}_{(0,2)}(x)$$

$$f_X(-\sqrt{t}) = 0 \qquad \forall t > 0$$
$$f_X(\sqrt{t}) = 0 \qquad \sqrt{t} \geq 2 \qquad f_X(\sqrt{t}) = 0 \qquad t \geq 2^2$$
$$f_X(\sqrt{t}) = \frac{1}{2} \qquad 0 < \sqrt{t} < 2 \qquad f_X(\sqrt{t}) = \frac{1}{2} \qquad 0 < t < 2^2$$

$$f_Z(t) = \begin{cases} 0 & t \leq 0 \lor t \geq 2^2 \\ \frac{1}{2\sqrt{t}}\frac{1}{2} & t \in (0, 2^2) \end{cases}$$

$$\mathbb{E}[Z] = \int_{\mathbb{R}} t f_Z(t)\,dt = \int_0^{2^2} t \frac{1}{2 \cdot 2\sqrt{t}}\,dt = \frac{1}{2 \cdot 2}\int_0^{2^2} t^{1/2}\,dt$$

$$= \frac{1}{\cancel{2}2}\frac{\cancel{2}}{3} t^{3/2}\Big|_{t=0}^{t=2^2} = \frac{2^2}{3}$$

$$\mathbb{E}[Z^2] = \int_{\mathbb{R}} t^2 f_Z(t)\,dt = \int_0^{2^2} t^2 \frac{1}{2 \cdot 2\sqrt{t}}\,dt = \frac{1}{2 \cdot 2}\int_0^{2^2} t^{3/2}\,dt$$

$$= \frac{1}{2a} \frac{2}{5} t^{5/2} \Big|_{t=0}^{t=a^2} = \frac{a^4}{5}$$

$$Var[Z] = E[Z^2] - (E[Z])^2 = \frac{a^4}{5} - \left(\frac{a^2}{3}\right)^2 = a^4\left(\frac{1}{5} - \frac{1}{9}\right)$$

$$= \frac{4}{45} a^4$$

$$F_Z(t) = \int_{-\infty}^{t} f_Z(t)\,dt \qquad f_Z(t) = \begin{cases} 0 & t \le 0 \lor t > a^2 \\ \frac{1}{2a\sqrt{t}} & t \in (0, a^2) \end{cases}$$

$F_Z(t) = 0 \qquad t < 0$

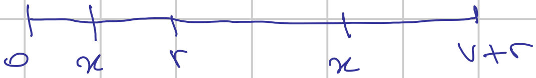
$F_Z(t) = 1 \qquad t > a^2$

$$t \in (0, a^2) \quad F_Z(t) = \int_0^t \frac{1}{2a\sqrt{u}}\,du = \frac{1}{2a}\int_0^t u^{-1/2}\,du =$$

$$= \frac{1}{\cancel{2}a} \cancel{2} u^{1/2} \Big|_{u=0}^{u=t} = \frac{\sqrt{t}}{a}$$

? t_M ? $\begin{cases} t \in (0, a^2) \\ \frac{\sqrt{t}}{a} = \frac{1}{2} \end{cases}$ $\begin{cases} t \in (0, a^2) \\ \sqrt{t} = \frac{a}{2} \end{cases}$ $\boxed{t = \frac{a^2}{4}}$

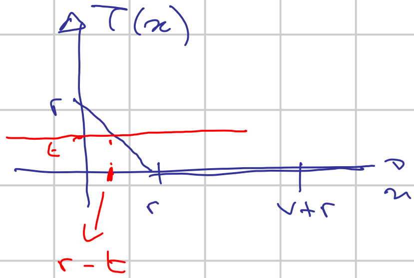
Semaforo che è rosso per r secondi
verde per v secondi
e poi le cose si ripete ciclicamente



$x \in (r, v+r) \quad T(x) = 0$

$x \in (0, r) \quad T(x) = r - x$

$$T(x) = \begin{cases} r - x & x \in (0, r] \\ 0 & x \in (r, v+r] \end{cases}$$

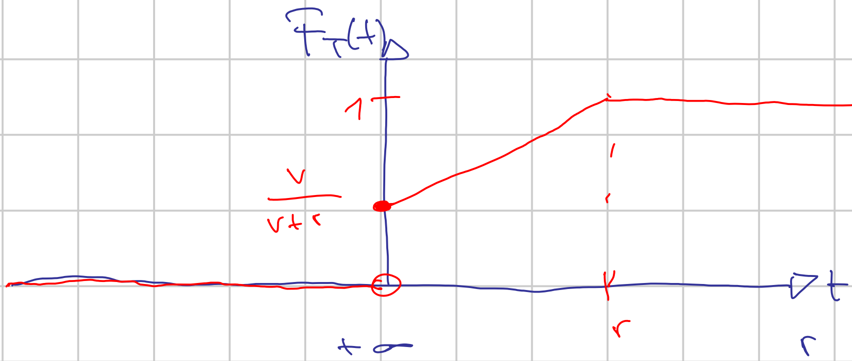


$\Omega = [0, v+r] \qquad \mathcal{E} = \mathcal{B}([0, v+r])$

$T(\Omega) = [0, r] \qquad \mathbb{P}(A) = \dfrac{\mathcal{L}^1(A)}{v+r} \quad \forall A \in \mathcal{B}([0,v+r])$

