V.A. vettoriali 2

Titolo nota 21/11/2014

$$\mathcal{L}^1(\Omega, \mathbb{P}) = \{ X : \Omega \to \mathbb{R} : \mathbb{E}[|X|] < +\infty \}$$
$$\mathcal{L}^2(\Omega, \mathbb{P}) = \{ X : \Omega \to \mathbb{R} : \mathbb{E}[X^2] < +\infty \}$$

$$\mathcal{L}^2(\Omega, \mathbb{P}) \subseteq \mathcal{L}^1(\Omega, \mathbb{P})$$

$$\underline{\Phi} : (X, Y) \in \mathcal{L}^2(\Omega, \mathbb{P}) \times \mathcal{L}^2(\Omega, \mathbb{P}) \longmapsto \mathbb{E}[XY] \in \mathbb{R}$$

$$\Psi(X) = \sqrt{\Phi(X, X)}$$

<u>PROPR</u> Ψ è una seminorma su $\mathcal{L}^2(\Omega, \mathbb{P})$

cioè $\Psi(X) \geq 0 \quad \forall X \in \mathcal{L}^2(\Omega, \mathbb{P})$

$\Psi(X) = 0 \implies X = 0$ q.c.

$\Psi(\alpha X) = |\alpha| \, \Psi(X) \quad \forall X$ e $\forall \alpha \in \mathbb{R}$

DISUGUAGLIANZA TRIANGOLARE $\to$ $\Psi(X+Y) \leq \Psi(X) + \Psi(Y)$

<u>DIM</u> $\Psi(X+Y) = \sqrt{\mathbb{E}[(X+Y)^2]}$

$$\underline{\mathbb{E}[(X+Y)^2]} = \mathbb{E}[X^2 + 2XY + Y^2] =$$
$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2]$$
$$\leq \mathbb{E}[X^2] + 2\sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]} + \mathbb{E}[Y^2]$$
$$= \underline{\left(\sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]}\right)^2}$$

$$\Psi(X+Y) \leq \Psi(X) + \Psi(Y) \quad \boxtimes$$

Date X e Y v.a. su $(\Omega, \mathcal{E}, \mathbb{P})$ di quadrato sommabile definisco COVARIANZA di X e Y

$$\mathrm{Cov}(X, Y) = \mathbb{E}[\underbrace{(X - \mathbb{E}[X])}_{\in \mathcal{L}^2(\Omega, \mathbb{P})}\underbrace{(Y - \mathbb{E}[Y])}_{\in \mathcal{L}^2(\Omega, \mathbb{P})}] \in \mathbb{R}$$

PROPRIETÀ DELLA COVARIANZA

$\mathrm{Cov} : \mathcal{L}^2(\Omega, \mathbb{P}) \times \mathcal{L}^2(\Omega, \mathbb{P}) \to \mathbb{R}$

- è bilineare simmetrica
- $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $\mathrm{Cov}(\alpha X + \beta Y, Z) = \alpha \mathrm{Cov}(X, Z) + \beta \mathrm{Cov}(Y, Z)$
- $\mathrm{Cov}(X, X) = \mathrm{Var}[X]$

$$\mathrm{Cov}(X, Y) = \int_{\mathbb{R}^2} xy \, \mathbb{P}_{X,Y}(dx\,dy) - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \int_{\mathbb{R}^2} (x - \mathbb{E}[X])(y - \mathbb{E}[Y]) \, \mathbb{P}_{X,Y}(dx\,dy)$$

- $|\mathrm{Cov}(X, Y)| \leq \sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}$
- $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X, Y)$

$$\mathrm{Var}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathrm{Var}[X_i] + 2 \sum_{1 \leq i < j \leq N} \mathrm{Cov}(X_i, X_j)$$

$$-1 \leq \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \leq 1$$

si dice COEFFICIENTE di CORRELAZIONE di X e Y

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$(\Omega, \mathcal{E}, \mathbb{P})$ spazio probabilizzato

$X, Y : \Omega \to \mathbb{R}$ v.a.

$(X, Y) : \Omega \to \mathbb{R}^2$ si dice che (X, Y) ha distribuzione assolutamente continua se $\exists f : \mathbb{R}^2 \to \overline{\mathbb{R}}_+$ SOMMABILE secondo Lebesgue t.c.

$$\mathbb{P}_{X,Y}(A) = \int_A f(x,y) \, dx\,dy \quad \forall A \in \mathcal{B}(\mathbb{R}^2)$$

