Titolo nota 19/11/2014

$\{a_n\}_{n=0}^{\infty}$ successione in $\mathbb{R}$. $x_0 \in \mathbb{R}$

SERIE DI POTENZE DI CENTRO x_0 E COEFFICIENTI a_n

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$y = x - x_0$ $\quad \sum_{n=0}^{\infty} a_n y^n$

$\sum_{n=0}^{\infty} a_n x^n$ converge sicuramente in $x = 0$

$S_n(x) = \sum_{k=0}^{n} a_k x^k$ $\quad S_n(0) = a_0 \quad \forall n \in \mathbb{N}$

TEO Se $\exists\, y \neq 0$ t.c. la serie numerica $\sum_{n=0}^{\infty} a_n y^n$ converge, allora

$\forall r \in (0, |y|)$ la serie $\sum a_n x^n$ converge totalmente in $[-r, r]$.

$$r := \sup \left\{ |x| : \sum_{n=0}^{\infty} a_n x^n \text{ converge} \right\}$$

Se $|x| < r \Rightarrow \sum_{n=0}^{\infty} a_n x^n$ converge

Se $|x| > r \Rightarrow \sum_{n=0}^{\infty} a_n x^n$ NON converge

$\sum_{n=0}^{\infty} x^n$ $\quad r = 1$ $\quad (-1, 1)$ converge

-1 NON ESISTE

1 DIVERGE A $+\infty$

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$c_n = \frac{x^n}{n}$ $\quad |c_n| = \frac{|x|^n}{n}$ $\quad \sqrt[n]{|c_n|} = \frac{|x|}{\sqrt[n]{n}} \longrightarrow |x|$

Converge $\forall x \in (-1,1)$

Per $|x| > 1$ $\lim_{n\to\infty} c_n = 0$ NON E' SODDISFATTA

$\Rightarrow$ per $|x| > 1$ la serie non converge

$\Rightarrow r = 1$

$x = 1$ $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{SERIE ARMONICA diverge}}$ $\qquad x = -1$ $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{\text{CONVERGE PER LEIBNIZ}}$

$\sum_{n=0}^{\infty} (-1)^n a_n$ $\qquad a_n > a_{n+1} > 0$

$\lim a_n = 0$

$[-1, 1)$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$c_n = \frac{x^n}{n^2}$ $\quad \sqrt[n]{|c_n|} = \frac{|x|}{\sqrt[n]{n^2}} \longrightarrow |x|$

Converge se $|x| < 1$

Se $|x| > 1$ $\lim_{n\to\infty} c_n = 0$ NON è soddisfatta

$\Rightarrow r = 1$

$x = 1$ $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{SERIE ARMONICA GENERALIZZATA } \alpha = 2}$ $\qquad x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

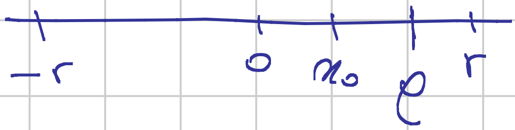
La serie $\sum \frac{x^n}{n^2}$ converge SSE $x \in [-1, 1]$

TEOREMA Se $\rho \in (0,r)$, allora la serie $\sum_{n=0}^{\infty} a_n x^n$ converge Totalmente in $[-\rho,\rho]$.

$\Rightarrow f : x \in (-r,r) \longmapsto \sum_{n=0}^{\infty} a_n x^n \in \mathbb{R}$
è una funzione continua

Fisso $x_0 \in (-r,r)$, per $x_0 \in [0,r)$

$x_0 \in [-\rho,\rho]$



TEO Se r è il raggio di convergenza della serie di potenze $\sum_{n=0}^{\infty} a_n x^n$ e $r>0$ o $r=+\infty$, allora la funzione

$$f : x \in (-r,r) \longmapsto \sum_{n=0}^{\infty} a_n x^n \in \mathbb{R}$$

ammette derivate di qualsiasi ordine e

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n\, n(n-1)\cdots(n-k+1)\, x^{n-k}$$

