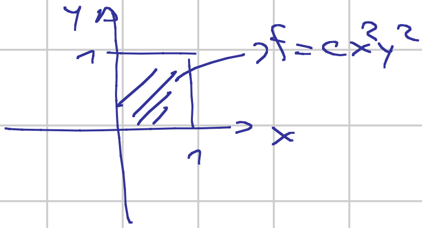
(X,Y) v.a. con densità congiunta

$$f(x,y) = \begin{cases} c x^2 y^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

Determinare il valore di c e la distribuzione della v.a. $(Z,T) := (3X, XY)$.



$$1 = \int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{[0,1]^2} c x^2 y^2\,dx\,dy =$$

$$= c\left(\int_0^1 x^2\,dx\right)\cdot\left(\int_0^1 y^2\,dy\right) = c\left(\int_0^1 x^2\,dx\right)^2 = c\left(\frac{x^3}{3}\Big|_0^1\right)^2 = \frac{c}{9}$$

$$\Rightarrow \boxed{c = 9}$$

$$(Z,T) = (3X, XY) = \varphi \circ (X,Y)$$

$$\longrightarrow \varphi : (x,y) \in \mathbb{R}^2 \longmapsto (3x, xy)$$

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$$\int_{\mathbb{R}^k} \psi(s_1 \dots s_k)\, \mathbb{P}_{\varphi\circ \underline{X}}(ds_1 \dots ds_k) = \int_{\mathbb{R}^N} \psi\circ\varphi(t_1 \dots t_N)\, \mathbb{P}_{\underline{X}}(dt_1 \dots dt_N)$$

$\underline{X} : \Omega \to \mathbb{R}^N$ $\quad\varphi : \mathbb{R}^N \to \mathbb{R}^k$ boreliana

$\psi : \mathbb{R}^k \to \mathbb{R}_+$ boreliana

$$\int_{\mathbb{R}^2} \psi(s_1, s_2)\, \mathbb{P}_{\varphi\circ(X,Y)}(ds_1, ds_2) = \int_{\mathbb{R}^2} \psi(3t_1, t_1 t_2)\, \mathbb{P}_{(X,Y)}(dt_1\,dt_2) =$$

$$= \int_{\mathbb{R}^2} \psi(3t_1, t_1 t_2)\, f(t_1, t_2)\,dt_1\,dt_2 =$$

$$= \int_{[0,1]^2} \psi(\underline{3t_1, t_1 t_2})\, 9 \underline{t_1^2\, t_2^2}\,dt_1\,dt_2$$

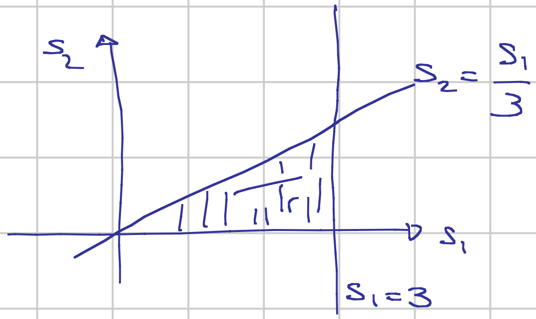
$$(t_1, t_2) \in [0,1]^2$$

$$s_1 = 3t_1 \qquad s_2 = t_1 t_2$$

$$t_1 = \frac{s_1}{3} \qquad t_2 = \frac{s_2}{t_1} = \frac{3s_2}{s_1}$$

$$\begin{cases} 0 \le \frac{s_1}{3} \le 1 \\ 0 \le \frac{3s_2}{s_1} \le 1 \end{cases} \qquad \begin{cases} 0 \le s_1 \le 3 \\ 0 \le 3s_2 \le s_1 \end{cases}$$

$$0 \le s_1 \le 3$$
$$0 \le s_2 \le \frac{s_1}{3}$$



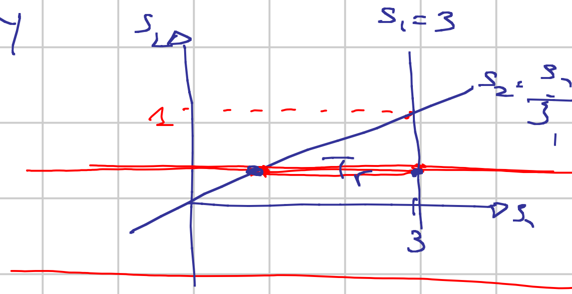
$$J = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{3s_2}{s_1^2} & \frac{3}{s_1} \end{pmatrix} \qquad |\det J| = \left|\frac{1}{3}\cdot\frac{3}{s_1} - 0\right| = \frac{1}{s_1}$$