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$$f(x,y) = \begin{cases} \dfrac{2xy}{(x^2+y^2)^{3/2}} & x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1 \\ 0 & \text{altrimenti} \end{cases}$$

$$1 = \mathbb{P}_{X,Y}(\mathbb{R}^2) = \int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_D f(x,y)\,dx\,dy =$$

$$= \int_D \frac{2xy}{(x^2+y^2)^{3/2}}\,dx\,dy \qquad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \qquad r \in (0,1)$$

$$\theta \in \left(0, \tfrac{\pi}{2}\right)$$

$$= \int_0^1 dr \int_0^{\pi/2} \frac{2\cancel{r^2}\cos\theta\sin\theta}{\cancel{r^3}}\,\cancel{r}\,d\theta \qquad |\det J| = r$$

$$= \left(\int_0^1 dr\right)\left(\int_0^{\pi/2} 2\cos\theta\sin\theta\,d\theta\right) = 1 \cdot \sin^2\theta \Big|_{\theta=0}^{\theta=\pi/2} = 1$$

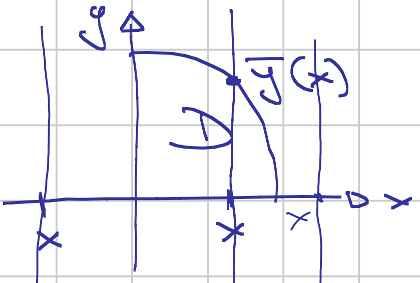
b) Distribuzione di X?

$A \in \mathcal{B}(\mathbb{R}) \qquad \mathbb{P}_X(A) = \mathbb{P}(X \in A) = \mathbb{P}(X \in A, Y \in \mathbb{R}) =$

$$= \int_{A \times \mathbb{R}} f(x,y)\,dx\,dy =$$

$$= \int_A \underbrace{\left(\int_{\mathbb{R}} f(x,y)\,dy\right)}_{g(x)} dx$$

$\forall A \in \mathcal{B}(\mathbb{R}) \qquad \mathbb{P}_X(A) = \int_A g(x)\,dx \Rightarrow \mathbb{P}_X = g(x)\,dx$



$$g(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$x < 0 \Rightarrow g(x) = 0$

$x > 1 \Rightarrow g(x) = 0$

$$x \in (0,1) \quad g(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_{(-\infty,0)} 0 + \int_0^{\bar{y}(x)} f(x,y)\,dy + \int_{(\bar{y}(x),+\infty)} 0$$

$$= \int_0^{\bar{y}(x)} f(x,y)\,dy$$

$$\begin{cases} x^2 + y^2 = 1 \\ y \geq 0 \end{cases} \qquad \begin{cases} y^2 = 1 - x^2 \\ y \geq 0 \end{cases} \qquad \bar{y}(x) = \sqrt{1-x^2}$$

$$g(x) = \int_0^{\sqrt{1-x^2}} \frac{2xy}{(x^2+y^2)^{3/2}}\,dy = \int_0^{\sqrt{1-x^2}} x \; 2y\,(x^2+y^2)^{-3/2}\,dy$$

$$= x \cdot \frac{1}{1-3/2} (x^2+y^2)^{1-3/2} \Big|_{y=0}^{y=\sqrt{1-x^2}}$$

$$= -2x \left(1 - (x^2)^{-1/2} \right) = 2x \left(\frac{1}{x} - 1 \right) =$$

$$= \frac{2x(1-x)}{x} = 2(1-x)$$

$$\boxed{g(x) = 2(1-x)\, \mathbb{1}_{(0,1)}(x)}$$

$\mathbb{E}[X] = \int_{\mathbb{R}} x\, g(x)\, dx$ se $\int_{\mathbb{R}} |x|\, g(x)\, dx$ è finito

$$\mathbb{E}[X] = \int_{\mathbb{R}} x\, 2(1-x)\, \mathbb{1}_{(0,1)}(x)\, dx = \int_0^1 2x(1-x)\, dx$$

$$= \int_0^1 (2x - 2x^2)\, dx = \left(x^2 - \frac{2}{3} x^3 \right) \Big|_{x=0}^{x=1} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 g(x)\, dx = \int_{\mathbb{R}} x^2\, 2(1-x)\, \mathbb{1}_{(0,1)}(x)\, dx$$

$$= \int_0^1 2x^2(1-x)\, dx = \int_0^1 (2x^2 - 2x^3)\, dx =$$

$$= \left(\frac{2}{3} x^3 - \frac{1}{2} x^4 \right) \Big|_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} = \frac{1}{18}$$

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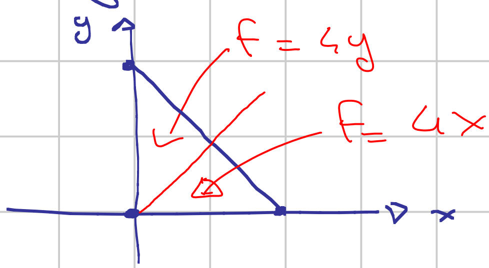
X, Y con distribuzione congiunta A.C.

$$f(x,y) = \begin{cases} 4 \max\{x,y\} & (x,y) \in T \\ 0 & \text{altrimenti} \end{cases}$$