$\forall k \in \mathbb{N}\ \forall x \in (-r,r)$

DIM $\sum_{n=0}^{\infty} a_n x^n$

Considero la "serie delle derivate"

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

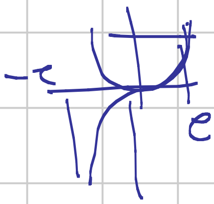
$0<\rho<r$ e sia $y \in (\rho,r)$. So che $\sum_{n=0}^{\infty} a_n y^n$ converge

$\Rightarrow \lim_{n\to+\infty} a_n y^n = 0 \Rightarrow$

$\exists M \in \mathbb{R} : |a_n y^n| \le M \quad \forall n \in \mathbb{N}$

$\|(n+1) a_{n+1} x^n\|_{\infty,(-\rho,\rho)} = (n+1)|a_{n+1}|\rho^n =$

$$\|F\|_{\infty, I} = \sup\{|f(x)|,\ x \in I\}$$



$$= \frac{(n+1)}{|y|} |a_{n+1} y^{n+1}| \frac{e^n}{|y|^n} \le$$

$$\le \frac{n+1}{|y|} M \left(\frac{e}{|y|}\right)^n$$

$$\sqrt[n]{\|a_{n+1}(n+1)x^n\|_{\infty,(-e,e)}} \le \sqrt[n]{\frac{M}{|y|}} \sqrt[n]{n+1} \frac{e}{|y|} \to \frac{e}{|y|} < 1$$

$\sum (n+1) a_{n+1} x^n$ converge totalmente in $(-e,e)$

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ limite uniforme di $S_n(x) = \sum_{k=0}^{n} a_k x^k$ in $(-e,e)$

$g(x) := \sum_{n=1}^{\infty} n a_n x^{n-1}$ limite uniforme di [redacted] $:= \sum_{k=1}^{n} k a_k x^{k-1} =$

$= \sum_{k=1}^{n} \frac{d}{dx}(a_k x^k)$

$= \frac{d}{dx} \sum_{k=1}^{n} a_k x^k$

$=$ [redacted]

$\{f_n\}$ funzioni $C^1(a,b)$
$f_n \to f$ uniformemente
$f_n' \to g$ uniformemente
$\Rightarrow f$ è $C^1(a,b)$
e $f'(x) = g(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad x \in (-r,r)$$

$$f'(x) = \sum_{n=1}^{\infty} \text{[redacted]} \qquad x \in (-r,r)$$

$$= \sum_{n=0}^{\infty} \text{[redacted]}\, x^n$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad x \in (-r,r)$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} \qquad x \in (-r,r)$$

Supponiamo che

$$f^{(k)}(x) = \sum^{\infty} n(n-1) \cdots (n-k+1) a_n x^{n-k} \qquad x \in (-r, r)$$

$$f^{(k+1)}(x) = \sum_{n=k+1}^{\infty} n(n-1) \cdots (n-k+1)(n-k) a_n x^{n-(k+1)}$$

$x = 0 \qquad f^{(k)}(0) \qquad 0^{n-k} = 0 \quad n-k > 0$

$\qquad\qquad\qquad\qquad\qquad 0^0 = 1 \quad n-k = 0$

$$f^{(k)}(0) = k!\, a_k \qquad \forall k \in \mathbb{N}$$

$$a_k = \frac{f^{(k)}(0)}{k!} \qquad \forall k \in \mathbb{N}$$

$$S_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

POLINOMIO DI McLAURIN DI f IN $x = 0$

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$$\sum_{k=0}^{\infty} k x^k$$

