Titolo nota 18/11/2014

$\mathbb{P}_X = G(p) \quad p \in (0,1) \quad \mathbb{P}(X=k) = p(1-p)^{k-1} \quad k = 1, 2, \ldots$

$Y(\Omega) = [0,1]$

$\forall t \in [0,1] \quad \forall k = 1, 2, 3, \ldots$

$\mathbb{P}(Y \le t \mid X = k) = t^k$

$F_Y(t) = 0 \quad \forall t < 0$

$F_Y(t) = 1 \quad \forall t \geq 1$

$t \in [0, 1) \quad \{Y \le t\}$

$\{X = k\} \quad k = 1, 2, 3, \ldots$

$\mathbb{P}(Y \le t) = \sum_{k=1}^{\infty} \mathbb{P}(Y \le t, X = k)$ TEO DELLE PROB. TOTALI

$$= \sum_{k=1}^{\infty} \mathbb{P}(Y \leq t \mid X = k)\, \mathbb{P}(X = k)$$

$$= \sum_{k=1}^{\infty} t^k p(1-p)^{k-1} =$$

$$= pt \sum_{k=1}^{\infty} \big(t(1-p)\big)^{k-1} = pt \sum_{j=0}^{\infty} \big(t(1-p)\big)^{j} \qquad j = k-1$$

$$= pt \frac{1}{1 - t(1-p)}$$

$$F_Y(t) = \begin{cases} 0 & t < 0 \\ \dfrac{pt}{1-t(1-p)} & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$

$$F_Y(1^-) = \frac{p}{1-(1-p)} = 1$$

$$F_Y(t) = \int_{-\infty}^{t} f(x)\,dx \qquad f(t) = \begin{cases} 0 & t<0,\ t>1 \\ \dfrac{p}{(1-t(1-p))^2} & t\in(0,1) \end{cases}$$

$$p\,\frac{\left(1-\cancel{t(1-p)}-\cancel{t(p-1)}\right)}{(1-t(1-p))^2} =$$

$$E[Y] = \int_0^{+\infty}(1-F_Y(t))\,dt = \int_0^1 \left(1-\frac{pt}{1-t(1-p)}\right)dt$$

CAVALIERI

oppure $E[Y] = \int_0^1 \frac{pt}{(1-t(1-p))^2}\,dt$

$$\int_0^1 \left(1+\frac{p}{1-p}\,\frac{(1-p)t}{t(1-p)-1}\right)dt =$$

$$\int_0^1 \left(1 + \frac{p}{1-p} \quad \frac{t(1-p)-1+1}{t(1-p)-1} \right) dt$$

$$= \left(t + \frac{p}{1-p} \left(t + \frac{1}{1-p} \log \left| t(1-p) - 1 \right| \right) \right) \Bigg|_{t=0}^{t=1} =$$

$$= 1 + \frac{p}{1-p} + \frac{p}{(1-p)^2} \log(p) = \frac{1}{1-p} + \frac{p}{(1-p)^2} \log(p)$$

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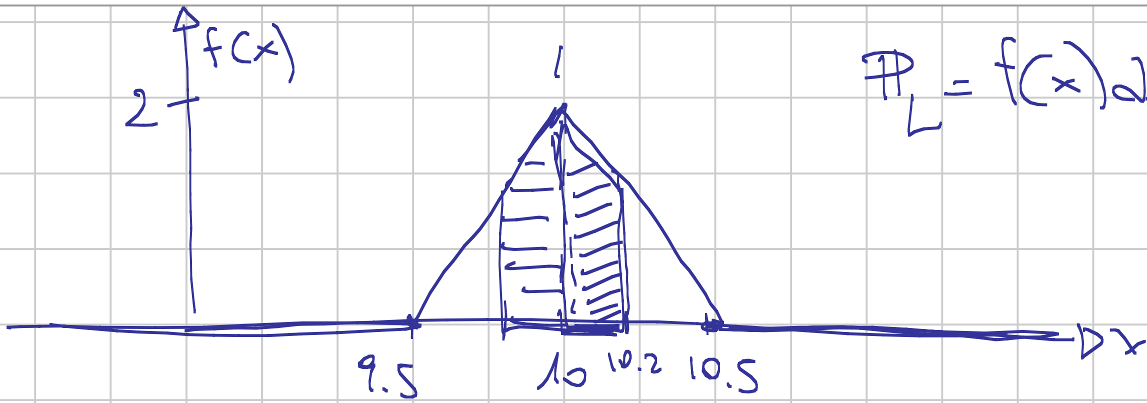