$$\int_{T_r} \psi(s_1, s_2)\, 9s_2^2 \cdot \frac{1}{s_1}\, ds_1\, ds_2 =$$
$$= \int_{\mathbb{R}^2} \psi(s_1, s_2)\, \frac{9s_2^2}{s_1} \mathbb{1}_{T_r}(s_1, s_2)\, ds_1\, ds_2$$

$$f_{Z,T}(s_1, s_2) = \frac{9s_2^2}{s_1} \mathbb{1}_{T_r}(s_1, s_2)$$

$$(Z, T) = (3X, \underbrace{XY}) \qquad T = XY$$

$$f_T(s_2) = \int_{\mathbb{R}} f_{Z,T}(s_1, s_2)\, ds_1$$



$f_T(s_2) = 0 \quad \forall s_2 < 0 \;\; e \;\; \forall s_2 > 1$

$s_2 \in (0, 1)$

$$f_T(s_2) = \int_{3s_2}^{3} \frac{9s_2^2}{s_1}\, ds_1 = 9s_2^2 \log|s_1| \Big|_{s_1 = 3s_2}^{s_1 = 3}$$

$$= 9s_2^2\left(\log(3) - \log(3s_2)\right) = 9s_2^2 \log \frac{\cancel{3}\,1}{\cancel{3}s_2} = -9s_2^2 \log s_2$$

$$f_T(s_2) = \begin{cases} -9s_2^2 \log(s_1) & s_2 \in (0,1), \\ 0 & \text{altrimenti.} \end{cases}$$

$\varphi : (x,y) \in \mathbb{R}^2 \mapsto xy \in \mathbb{R}$ $\qquad$ $\psi : \mathbb{R} \to \mathbb{R}_+$ boreliana

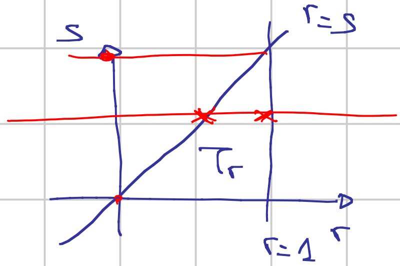
$$\int_{\mathbb{R}} \psi(s)\, \mathbb{P}_{\varphi(X,Y)}(dt_1 dt_2) = \int_{\mathbb{R}^2} \psi(\varphi(t_1,t_2))\, \mathbb{P}_{X,Y}(dt_1,dt_2) =$$

$$= \int_{\mathbb{R}^2} \psi(t_1 t_2)\, 9 t_1^2 t_2^2\, \mathbb{1}_{[0,1]^2}(t_1,t_2)\, dt_1 dt_2 =$$

$$= \int_{[0,1]^2} \psi(t_1 t_2)\, 9 t_1^2 t_2^2\, dt_1 dt_2$$

$s = t_1 t_2 \qquad t_1 = r$

$r = t_1 \qquad t_2 = \frac{s}{r}$



$$\begin{matrix} 0 \le t_1 \le 1 \\ 0 \le t_2 \le 1 \end{matrix} \qquad \begin{cases} 0 \le r \le 1 \\ 0 \le \frac{s}{r} \le 1 \end{cases} \qquad \begin{cases} 0 \le r \le 1 \\ 0 \le s \le r \end{cases}$$

$$J = \begin{pmatrix} 1 & 0 \\ -\frac{s}{r^2} & \frac{1}{r} \end{pmatrix} \qquad |\det J| = \frac{1}{r}$$

$$= \int_{T_r} \psi(s)\, 9 s^2 \frac{1}{r}\, dr\, ds = \int_0^1 \psi(s) \left(\int_s^1 \frac{9 s^2}{r}\, dr \right) ds$$

$$= \int_0^1 \psi(s) \underbrace{\left(9 s^2 \int_s^1 \frac{1}{r}\, dr \right)}\, ds$$

$$\int_0^1 \psi(s)\, 9 s^2 \left(\log|r| \Big|_{r=s}^{r=1} \right) ds = \int_0^1 \psi(s)\, 9 s^2 (-\log(s))\, ds$$

$$= \int_{\mathbb{R}} \psi(s) g(s)\, ds \qquad g(s) = \begin{cases} -9 s^2 \log(s) & s \in (0,1) \\ 0 & \text{altrimenti.} \end{cases}$$