Fisso $x \in \mathbb{R}$ $\qquad \lim_{k \to \infty} k x^k = 0$ SSE $\underline{x \in (-1, 1)}$

$$\sqrt[k]{|k x^k|} = \sqrt[k]{k}\, |x| \longrightarrow |x| < 1$$

$\forall x \in (-1,1)$ ho convergenza assoluta

L'insieme di convergenza è l'aperto $(-1, 1)$

$$\sum_{k=0}^{\infty} k x^k = \sum_{k=1}^{\infty} k x^k = \boxed{\sum_{k=1}^{\infty} k x^{k-1}}\, x =$$

$$= x \sum_{k=1}^{\infty} \frac{d}{dx} x^k = x \frac{d}{dx} \sum_{k=1}^{\infty} x^k = $$

$$= x \frac{d}{dx}\left(\frac{1}{1-x} - 1\right) = x \frac{d}{dx} \frac{1}{1-x} = \frac{x}{(1-x)^2}$$

TEOREMA Sia $\sum_{n=0}^{\infty} a_n x^n$ una serie di potenze.
Supponiamo che esiste il $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$
1) Se $L=0$ $\Rightarrow$ la serie converge $\forall x \in \mathbb{R}$ cioè il raggio di convergenza è $+\infty$
2) Se $L \in (0,+\infty)$ $\Rightarrow$ il raggio di convergenza è $\frac{1}{L}$
3) Se $L=+\infty$ $\Rightarrow$ la serie converge solo se $x=0$ cioè il raggio di convergenza è 0.

DIM Fisso $x \in \mathbb{R}$ e considero $\sum_{n=0}^{\infty} |a_n x^n|$
CRITERIO DELLA RADICE
$$\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|}$$
1) $\sqrt[n]{|a_n|} \to 0$ $\quad |x|\sqrt[n]{|a_n|} \to 0 \quad \forall x \in \mathbb{R}$
$\Rightarrow$ la serie $\sum a_n x^n$ converge assolutamente $\forall x \in \mathbb{R}$
2) $\sqrt[n]{|a_n|} \to L \in (0,+\infty)$
$|x|\sqrt[n]{|a_n|} \to L|x|$
$L|x| < 1$ sse $|x| < \frac{1}{L}$
$L|x| > 1$ sse $|x| > \frac{1}{L}$
Converge $\forall x \in (-\frac{1}{L}, \frac{1}{L})$ $\Rightarrow r = \frac{1}{L}$
non converge $\forall x : |x| > \frac{1}{L}$
3) $\sqrt[n]{|a_n|} \to +\infty$
$$\sqrt[n]{|a_n x^n|} = |x|\sqrt[n]{|a_n|} \to \begin{cases} 0 & x=0 \\ +\infty & x \neq 0 \end{cases}$$

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$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = P_n(x,0)$$
$$f(x) = e^x$$

f derivabile infinite volte in un intervallo aperto
$n \in \mathbb{N}$ $\quad P_n(x,x_0)$ $\quad x_0 \in (a,b)$
$$f(x) = P_n(x,x_0) + R_n(x,x_0)$$
$$\lim_{n\to\infty} P_n(x,x_0) = f(x) \text{ sse } \lim_{n\to\infty} R_n(x,x_0) = 0$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad x \in ?$$

$$a_n = \frac{1}{n!} \qquad \sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}}$$

$$b_n > 0 \qquad \frac{b_{n+1}}{b_n} \qquad \sqrt[n]{b_n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \overset{\lim}{=} \frac{1}{(n+1)!} \cdot n! = \lim_{n\to\infty} \frac{1}{n+1} = 0 \qquad = 0 \quad r = +\infty$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \forall x \in \mathbb{R}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \forall x \in (-1,1)$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k \qquad x \in (-1,1)$$

$$\frac{1}{1+x^2} = \frac{1}{1+y}\Big|_{y=x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \qquad x \in (-1,1)$$

$$\int_0^x \frac{1}{1+t^2}\,dt = \operatorname{artg}(t)\Big|_{t=0}^{t=x} = \operatorname{arTg}(x)$$

$$\operatorname{arTg}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k}\,dt \qquad \underline{x \in (-1,1)}$$

$$\sum_{n=0}^{\infty} a_n x^n \qquad x \in (-r,r) \qquad S_n(x) = \sum_{k=0}^{n} a_k x^k$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \qquad x \in (-r,r) \qquad \sigma_n(x) = \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}$$

