LEGGE DEI GRANDI NUMERI

Titolo nota 04/12/2014

Sia $\{X_n\}_{n\in\mathbb{N}}$ successione di v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$.

Considero $S_n := \sum_{k=1}^{n} X_k \quad n \in \mathbb{N}$

Supponiamo che

1) le $\{X_n\}_{n\in\mathbb{N}}$ siano a due a due scorrelate

2) tutte le $\{X_n\}_{n\in\mathbb{N}}$ abbiano lo stesso valore atteso finito : $\exists\, E \in \mathbb{R} : \mathbb{E}[X_n] = E \quad \forall n \in \mathbb{N}$

3) le varianze $Var[X_n]$ $n \in \mathbb{N}$ siano equilimitate cioè $\exists\, C \in \mathbb{R}$ t.c. $Var[X_n] \leq C^2 \quad \forall n \in \mathbb{N}$

Allora

$$\int_\Omega \left| \frac{S_n(\omega)}{n} - E \right|^2 \mathbb{P}(d\omega) \leq \frac{C^2}{n} \qquad \forall n \in \mathbb{N}$$

In particolare

$$\mathbb{P}\left(\left| \frac{S_n}{n} - E \right| \geq \delta \right) \leq \frac{C^2}{n\delta^2} \qquad \forall \delta > 0 \ \forall n \in \mathbb{N}$$

DIM

$$S_n := \sum_{k=1}^{n} X_k \Rightarrow \mathbb{E}[S_n] = \sum_{k=1}^{n} \mathbb{E}[X_i] = nE$$

$$\mathbb{E}\left[\frac{S_n}{n}\right] = E \qquad \forall n \in \mathbb{N}$$

$$Var[S_n] = Var\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} Var[X_k] + 2\sum_{i<j} \underbrace{Cov(X_i, X_j)}_{=0}$$

$$\Rightarrow Var[S_n] = \sum_{k=1}^{n} Var[X_k] \leq \sum_{k=1}^{n} C^2 = C^2 n$$

$$Var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} Var[S_n] \leq \frac{C^2}{n}$$

$$\int_\Omega \left| \frac{S_n(\omega)}{n} - E \right|^2 \mathbb{P}(d\omega) = \mathbb{E}\left[\left(\frac{S_n}{n} - E\right)^2\right] = Var\left[\frac{S_n}{n}\right] \leq$$

$$\leq \frac{C^2}{n}$$

Sia $\delta > 0$

$$\mathbb{P}\left(\left| \frac{S_n}{n} - E \right| \geq \delta \right) \leq \frac{Var\left[\frac{S_n}{n}\right]}{\delta^2} \leq \frac{C^2}{n\delta^2} \qquad \forall n \in \mathbb{N}$$

$\frac{S_n}{n}$

$x_1 \quad x_2 \quad \dots \quad x_n$

$x_i = X_i(\omega) \qquad \omega \in \Omega \qquad (\Omega, \mathcal{E}, \mathbb{P})$

$\sum_{i=1}^{n} x_i$

$\Omega = \{0,1\}^\infty \qquad X_i(\omega) = \omega_i$