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$$f(x,y) = \begin{cases} c(x-y)^2 & 0 \le x, y \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

Trovare c e la distribuzione congiunta di $(X^2, X^2 Y^2)$

$$1 = \int_{\mathbb{R}^2} f(x,y)\, dx dy = c \int_{[0,1]^2} (x-y)^2\, dx dy =$$

$$= c \int_0^1 \left(\int_0^1 (x-y)^2 \, dx \right) dy = c \int_0^1 \frac{(x-y)^3}{3} \Big|_{x=0}^{x=1} dy =$$

$$= \frac{c}{3} \int_0^1 \left((1-y)^3 + y^3 \right) dy = \frac{c}{12} \left(-(1-y)^4 + y^4 \right) \Big|_{y=0}^{y=1}$$

$$= \frac{c}{12} (1+1) = \frac{c}{6} \quad \Rightarrow \boxed{c = 6}$$

$T, Z = (X^2, X^2Y^2)$ $\quad f_0(X,Y) \quad f(x,y) = (x^2, x^2y^2)$

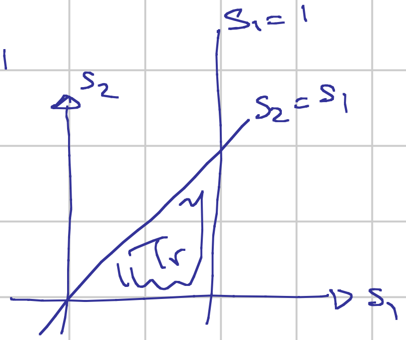
$$\int_{\mathbb{R}^2} \varphi(s_1, s_2) \, \mathbb{P}_{f_0(X,Y)}(ds_1 \, ds_2) = \int_{\mathbb{R}^2} \varphi(f(t_1,t_2)) \, \mathbb{P}_{(X,Y)}(dt_1 \, dt_2) =$$

$$= \int_{\mathbb{R}^2} \varphi(t_1^2, t_1^2 t_2^2) \, 6 (t_1 - t_2)^2 \, \mathbb{1}_{[0,1]^2}(t_1, t_2) \, dt_1 dt_2 =$$

$s_1 = t_1^2$ $\qquad t_1 = \sqrt{s_1}$

$s_2 = t_1^2 t_2^2$ $\qquad t_2 = \frac{\sqrt{s_2}}{t_1} = \frac{\sqrt{s_2}}{\sqrt{s_1}}$

$$\begin{cases} 0 \le t_1 \le 1 \\ 0 \le t_2 \le 1 \end{cases} \qquad \begin{cases} 0 \le \sqrt{s_1} \le 1 \\ 0 \le \frac{\sqrt{s_2}}{\sqrt{s_1}} \le 1 \end{cases}$$



$$\begin{cases} 0 \le s_1 \le 1 \\ 0 \le s_2 \le s_1 \end{cases}$$

$t_1 = s_1^{1/2}$

$t_2 = s_2^{1/2} s_1^{-1/2}$

$$J = \begin{pmatrix} \frac{1}{2} s_1^{-1/2} & 0 \\ \cancel{\ast} & \frac{1}{2} s_2^{-1/2} s_1^{-1/2} \end{pmatrix}$$

$$|\det J| = \frac{1}{4} s_2^{-1/2} s_1^{-1}$$

$$= \int_{Tr} \varphi(s_1, s_2) \underbrace{6 \left(s_1^{1/2} - s_2^{1/2} s_1^{-1/2} \right)^2 \frac{1}{4} s_2^{-1/2} s_1^{-1}}_{\frac{3}{2} s_1^{-2} \left(s_1 - s_2^{1/2} \right)^2 s_2^{-1/2}} ds_1 \, ds_2$$

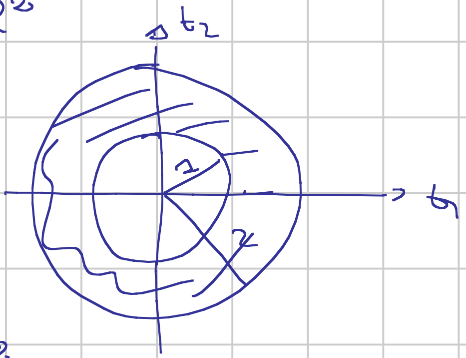
(X,Y)

$$f(x,y) = \begin{cases} \frac{3}{14\pi}\sqrt{x^2+y^2} & 1 \le x^2+y^2 \le 4 \\ 0 & \text{altrimenti} \end{cases}$$