$f_n \to f$ uniformemente

$$F_n(x) = F_n(a) + \int_a^x f_n(t)\,dt$$

$\{F_n\}$ conv uniformemente e

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

$$\operatorname{arctg}(x) = \sum_{k=0}^{\infty} \int_0^x (-1)^k t^{2k}\,dt = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{2k+1}\Bigg|_{t=0}^{t=x}$$

$$\operatorname{arctg}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad x \in (-1,1)$$

$x = 1$ $\quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ converge per Leibnitz

$x = -1$ $\quad \sum_{k=0}^{\infty} \frac{(-1)^k(-1)}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ converge per Leibnitz

$$\operatorname{arctg}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad x \in [-1,1] \ ?$$

TEOREMA DI ABEL se $\sum a_n x^n$ converge in $y \neq 0$
allora converge:
uniformemente in $[-r, y]$ $\forall r \in (0,y)$, se $y > 0$
uniformemente in $[y, r]$ $\forall r \in (0,-y)$ se $y < 0$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad x \in (-1,1)$$

$$\log(1+x) = \log|1+t|\Big|_{t=0}^{t=x} = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\,dt =$$

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1}\Bigg|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \qquad x \in (-1,1]$$

$x = 1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converge per Leibnitz

$x = -1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1} = -\infty$

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \qquad x \in (-1, 1]$$

$$\log(1-x) = \log(1+(-x)) = \sum_{n=0}^{\infty} \frac{(-1)^n (-x)^{n+1}}{n+1} =$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1} \qquad x \in [-1, 1)$$

$$\log \frac{1+x}{1-x} = \log(1+x) - \log(1-x) = \qquad x \in (-1,1)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{n+1} x^{n+1} \qquad n = 2k$$

$$= \sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1}$$

$$g : x \in (-1, 1) \longmapsto \frac{1+x}{1-x} \in \mathbb{R}$$

$$\lim_{x \to 1^-} \frac{1+x}{1-x} = +\infty \qquad \lim_{x \to -1^+} \frac{1+x}{1-x} = 0$$

$$\log \frac{1+x}{1-x} \qquad \log(2) \qquad \frac{1+x}{1-x} = 2$$

$$1 + x = 2 - 2x \qquad 3x = 1 \qquad x = \frac{1}{3}$$

$$\frac{1+x}{1-x} = 3 \qquad 1 + x = 3 - 3x$$

$$4x = 2 \qquad x = \frac{1}{2}$$

$$\log(3) = \log \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{2}{2k+1} \left(\frac{1}{2}\right)^{2k+1} = \sum_{n=0}^{\infty} \frac{1}{2k+1} \left(\frac{1}{2}\right)^{2k}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad x \in \mathbb{R}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \qquad \sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} \qquad e^{x} - e^{-x} = \sum_{n=0}^{\infty} \left(\frac{1}{n!} - \frac{(-1)^{n}}{n!} \right) x^{n} \qquad \cancel{n=2k}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} x^{n} \qquad n = 2k+1$$

$$\sum_{k=0}^{\infty} \frac{2}{(2k+1)!} x^{2k+1}$$

$$\left.\begin{aligned} \cosh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \\ \sinh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \end{aligned}\right) \quad \forall x \in \mathbb{R}$$