$\mathbb{P}_L = f(x)dx$

$$\mathbb{P}(9.8 \le L \le 10{,}2) = F_L(10{,}2) - F_L(9.8) =$$

$$= \int_{9.8}^{10.2} f(x)dx = 2\int_{10}^{10.2} f(x)dx$$

$(10, 2) \quad (10.5, 0) \qquad \frac{y-0}{2-0} = \frac{x-10.5}{10.-10.5}$

$$\frac{y}{2} = -2(x-10.5)$$

$x = 10.2 \qquad y = -4(10.2-10.5) = \frac{12}{10} = 1.2$

$$\mathbb{P}(9.8 \le L \le 10{,}2) = \cancel{2} \cdot \frac{1}{\cancel{2}} \frac{2}{10}\left(2 + \frac{12}{10}\right) =$$

$$= \frac{2}{10} \cdot \frac{32}{10} = \frac{64}{100} = 0.64$$

$B(20, 0.64)$

$$B(20, 0.64)(\{20\}) + B(20, 0.64)(\{19\})$$

$$= \binom{20}{20}(0.64)^{20}(1-0.64)^{0} + \binom{20}{19}(0.64)^{19}(1-0.64)^{1}$$

$$= (0.64)^{19}(0.64 + 20 \cdot 0.36) = 7.84(0.64)^{19}$$

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$$\mathbb{P}_X = N(\mu, \sigma^2) \qquad (\Omega, \mathcal{E}, \mathbb{P})$$

$$Y : \Omega \to \mathbb{R} \qquad Y(\omega) = \begin{cases} -1 & X(\omega) \le \mu - \sigma \\ 0 & \mu - \sigma < X(\omega) < \mu + \sigma \\ 1 & X(\omega) \ge \mu + \sigma \end{cases}$$

$$\mathbb{P}(Y = -1) = \mathbb{P}(X \le \mu - \sigma) = (\star)$$

X v.a. gaussiana di parametri μ e σ^2

$$X = \mu + \sigma X_0 \qquad \mathbb{P}_{X_0} = N(0,1)$$

$$(\star)\ \mathbb{P}(\mu + \sigma X_0 \le \mu - \sigma) = \mathbb{P}(X_0 \le -1) = \Phi(-1)$$

$$\mathbb{P}(Y=1) = \mathbb{P}(X \ge \mu + \sigma) = \mathbb{P}(\cancel{\mu} + \sigma X_0 \ge \cancel{\mu} + \sigma)$$

$$= \mathbb{P}(X_0 \ge 1) = 1 - \mathbb{P}(X_0 < 1) =$$

$$= 1 - \mathbb{P}(X_0 \le 1)$$

$$= 1 - \Phi(1)$$

$$\mathbb{P}(Y=1) = 1 - \Phi(1)$$

$$\mathbb{P}(Y=-1) = \Phi(-1) = 1 - \Phi(1)$$

$$\mathbb{P}(Y=0) = 1 - \mathbb{P}(Y=1) - \mathbb{P}(Y=-1) =$$

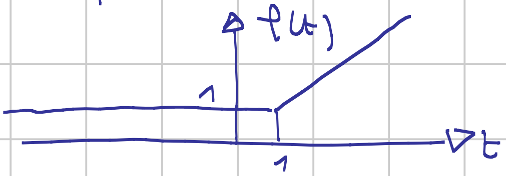
$$= 1 - 2(1 - \Phi(1)) =$$

$$= 2\Phi(1) - 1$$

$$\mathbb{E}[Y] = 1 \cdot \mathbb{P}(Y=1) + 0 \cdot \mathbb{P}(Y=0) - 1\,\mathbb{P}(Y=-1)$$
$$= 0$$
$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] = 1^2\,\mathbb{P}(Y=1) + 0^2\,\mathbb{P}(Y=0) +$$
$$+ (-1)^2\,\mathbb{P}(Y=-1)$$
$$= 2\left(1 - \Phi(1)\right)$$

