Titolo nota 15/12/2014

X_1, X_2, X_3 v.a. indipendenti

$\mathbb{P}_{X_1} = \mathbb{P}_{X_2} = \mathbb{P}_{X_3} = \exp(\lambda)$

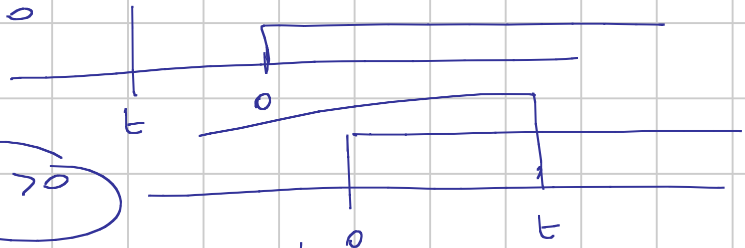
$S_2 := X_1 + X_2 \qquad f_{S_2}(t) = \int_{\mathbb{R}} f_{X_1}(t-y) \underbrace{f_{X_2}(y)}\, dy$

$f_{X_1}(x) = f_{X_2}(x) = \lambda e^{-\lambda x} \mathbb{1}_{(0,+\infty)}(x)$

$f_{S_2}(t) = \int_0^{+\infty} f_{X_1}(t-y)\, \lambda e^{-\lambda y}\, dy$

$f_{X_1}(t-y) \neq 0 \quad$ sse $\quad t-y>0 \quad$ sse $\quad y<t$

$t<0$



$\Rightarrow f_{S_2}(t) = 0$

$t>0$

$f_{S_2}(t) = \int_0^t \lambda e^{-\lambda(t-\cancel{y})}\, \cancel{\lambda e^{-\lambda y}}\, dy = t\lambda^2 e^{-\lambda t}$

$f_{S_2}(t) = \lambda(\lambda t) e^{-\lambda t} \mathbb{1}_{(0,+\infty)}(t)$

$\overline{X}_2 = \frac{X_1+X_2}{2} = \frac{S_2}{2} \qquad \mathbb{P}_X = f(x)\,dx$

$\mathbb{P}_{aX+b} = g(x)\,dx$

$f_{\overline{X}_2}(t) = 2\lambda(2\lambda t) e^{-2\lambda t} \mathbb{1}_{(0,+\infty)}(2t) \qquad g(x) = \frac{1}{|a|} f\left(\frac{x-b}{a}\right)$

$= 2\lambda(2\lambda t) e^{-2\lambda t} \mathbb{1}_{(0,+\infty)}(t)$

$S_3 := X_1 + X_2 + X_3 = (X_1 + X_2) + X_3 = S_2 + X_3$

$f_{S_3}(t) = \int_{\mathbb{R}} f_{S_2}(t-y) f_{X_3}(y)\, dy$

$$= \int_0^{+\infty} f_{S_2}(t-y)\, \lambda e^{-\lambda y}\, dy =$$

$t-y>0 \quad y<t \qquad t<0 \quad f_{S_3}(t)=0$

$$t>0 \quad f_{S_3}(t) = \int_0^t \lambda(\lambda(t-y))\, e^{-\lambda(t-\cancel{y})} \lambda \cancel{e^{-\lambda y}}\, dy$$

$$= \lambda^3 e^{-\lambda t} \int_0^t (t-y)\, dy = \lambda^3 e^{-\lambda t}\left(-\frac{1}{2}(t-y)^2\Big|_{y=0}^{t}\right) =$$

$$= \frac{\lambda^3 t^2}{2} e^{-\lambda t} = \frac{\lambda(\lambda t)^2}{2} e^{-\lambda t}$$

$$f_{S_3}(t) = \frac{\lambda(\lambda t)^2}{2} e^{-\lambda t}\, \mathbb{1}_{(0,+\infty)}(t)$$

$$\bar{X}_3 = \frac{X_1+X_2+X_3}{3} = \frac{S_3}{3} \qquad b=0 \quad a=\frac{1}{3}$$

$$f_{\bar{X}_3}(t) = 3\,\frac{\lambda}{2}\,(3\lambda t)^2\, e^{-3\lambda t}\, \mathbb{1}_{(0,+\infty)}(3t)$$

$$= \frac{3\lambda(3\lambda t)^2}{2} e^{-3\lambda t}\, \mathbb{1}_{(0,+\infty)}(t)$$

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$\mathbb{P}_X = U([0,3]) \qquad \mathbb{P}_Y = B(3,p)$

X e Y indipendenti

$\mathbb{P}(X+Y\le 3) \qquad \mathbb{P}(XY)\le 3$

$\{X+Y\le 3\} \qquad Y(\Omega) = \{0,1,2,3\}$

$$\{X+Y\le 3\} = \bigcup_{k=0}^{3} \{X+k\le 3, Y=k\}$$

$$\mathbb{P}(X+Y\le 3) = \sum_{k=0}^{3} \mathbb{P}(\underbrace{X\le 3-k}, \underbrace{Y=k}) =$$

$$= \sum_{k=0}^{3} \mathbb{P}(X\le \underbrace{3-k}_{t})\,\mathbb{P}(Y=k)$$

$$= \sum_{k=0}^{3} \frac{3-k}{3}\binom{3}{k} p^k (1-p)^{3-k} \overset{?}{=} 1-p$$