$$F_T(t) = \mathbb{P}(T \le t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge r \\ \dfrac{(v+\cancel{r}) - (\cancel{r} - t)}{v+r} & t \in [0, r) \end{cases}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \dfrac{v+t}{v+r} & t \in [0, r) \\ 1 & t \ge r \end{cases}$$



$$\mathbb{E}[T] = \int_0^{+\infty} (1 - F_T(t))\,dt = \int_0^r \left(1 - \frac{v+t}{v+r}\right) dt =$$

$$= \int_0^r \frac{r-t}{v+r}\,dt = \frac{-1}{2(v+r)} (t-r)^2 \Big|_{t=0}^{t=r} = \frac{r^2}{2(v+r)}$$

$$\mathbb{E}[T^2] = \int_0^{+\infty} (1 - F_{T^2}(t))\,dt =$$

$t > 0 \quad \mathbb{P}(T^2 \leq t) = \mathbb{P}(-\sqrt{t} \leq T \leq \sqrt{t}) =$

$= \mathbb{P}(T \leq \sqrt{t}) - \mathbb{P}(T < -\sqrt{t}) =$

$= F_T(\sqrt{t}) - \underbrace{F_T(-\sqrt{t})}_{=0} + \underbrace{\mathbb{P}(T = -\sqrt{t})}_{=0}$

$$\mathbb{E}[T^2] = \int_0^{+\infty} \left(1 - F_T(\sqrt{t})\right) dt$$

$$t > 0 \quad F_T(\sqrt{t}) = \begin{cases} \dfrac{v+\sqrt{t}}{v+r} & \sqrt{t} \in (0, r) \\ 1 & \sqrt{t} \geq r \end{cases}$$

$$= \begin{cases} \dfrac{v+\sqrt{t}}{v+r} & t \in (0, r^2) \\ 1 & t \geq r^2 \end{cases}$$

$$\mathbb{E}[T^2] = \int_0^{r^2} \left(1 - \frac{v+\sqrt{t}}{v+r}\right) dt = \int_0^{r^2} \frac{r-\sqrt{t}}{v+r} \, dt$$

$$= \frac{rt}{v+r} - \frac{2}{3} \frac{t^{3/2}}{v+r} \Bigg|_{t=0}^{t=r^2} = \frac{r^3}{v+r} - \frac{2}{3} \frac{r^3}{v+r} = \frac{r^3}{3(v+r)}$$

DISTRIBUZIONI DISCRETE

DELTA DI DIRAC

Fissato $x_0 \in \mathbb{R}$ pongo $\delta_{x_0}(A) = \begin{cases} 1 & \text{se } A \ni x_0 \\ 0 & \text{se } A \not\ni x_0 \end{cases}$

per ogni $A \in \mathcal{B}(\mathbb{R})$.

Se X è una v.a. allora $\mathbb{P}_X = \delta_{x_0}$ sse $X(\omega) = x_0$ $\mathbb{P}$-q.c.

Inoltre $\mathbb{E}[X] = x_0$, $Var[X] = 0$.

DISTRIBUZIONE DI BERNOULLI DI PARAMETRO $p \in [0,1]$

È la misura di probabilità $B(p)$ concentrata su $\{0,1\}$ t.c.

$$B(p)(\{0\}) = 1-p$$
$$B(p)(\{1\}) = p$$

$\Rightarrow$ Per ogni $A \in \mathcal{B}(\mathbb{R})$ si ha $B(p)(A) = \begin{cases} 1 & A \supset \{0,1\} \\ p & 1 \in A \quad 0 \notin A \\ 1-p & 0 \in A \quad 1 \notin A \\ 0 & 0 \notin A, \ 1 \notin A \end{cases}$

ovvero

$$B(p) = (1-p)\delta_0 + p\,\delta_1$$

Se X è una v.a. t.c. $\mathbb{P}_X = B(p)$, allora

$$\mathbb{E}[X] = 0 \cdot (1-p) + 1 \cdot p = p$$
$$\mathbb{E}[X^2] = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$
$$\Rightarrow Var[X] = p - (p)^2 = p(1-p)$$

Possiamo costruire una v.a. X t.c. $\mathbb{P}_X = B(p)$ pensando al lancio di una moneta su cui esce Testa con probabilità p

$\Omega = \{0,1\}$ $\quad \mathcal{E} = \mathcal{P}(\Omega)$ $\quad$ (1 = successo = Testa)