$\mathbb{P}_X = U((0,2))$ $\qquad Y = \max\{1, X\}$

$f(t) = \max\{1, t\} \quad \Rightarrow \quad Y = f \circ X$



$t \in \mathbb{R}$

$F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(\max\{1, X\} \le t) =$

$= \mathbb{P}(1 \le t, X \le t)$

$= \begin{cases} \mathbb{P}(\emptyset) = 0 & t < 1 \\ \mathbb{P}(X \le t) & t \ge 1 \end{cases} = \begin{cases} 0 & t < 1 \\ F_X(t) & t \ge 1 \end{cases}$

$\mathbb{P}_X = U(0,2) \qquad F_X(t) = \begin{cases} 0 & t < 0 \\ t/2 & 0 \le t < 2 \\ 1 & t \ge 2 \end{cases}$

$F_Y(t) = \begin{cases} 0 & t < 1 \\ \frac{t}{2} & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}$

$F_Y(1) - F_Y(1^-) = \frac{1}{2} - 0 = \frac{1}{2}$

$\mathbb{P}_Y = \frac{1}{2}\delta_1 + g(t)\,dt \qquad g(t) = \frac{1}{2}\mathbb{1}_{(1,2)}(t)$

$Y = f \circ X \qquad f(t) = \max\{1, t\}$

ψ funzione Borel misurabile non negativa

$\int_{\mathbb{R}} \psi(s)\, \mathbb{P}_Y(ds) = \int_{\mathbb{R}} \psi(f(t))\, \mathbb{P}_X(dt) =$

$= \int_{\mathbb{R}} \psi(\max\{1, t\})\, f(t)\,dt =$

$\hookrightarrow$ densità di $X = \frac{1}{2}\mathbb{1}_{(0,2)}(t)$

$= \int_0^2 \psi(\max\{1, t\})\, \frac{1}{2}\,dt =$

$$= \int_0^1 \frac{1}{2} \varphi(1)\, dt \quad + \int_1^2 \frac{1}{2} \varphi(t) dt =$$

$$= \frac{1}{2} \varphi(1) + \int_{\mathbb{R}} \varphi(t) \frac{1}{2} \mathbb{1}_{(1,2)}(t) dt$$

$$\mathbb{P}_Y = \frac{1}{2} \delta_1 + \frac{1}{2} \mathbb{1}_{(1,2)}(t) dt$$

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$X(\Omega) = Y(\Omega) = \mathbb{N}_0$

$$\mathbb{P}(Y=i \mid X+Y=k) = \begin{cases} \binom{k}{i} p^i (1-p)^{k-i} & i=0, \dots k \\ 0 & i>k \end{cases}$$

$\forall k \in \mathbb{N}_0$

$\mathbb{P}_{X+Y} = \text{Poiss}(\lambda)$

$(X+Y)(\Omega) = \mathbb{N}_0$

$\{X+Y=k\} \quad k \in \mathbb{N}_0$

TEO DELLE PROBABILITÀ TOTALI:

$$\mathbb{P}(Y=i) = \sum_{k=0}^{\infty} \mathbb{P}(Y=i, X+Y=k) =$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Y=i \mid X+Y=k)\, \mathbb{P}(X+Y=k)$$

$$= \sum_{k=i}^{+\infty} \binom{k}{i} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=i}^{+\infty} \frac{\cancel{k!}}{i!(k-i)!} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{\cancel{k!}} =$$