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$$\cos(x) \qquad P_{2n+1}(x,0) = P_{2n}(x,0) = \sum_{k=0}^{n} \frac{(-1)^{k}}{(2k)!} x^{2k}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} x^{2k} \qquad R = ?$$

$$\sum_{n=0}^{\infty} a_n x^{n} \qquad a_n = \begin{cases} 0 & n \text{ dispari} \\ \frac{(-1)^{k}}{(2k)!} & n \text{ pari} = 2k \end{cases}$$

$$\sqrt[n]{|a_n|} \qquad \frac{a_{n+1}}{a_n}$$

$$y = x^{2} \qquad \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} y^{k} \qquad a_k = \frac{(-1)^{k}}{(2k)!} \neq 0 \quad \forall k \in \mathbb{N}$$

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{(2(k+1))!} \, 2k! = \frac{1}{(2k+2)(2k+1)} \to 0$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} y^{k} \quad \text{converge } \forall y \in \mathbb{R}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} x^{2k} \quad \text{converge } \forall x \in \mathbb{R}$$

$$\sin(x) = -\frac{d}{dx} \cos(x)$$

$$\sin(x) = \sin(x) - \sin(0) = \int_0^{x} \cos(t)\,dt = \int_0^{x} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} t^{2k}\,dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{t^{2k+1}}{2k+1}\Bigg|_{t=0}^{t=x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}$$

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$(1+x)^\alpha$

$\alpha = n \in \mathbb{N} \qquad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \forall x \in (-1,1)$$

$$\binom{\alpha}{0} := 1$$

$$n \geq 1 \qquad \binom{\alpha}{n} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!} =$$

$$= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

$$\alpha = N \qquad \frac{N(N-1)\cdots(N-n+1)}{n!} = \frac{N!}{n!(N-n)!}$$

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha-1)\cdots(\overbrace{\alpha-(n+1-1)}^{\alpha-n})}{(n+1)!} \cdot \frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)}\right| =$$

$$= \left|\frac{1}{n+1}(\alpha-n)\right| = \frac{n-\alpha}{n+1} \longrightarrow 1 \qquad R = 1$$

$$\forall x \in (-1,1) \qquad f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

$$(1+x)f'(x) = (1+x)\sum_{n=1}^{\infty} \binom{\alpha}{n} n x^{n-1} =$$

$$= \sum_{n=1}^{\infty} \binom{\alpha}{n} n x^{n-1} + \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^n$$

$k = n-1 \qquad\qquad k = n$

$$= \sum_{k=0}^{\infty} \binom{\alpha}{k+1}(k+1) x^k + \sum_{k=0}^{\infty} k \binom{\alpha}{k} x^k$$

$$= \sum_{k=0}^{\infty} \left\{ (k+1)\binom{\alpha}{k+1} + k\binom{\alpha}{k} \right\} x^k$$

$$\cancel{(k+1)}\frac{\alpha(\alpha-1)\cdots(\alpha-(k+1)+1)}{\cancel{(k+1)!}\ k!}+k\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$=\frac{1}{k!}\alpha(\alpha-1)\cdots(\alpha-k+1)\left(\alpha-\cancel{k}+\cancel{k}\right)=\alpha\binom{\alpha}{k}$$

$$(1+x)f'(x)=\sum_{k=0}^{\infty}\alpha\binom{\alpha}{k}x^k=\alpha\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k=\alpha f(x)$$

$$\forall x\in(-1,1)\qquad (1+x)f'(x)=\alpha f(x)$$

$$g: x\in(-1,1)\longmapsto(1+x)^{-\alpha}f(x)\in\mathbb{R}$$

$$g'(x)=-\alpha(1+x)^{-\alpha-1}f(x)+(1+x)^{-\alpha}f'(x)=$$
$$=(1+x)^{-\alpha-1}\left(-\alpha f(x)+(1+x)f'(x)\right)=0$$

$$g(x)=g(0)\qquad\forall x\in(-1,1)$$

$$g(0)=(1+0)^{-\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{n}x^n\Big|_{x=0}=1\cdot1=1$$

$$(1+x)^{-\alpha}\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n=1\qquad\forall x\in(-1,1)$$