$\mathbb{P}(\{0\}) = 1-p$, $\mathbb{P}(\{1\}) = p$, $X(\omega) = \omega$

DISTRIBUZIONE BINOMIALE DI PARAMETRI $n \in \mathbb{N}$ E $p \in [0,1]$

È la misura di probabilità concentrata su $\{0, 1, \ldots, n\}$ t.c.

$$B(n,p)(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \quad \forall k = 0, 1, \ldots, n-1, n$$

È una distribuzione di probabilità?

$$\sum_{k=0}^{n} B(n,p)(\{k\}) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+(1-p))^n = 1^n = 1 \quad n$$

Se X è una v.a. t.c. $P_X = B(n,p)$ $\Rightarrow$

$$E[X] = \sum_{k=0}^{n} k\, B(n,p)(\{k\}) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= (1-p)^n \sum_{k=0}^{n} \binom{n}{k} k\, x^k \Big|_{x=\frac{p}{1-p}} = (1-p)^n\, x \sum_{k=0}^{n} \binom{n}{k} k\, x^{k-1} \Big|_{x=\frac{p}{1-p}}$$

$$= (1-p)^n \frac{p}{1-p} \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} x^k \Big|_{x=\frac{p}{1-p}} =$$

$$= (1-p)^n \frac{p}{1-p} \frac{d}{dx} (1+x)^n \Big|_{x=\frac{p}{1-p}} =$$

$$= (1-p)^n \frac{p}{1-p}\, n\, (1+x)^{n-1} \Big|_{x=\frac{p}{1-p}} \qquad 1+x = \frac{p}{1-p} + 1 = \frac{1}{1-p}$$

$$= \cancel{(1-p)^n} \frac{p}{\cancel{1-p}}\, n\, \cancel{(1-p)^{-n+1}} = np$$

$$E[X^2] = \sum_{k=0}^{n} k^2 B(n,p)(\{k\}) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= (1-p)^n \sum_{k=0}^{n} \binom{n}{k} k^2 x^k \Big|_{x=\frac{p}{1-p}} =$$

$$= (1-p)^n \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^k \Big|_{x=\frac{p}{1-p}} + \underbrace{(1-p)^n \sum_{k=0}^{n} \binom{n}{k} k\, x^k}_{=np} \Big|_{x=\frac{p}{1-p}}$$

$$= (1-p)^n x^2 \frac{d^2}{dx^2} \sum_{k=0}^{n} \binom{n}{k} x^k \Big|_{x=\frac{p}{1-p}} + np$$

$$= (1-p)^n \left(\frac{p}{1-p}\right)^2 \frac{d^2}{dx^2} (1+x)^n \Big|_{x=\frac{p}{1-p}} + np$$

$$= (1-p)^n \frac{p^2}{(1-p)^2}\, n(n-1)\, (1+x)^{n-2} \Big|_{x=\frac{p}{1-p}} + np$$

$$= \cancel{(1-p)^{n-2}}\, p^2\, n(n-1)\, \cancel{(1-p)^{2-n}} + np$$

$$\Rightarrow Var[X] = p^2 n(\cancel{n}-1) + np - \cancel{(np)^2} = np(1-p)$$

Posso costruire una v.a. con $\mathbb{P}_X = B(n,p)$?

Lancio una stessa moneta (su cui esce Testa con probabilità p) n volte

$\Omega = \{0,1\}^n$ $\quad \mathcal{E} = \mathcal{P}(\Omega)$ e, per ogni $\omega \in \Omega$

$\mathbb{P}(\{\omega\}) = p^k (1-p)^{n-k}$ dove $k = \#$ di componenti di ω che valgono 1

$X(\omega) = \sum_{i=1}^{n} \omega_i$, v.a. che conta i successi ottenuti in n lanci

$\Rightarrow \mathbb{P}_X = B(n,p)$.

PROPRIETÀ DEGLI EVENTI RARI

Sia $\{p_n\}$ successione in $[0,1]$ t.c.

$$\exists \lim_{n \to \infty} n p_n = \lambda \in (0, +\infty)$$

Per $k \in \mathbb{N}$ considero $B(n,p)(\{k\})$

k è fissato $\Rightarrow \forall k \geq n$ ho $B(n,p)(\{k\}) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$

Calcolo

$$\lim_{n \to \infty} B(n,p)(\{k\}) = \lim_{n \to \infty} \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{n!}{k!(n-k)!} p_n^k (1-p_n)^{n-k} =$$

$$= \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} (n p_n)^k \left[(1-p_n)^{\frac{1}{p_n}} \right]^{p_n (n-k)}$$

$$\to \frac{1}{k!} \, 1 \, \lambda^k \left(e^{-1}\right)^{\lambda - 0} = e^{-\lambda} \frac{\lambda^k}{k!}$$