$j = k-i$

$k = i+j$

$$= \sum_{j=0}^{+\infty} \frac{1}{i!\,j!} p^i (1-p)^j e^{-\lambda} \lambda^i \cdot \lambda^j$$

$$= \frac{(p\lambda)^i}{i!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{((1-p)\lambda)^j}{j!} =$$

$$= \frac{(p\lambda)^i}{i!} \cancel{e^{-\lambda}} e^{\cancel{\lambda} - \lambda p} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

$$\mathbb{P}_Y = \text{Poiss}(\lambda p)$$

$$\{X=i, X+Y=k\} = \{Y=k-i, X+Y=k\}$$

$$\mathbb{P}(X=i) = \sum_{k=0}^{+\infty} \mathbb{P}(X=i, X+Y=k) =$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Y=k-i, X+Y=k)$$

$$= \sum_{k=0}^{\infty} \underbrace{\mathbb{P}(Y = k-i \mid X+Y=k)}_{=0 \text{ ~~se~~ } k-i > k} \mathbb{P}(X+Y=k)$$

$\{Y = k-i\}$ ha $\mathbb{P}$ nulla se $k-i \notin \mathbb{N}_0$

$k-i \geq 0$ sse $k \geq i$

$$= \sum_{k=i}^{\infty} \mathbb{P}(Y=k-i \mid X+Y=k)\mathbb{P}(X+Y=k)$$

$$= \sum_{k=i}^{+\infty} \binom{k}{k-i} p^{k-i} (1-p)^{i} e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k=i}^{+\infty} \frac{\cancel{k!}}{(k-i)!\,i!} p^{k-i}(1-p)^{i} e^{-\lambda} \frac{\lambda^k}{\cancel{k!}} =$$

$j = k-i$

$k = i+j$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{i!} p^{j} (1-p)^{i} e^{-\lambda} \lambda^{i} \cdot \lambda^{j}$$

$$= \frac{1}{i!} e^{-\lambda} (\lambda(1-p))^{i} \underbrace{\sum_{j=0}^{\infty} \frac{1}{j!} (p\lambda)^{j}}_{e^{\lambda p}}$$

$$= \frac{1}{i!} e^{-\lambda(1-p)} (\lambda(1-p))^{i}$$

$$\mathbb{P}_X = \text{Poiss}(\lambda(1-p))$$

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$a>0 \qquad \mathbb{P}_X = U((0,a)) \qquad Y := \sqrt{X}$

ψ Borel misurabile nonnegativa

$Y = \varphi \circ X \qquad \varphi(t) = \begin{cases} 0 & t \le 0 \\ \sqrt{t} & t > 0 \end{cases}, \quad \varphi(t) = \sqrt{|t|}$

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_Y(dt) = \int_{\mathbb{R}} \psi(\varphi(t))\, \mathbb{P}_X(dt) = \int_{\mathbb{R}} \psi(\varphi(t)) f(t)\, dt \quad (*)$$

$$f(t) = \begin{cases} \frac{1}{a} & t \in (0,a) \\ 0 & \text{altrimenti} \end{cases}$$

$$(*) = \int_0^a \frac{1}{a} \psi(\sqrt{t})\, dt \qquad s = \sqrt{t} \quad t = s^2$$

$$= \int_0^{\sqrt{a}} \frac{1}{a} \psi(s)\, 2s\, ds \qquad \begin{matrix} t=0 & s=0 \\ t=a & s=\sqrt{a} \\ & dt = 2s\,ds \end{matrix}$$

$$\mathbb{P}_Y = g(y)\,dy \qquad g(y) = \frac{2y}{a} \mathbb{1}_{(0,\sqrt{a})}(y)$$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} y\, g(y)\, dy = \int_0^{\sqrt{a}} \frac{2y^2}{a}\, dy =$$

$$= \frac{2}{3a} y^3 \Big|_{y=0}^{y=\sqrt{a}} = \frac{2}{3a} a^{3/2} = \frac{2}{3}\sqrt{a}$$

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} y^2 g(y)\, dy = \int_0^{\sqrt{a}} y^2 \cdot \frac{2y}{a}\, dy = \int_0^{\sqrt{a}} \frac{2}{a} y^3\, dy$$

$$= \frac{2}{a} \frac{y^4}{4} \Big|_{y=0}^{y=\sqrt{a}} = \frac{a}{2}$$

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{a}{2} - \frac{4}{9} a = \frac{a(9-8)}{18}$$

$$= \frac{a}{18}$$