$Z = X^2+Y^2 \qquad Z = \varphi \circ (X,Y) \qquad \varphi(x,y) = x^2+y^2$

$$\int_{\mathbb{R}} \psi(s)\, \mathbb{P}_{Z=\varphi\circ(X,Y)}(ds) = \int_{\mathbb{R}^2} \psi(\varphi(t_1,t_2))\, \mathbb{P}_{X,Y}(dt_1 dt_2) =$$

$$= \int_{\mathbb{R}^2} \psi(t_1^2+t_2^2)\, \frac{3}{14\pi}\sqrt{t_1^2+t_2^2}\; \mathbb{1}_C(t_1,t_2)\, dt_1 dt_2$$



$$= \int_1^2 dr \int_0^{2\pi} \psi(r^2)\, \frac{3}{14\pi}\, r \cdot r\, d\theta =$$

$$= \int_1^2 \psi(r^2)\, \frac{3}{14\pi}\, 2\pi r^2\, dr$$

$$= \int_1^2 \psi(r^2)\, \frac{3}{7} r^2\, dr$$

$s = r^2 \qquad r = \sqrt{s} \qquad dr = \frac{1}{2\sqrt{s}}\, ds$

$r=1 \quad s=1$

$r=2 \quad s=4$

$$= \int_1^4 \psi(s)\, \frac{3}{7}\, s\, \frac{1}{2\sqrt{s}}\, ds = \int_1^4 \psi(s)\, \frac{3}{14}\sqrt{s}\, ds$$

$$= \int_{\mathbb{R}} \psi(s)\, g(s)\, ds \qquad g(s) = \begin{cases} \frac{3}{14}\sqrt{s} & s \in (1,4) \\ 0 & \text{altrimenti} \end{cases}$$

Trovare r t.c. $\underbrace{\mathbb{P}(X^2+Y^2 \ge r^2)}_{1-F_Z(r^2)} = \frac{1}{2}$

$1 - F_Z(r^2) = \frac{1}{2} \qquad F_Z(r^2) = \frac{1}{2}$

$$F_Z(t) = \int_{-\infty}^{t} g(s)\,ds = \begin{cases} 0 & t<1 \\ \int_1^t \frac{3}{14}\sqrt{s}\,ds & t\in[1,4) \\ 1 & t\geq 4 \end{cases}$$

$$\begin{cases} \int_1^{r^2} \frac{3}{14}\sqrt{s}\,ds = \frac{1}{2} \\ r^2 \in [1,4) \end{cases}$$

$$\int_1^{r^2} \frac{3}{14} s^{1/2}\,ds = \frac{\cancel{3}}{\underset{7}{\cancel{14}}}\,\frac{\overset{1}{\cancel{2}}}{\cancel{3}}\, s^{3/2}\Big|_{s=1}^{s=r^2} = \frac{1}{2}$$

$$\frac{1}{7}(r^3-1) = \frac{1}{2} \qquad \frac{1}{7}r^3 = \frac{1}{2}+\frac{1}{7} = \frac{9}{14}$$

$$r^3 = \frac{9}{2} \qquad r = \sqrt[3]{\frac{9}{2}}$$

$$r^2<4 \qquad r^3<8$$

$$\mathbb{P}_X = U([0,\tfrac{1}{2}]) \qquad \mathbb{P}_Y = \mathrm{Ber}(p) \qquad p\in(0,1)$$

X, Y indipendenti

$$Z := \max\{X,Y\}$$

$$F_Z(t) = \mathbb{P}(Z\leq t) = \mathbb{P}(\max\{X,Y\}\leq t) =$$
$$= \mathbb{P}(X\leq t, Y\leq t) = \mathbb{P}(X\leq t)\mathbb{P}(Y\leq t) = F_X(t)F_Y(t)$$