$\mathbb{P}(XY \le 3)$

$\{XY \le 3\} = \bigcup_{k=0}^{3} \{X \cdot k \le 3, Y=k\}$

$$\mathbb{P}(XY \le 3) = \sum_{k=0}^{3} \mathbb{P}(kX \le 3, Y=k) =$$

$$= \sum_{k=0}^{3} \mathbb{P}(kX \le 3)\mathbb{P}(Y=k) =$$

$$= \mathbb{P}(Y=0) + \sum_{k=1}^{3} \mathbb{P}\left(X \le \frac{3}{k}\right)\mathbb{P}(Y=k)$$

$$= \binom{3}{0} p^0 (1-p)^3 + \sum_{k=1}^{3} \frac{3/k}{3} \binom{3}{k} p^k (1-p)^{3-k} =$$

$$= (1-p)^3 + \sum_{k=1}^{3} \frac{1}{k}\binom{3}{k} p^k (1-p)^{3-k} \overset{?}{=} 1 - \frac{5}{2}p^2 + \frac{11}{6}p^3$$

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$\lambda > 0 \quad p \in [0,1]$

$\mathbb{P}_Y = \text{Poiss}(\lambda) \qquad Y(\Omega) = \mathbb{N}_0$

$X(\Omega) = \{0,1\} \qquad \mathbb{P}(X=1 \mid Y=k) = p^k \quad \forall k \in \mathbb{N}_0$

Densità di X e di $Z := XY$

$\mathbb{P}(\cdot \mid Y=k) \qquad \mathbb{P}(X=1 \mid Y=k) = p^k$

$\qquad \mathbb{P}(X=0 \mid Y=k) = 1-p^k$

$$\mathbb{P}(X=1) = \sum_{k=0}^{\infty} \mathbb{P}(X=1, Y=k) = \sum_{k=0}^{\infty} \mathbb{P}(X=1 \mid Y=k)\mathbb{P}(Y=k)$$

$$= \sum_{k=0}^{\infty} p^k e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda p)^k}{k!} = e^{-\lambda}e^{\lambda p} = e^{-\lambda(1-p)}$$

$$\mathbb{P}_X = \text{Ber}\left(e^{-\lambda(1-p)}\right)$$

$Z(\Omega) = \mathbb{N}_0$

$k \in \mathbb{N}_0 \qquad \mathbb{P}(Z=k) = \mathbb{P}(XY=k)$

$\{XY=k\} = \{XY=k, X=0\} \cup \{XY=k, X=1\} \leftarrow$

$$= \bigcup_{j=0}^{\infty} \{XY=k, Y=j\}$$

$$\mathbb{P}(XY=k) = \mathbb{P}(\underbrace{XY=k}, X=0) + \mathbb{P}(XY=k, X=1)$$

$k=0$ $\quad \mathbb{P}(XY=0) = \mathbb{P}(X=0) + \mathbb{P}(Y=0, X=1) =$

$$= 1 - e^{-\lambda(1-p)} + \underbrace{\mathbb{P}(X=1 \mid Y=0)}_{p^0 = 1} \mathbb{P}(Y=0)$$

$$= 1 - e^{-\lambda(1-p)} + \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda(1-p)} + e^{-\lambda}$$

$k \geq 1$ $\quad \mathbb{P}(XY=k) = \mathbb{P}(Y=k, X=1)$

$$= \mathbb{P}(X=1 \mid Y=k)\, \mathbb{P}(Y=k)$$

$$= p^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \frac{(\lambda p)^k}{k!}$$

$\mathbb{P}_X = \mathbb{P}_Y = N(0,1)$ indipendenti

$Z := \operatorname{sgn}(X) \cdot |Y| \implies \mathbb{P}_Z = N(0,1)$

$$F_Z(t) = \mathbb{P}(Z \leq t) = \mathbb{P}(X>0, |Y| \leq t) + \mathbb{P}(X<0, -|Y| \leq t) + \overbrace{\mathbb{P}(\cancel{X=0})}^{=0}$$

$$= \underbrace{\mathbb{P}(X>0)}_{1/2} \mathbb{P}(|Y| \leq t) + \underbrace{\mathbb{P}(X<0)}_{1/2} \mathbb{P}(|Y| \geq -t)$$

(1) $t>0$ $\quad \frac{1}{2}\left(\mathbb{P}(-t \leq Y \leq t) + 1\right) =$

$$= \frac{1}{2}\left(\mathbb{P}(Y \leq t) - \mathbb{P}(Y \leq -t) + 1\right)$$

$$= \frac{1}{2}\left(\Phi(t) - \Phi(-t) + 1\right)$$

$$\boxed{\Phi(-t) = 1 - \Phi(t)} \qquad \frac{1}{2}\left(\Phi(t) - \cancel{1} + \Phi(t) + \cancel{1}\right) = \Phi(t)$$

(2) $t<0$ $\quad \frac{1}{2}\mathbb{P}(|Y| > -t) =$

$$= \frac{1}{2}\left(\mathbb{P}(Y > -t) + \mathbb{P}(Y < t)\right)$$

$$= \frac{1}{2}\left(1 - \Phi(-t) + \Phi(t)\right) = \frac{1}{2}\left(\Phi(t) + \Phi(t)\right) = \Phi(t)$$

$F_Z = \Phi \qquad \mathbb{P}_Z = N(0,1)$