$$f(x)=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n=(1+x)^{\alpha}\qquad\forall x\in(-1,1)$$

$$(1+x)^{-1/2}=\frac{1}{\sqrt{1+x}}=\sum_{n=0}^{\infty}\binom{-1/2}{n}x^n\qquad\forall x\in(-1,1)$$

$$\frac{-\frac12\left(-\frac12-1\right)\left(-\frac12-2\right)\cdots\left(-\frac12-n+1\right)}{n!}=\qquad -1-2n+2$$

$$-\frac12\left(-\frac32\right)\left(-\frac52\right)\cdots\frac{-(2n-1)}{2}\ \frac{1}{n!}$$

$$=\frac{(-1)^n(2n-1)!!}{2^n n!}=\frac{(-1)^n(2n-1)!!}{(2n)!!}$$

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} x^n \quad \forall x \in (-1,1)$$

$$(1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} (-x^2)^n \qquad -x^2 \in (-1,1)$$

$$\Updownarrow$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n} \qquad x \in (-1,1)$$

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin(x) - \cancel{\arcsin(0)} = 0$$

$$\arcsin(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} t^{2n}\, dt \quad \forall x \in (-1,1)$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{(2n+1)}$$

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$$\sum_{k=1}^{\infty} \frac{x^{2k-1}}{(k-1)!} =$$

$$\sum a_n x^n$$

$$x \neq 0 \qquad \frac{1}{x} \sum_{k=1}^{\infty} \frac{x^{2k}}{(k-1)!} \qquad y = x^2$$

$$\frac{1}{x} \boxed{\left(\sum_{k=1}^{\infty} \frac{y^k}{(k-1)!} \right)} \qquad \sum \frac{t^k}{k!}$$

$$j = k-1 \qquad \sum_{k=1}^{\infty} \frac{y^k}{(k-1)!} = \sum_{j=0}^{\infty} \frac{y^j \cdot y}{j!} = y \sum_{j=0}^{\infty} \frac{y^j}{j!}$$

$$\forall y \in \mathbb{R} \quad \sum_{j=0}^{\infty} \frac{y^j}{j!} = e^y$$

$$\frac{1}{x} y e^y = \frac{1}{x} x^2 e^{x^2} = x e^{x^2} \quad \forall x \in \mathbb{R}$$

$$\sum_{n=0}^{\infty} n^2 x^n$$

$a_n = n^2$

$\sqrt[n]{|a_n|} = \sqrt[n]{n^2} \longrightarrow 1$

$r = \frac{1}{1} = 1$

$I = (-1, 1)$

$$n^2 = n\cdot(n-1+1) = n(n-1) + n$$

$$\sum_{n=0}^{\infty} n(n-1)x^n + \sum_{n=0}^{\infty} n x^n$$

$r_1 = 1 \qquad r_2 = 1$

$\forall x \in (-1,1) \qquad \sum n^2 x^n = \underbrace{\sum n(n-1)x^n} + \sum n x^n$

$$\sum_{n=0}^{\infty} n(n-1)x^{n} = \sum_{n=2}^{+\infty} n(n-1)x^{n-2}\cdot x^2 =$$

$$= x^2 \sum_{n=2}^{\infty} \frac{d^2}{dx^2} x^n = x^2 \frac{d^2}{dx^2} \sum_{n=2}^{\infty} x^n =$$

$(1-x)^{-1}$

$(1-x)^{-2}$

$$= x^2 \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) =$$

$a + bx$

$$= x^2 \cdot 2(1-x)^{-3} = \frac{2x^2}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=1} n\cdot x^{n-1}\cdot x = x \sum_{n=1} \frac{d}{dx} x^n =$$

$$= x \frac{d}{dx} \sum_{n=1}^{\infty} x^n = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(2x+1-x)}{(1-x)^3}$$

$$= \frac{x(x+1)}{(1-x)^3} \qquad \forall x \in (-1,1)$$