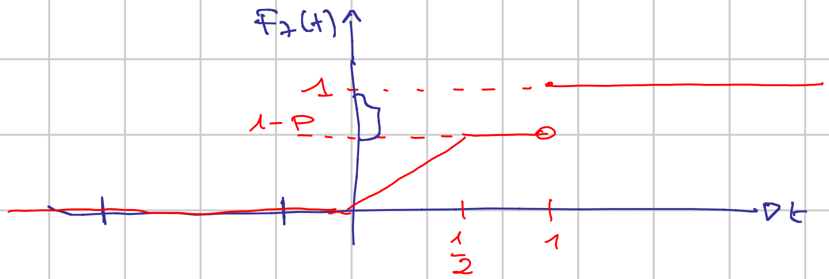
$t<0 \quad F_Z(t) = 0$

$0\leq t<\frac{1}{2} \quad F_X(t) = \frac{t}{1/2} = 2t \qquad F_Y(t) = 1-p$

$\qquad F_Z(t) = 2(1-p)t$

$\frac{1}{2}\leq t<1 \quad F_X(t) = 1 \qquad F_Y(t) = 1-p \qquad F_Z(t) = 1-p$

$t\geq 1 \quad F_X(t) = F_Y(t) = 1$



$$2(1-p)\mathbb{1}_{(0,\frac{1}{2})} + p\delta_1$$

$$E[Z] = \int_0^{+\infty}(1-F_Z(t))\,dt - \cancel{\int_{-\infty}^{0} F_Z(t)\,dt}$$

$$= \int_0^{1/2}(1-2(1-p)t)\,dt + \int_{\frac{1}{2}}^{1} (1-(1-p))\,dt$$

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$\mathbb{P}_X = \mathcal{B}(2, \frac{1}{2})$ $\quad\quad Y(\Omega) = \{1,2,3\}$

$\mathbb{P}(Y=j \mid X=0) = a(3-j)$

$\mathbb{P}(Y=j \mid X=1) = b|1-j|$ $\quad\quad \forall j = 1,2,3$

$\mathbb{P}(Y=j \mid X=2) = cj$

$\forall (h,k) \in (X,Y)(\Omega) \quad\quad \mathbb{P}(X=h, Y=k)$

$\mathbb{P}(Y=1 \mid X=0) + \mathbb{P}(Y=2 \mid X=0) + \mathbb{P}(Y=3 \mid X=0) = 1$

$a(3-1) + a(3-2) + a(3-3) = 1$

$2a + a = 1 \quad\quad a = \frac{1}{3}$

$\mathbb{P}(Y=1 \mid X=1) + \mathbb{P}(Y=2 \mid X=1) + \mathbb{P}(Y=3 \mid X=1) = 1$

$b|1-1| + b|1-2| + b|1-3| = 1$

$b + 2b = 1 \quad\quad b = \frac{1}{3}$

$\mathbb{P}(Y=1 \mid X=2) + \mathbb{P}(Y=2 \mid X=2) + \mathbb{P}(Y=3 \mid X=2) = 1$

$c \cdot 1 + c \cdot 2 + c \cdot 3 = 1$

$6c = 1 \quad\quad c = \frac{1}{6}$

$(X,Y)(\Omega) \subset X(\Omega) \times Y(\Omega) = \{0,1,2\} \times \{1,2,3\}$

$\mathbb{P}(X=0, Y=1) = \mathbb{P}(Y=1 \mid X=0)\,\mathbb{P}(X=0) = \frac{2}{3}\binom{2}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^2$

$= \frac{2}{3}\,\frac{1}{4} = \frac{1}{6}$

$\mathbb{P}(X=0, Y=2) = \mathbb{P}(Y=2 \mid X=0)\,\mathbb{P}(X=0) = \frac{1}{3}\,\frac{1}{4} = \frac{1}{12}$

$\mathbb{P}(X=0, Y=3) = \mathbb{P}(Y=3 \mid X=0)\,\mathbb{P}(X=0) = 0$

$\mathbb{P}_X = \exp(\lambda)$ $\qquad$ $\mathbb{P}_Y = Ber(p)$

X, Y indipendenti

$Z := X + Y$

$$F_Z(t) = \mathbb{P}(Z \le t) = \mathbb{P}(X + Y \le t) =$$

$$= \mathbb{P}(X + Y \le t, Y = 0) + \mathbb{P}(X + Y \le t, Y = 1)$$

$$= \mathbb{P}(X \le t, Y = 0) + \mathbb{P}(X \le t-1, Y = 1)$$

$$= \mathbb{P}(X \le t)\underbrace{\mathbb{P}(Y = 0)}_{1-p} + \mathbb{P}(X \le t-1)\underbrace{\mathbb{P}(Y = 1)}_{p}$$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda t} & t \ge 0 \end{cases}$$

$t < 0$ $\qquad$ $F_Z(t) = 0$

$0 \le t < 1$ $\qquad$ $F_Z(t) = (1-p)(1 - e^{-\lambda t})$

$1 \le t$ $\qquad$ $F_Z(t) = (1-p)(1 - e^{-\lambda t}) + p(1 - e^{-\lambda(t-1)})$

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{1}{\lambda} + p$$