T = Triangolo di vertici $(0,0)$, $(1,0)$ e $(0,1)$



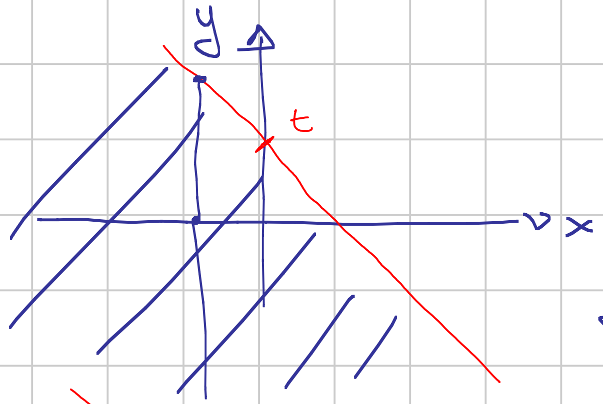
$$Z := X + Y$$

$$\mathbb{P}_{X+Y}(A) = \mathbb{P}_{X,Y}\left(\{ (x,y) \in \mathbb{R}^2 : x+y \in A \} \right) =$$

$$= \int_{\{(x,y)\in\mathbb{R}^2:\ x+y\in A\}} f(x,y)\,dx\,dy$$

$F_{X+Y}(t) = \mathbb{P}_{X+Y}(A)$ con $A = (-\infty, t]$

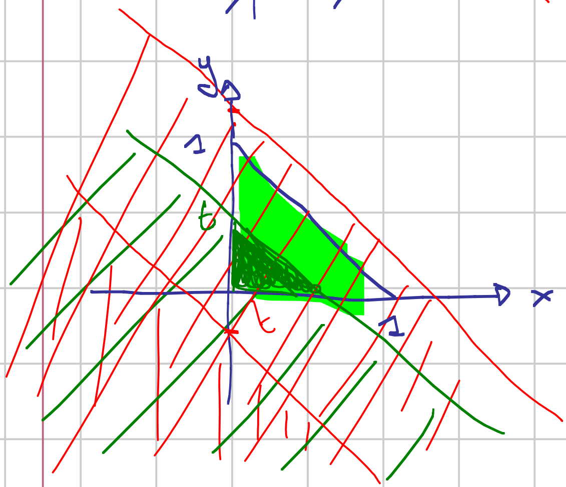
$$= \int_{\{(x,y)\in\mathbb{R}^2:\ x+y\le t\}} f(x,y)\,dx\,dy$$



$x+y=t \quad y=t-x$

$\forall x\in\mathbb{R} \quad y\in(-\infty, t-x)$

$$F_{X+Y}(t) = \int_{-\infty}^{+\infty} dx \left(\int_{-\infty}^{t-x} f(x,y)\,dy\right)$$

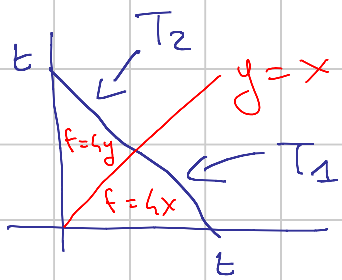


$F_{X+Y}(t) = 0 \qquad t\le 0$

$\qquad\qquad\quad 1 \qquad t\ge 1$

$t\in(0,1)$

$$F_{X+Y}(t) = \int_{T_1} 4x\,dx\,dy + \int_{T_2} 4y\,dx\,dy$$



FINIRE PER ESERCIZIO

$$F_{X+Y}(t) = \begin{cases} 0 & t<0 \\ t^3 & 0\le t<1 \\ 1 & t\ge 1 \end{cases}$$

$$f_{X+Y}(t) = \begin{cases} 0 & t<0,\ t>1 \\ 3t^2 & t\in(0,1) \end{cases}$$

$\psi : \mathbb{R}\to\mathbb{R}$ Borel misurabile nonnegative

$\varphi : (x,y)\in\mathbb{R}^2 \longmapsto x+y\in\mathbb{R}$

$$\int_{\mathbb{R}} \psi(t)\,\mathbb{P}_{\varphi\circ(X,Y)}(dt) = \int_{\mathbb{R}^2} \psi(\varphi(x,y))\,\mathbb{P}_{X,Y}(dx\,dy)$$

$$\underline{\int_{\mathbb{R}} \psi(t)\,\mathbb{P}_{X+Y}(dt)} = \int_{\mathbb{R}^2} \psi(x+y)\,f(x,y)\,dx\,dy =$$

$u = x$ $\qquad J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\qquad \det J = 1$ $\qquad x = u$

$v = x + y$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y = v - u$

$$= \int_{\mathbb{R}^2} \varphi(v) f(u, v-u)\, du\, dv = \int_{\mathbb{R}} \varphi(v) \left(\int_{\mathbb{R}} f(u, v-u)\, du \right) dv$$

$\mathbb{P}_{X+Y} = g(v)\, dv \qquad g(v) = \int_{\mathbb{R}} f(u, v-u)\, du$