A connection between Lorentzian distance and mechanical least action

Ettore Minguzzi

Università Degli Studi Di Firenze

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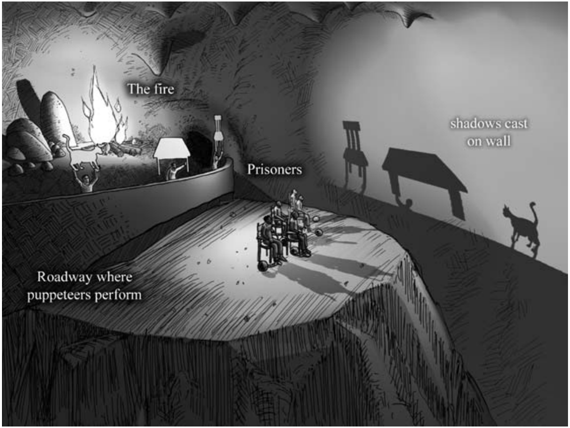
Plato's allegory of the cave

[...] they see only their own shadows, or the shadows of one another, which the fire throws on the opposite wall of the cave [...]. To them, I said, the truth would be literally nothing but the shadows of the images.

Plato, The allegory of the cave,
Book VII of the Republic



Useful for clarifying

- the connection between relativistic and non-relativistic (classical) physics,
- the causality properties of gravitational waves,
- the problem of existence of solutions to the Hamilton-Jacobi equation and other problems in Lagrangian mechanics.

These apparently independent problems are strongly related as the study of lightlike dimensional reduction proves.

- Let $E = T \times Q$, $T = \mathbb{R}$, be a classical $d+1$-dimensional extended configuration space of coordinates (t, q). Let a_t be a positive definite time dependent metric on S, and b_t a time dependent 1-form field on S. Let $V(t, q)$ be a time dependent scalar field on S.
- In 1929 Eisenhart pointed out that the trajectories of a Lagrangian problem

$$L(t, q, \dot{q}) = \frac{1}{2} a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q),$$

$$\delta \int_{t_0}^{t_1} L(t, q, \dot{q}) \mathrm{d}t = 0, \qquad q(t_0) = q_0, \; q(t_1) = q_1$$

may be obtained as the projection of the *spacelike* geodesics of a $d+2$-dimensional manifold $M = E \times Y$, $Y = \mathbb{R}$, of metric

$$\mathrm{d}s^2 = a_t - \mathrm{d}t \otimes (\mathrm{d}y - b_t) - (\mathrm{d}y - b_t) \otimes \mathrm{d}t - 2V \mathrm{d}t^2,$$

- The considered Lagrangian is the most general which comes from Newtonian mechanics by considering holonomic constraints.
- Eisenhart had in mind Jacobi's metric $(E - V)a$, and Jacobi's action principle which holds for time independent Lagrangians and $b = 0$.

- The Eisenhart metric takes its simplest and most symmetric form in the case of a free particle in Euclidean space $a_{bc} = \delta_{bc}$, $b_c = 0$, $V = const.$ Remarkably, in this case the Eisenhart metric becomes the Minkowski metric.
- The Eisenhart metric is Lorentzian but Eisenhart did not give to this fact a particular meaning.

Constructing the Brinkmann - Eisenhart spacetime

- The vector field $n = \partial/\partial y$ is covariantly constant and lightlike.

It is better to proceed in steps starting from (M, g)

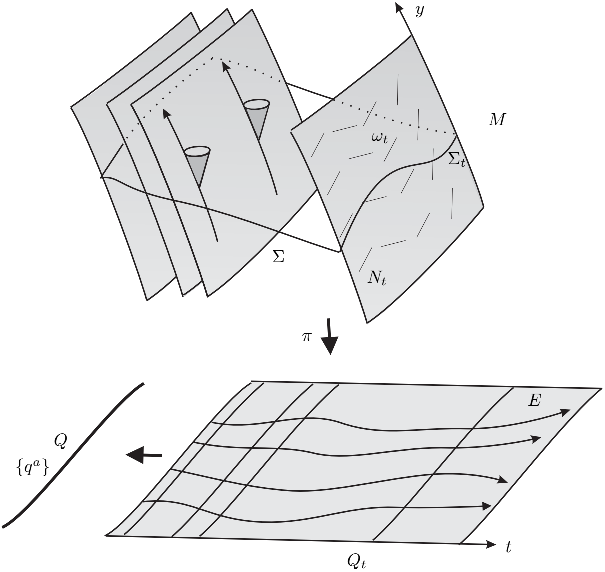
- Assume n is Killing and lightlike in such a way that the quotient projection $\pi : M \to E$ gives a principal bundle over $\mathbb{R}$.
- n is twist-free, $n \wedge \mathrm{d}n = 0$, $\Rightarrow$ E foliates into simultaneity slices, $n = -\psi \mathrm{d}t$.
- n is covariantly constant $\Rightarrow$ the foliation takes a natural 'time parameter' - the *classical time* $n = -\mathrm{d}t$.
- Connection on principal bundles $\pi_t : N_t \to Q_t \Rightarrow$ Newtonian flow on E.
- Curvature $\Rightarrow$ Coriolis forces.
- If the curvature Ω_t vanishes the observers are non-rotating.

The Brinkmann-Eisenhart metric

$$\mathrm{d}s^2 = a_t - \mathrm{d}t \otimes (\mathrm{d}y - b_t) - (\mathrm{d}y - b_t) \otimes \mathrm{d}t - 2V \mathrm{d}t^2,$$

on $M = E \times Y$, $Y = \mathbb{R}$, describes the most general spacetime with a covariantly constant lightlike field (Bargmann structure, generalized wave metric). Brinkmann proved this result locally. The space metric a and 1-form field b are fixed only if the coordinate system is fixed and this is done by choosing a Newtonian flow on E which defines the space Q.

A figure



- Let n be a lightlike Killing vector field on the spacetime (M, g). In any spacetime dimension $n \wedge \mathrm{d}n = 0$ if and only if $R_{\mu\nu}n^\mu n^\nu = 0$.
- Define the *Newtonian flow* as a vector field v on E such that $\mathrm{d}t[v] = 1$. The Newtonian flows and the connections ω_t on the bundles $\pi_t : N_t \to Q_t$ are in one-to-one relation through the formula $\omega_t(\cdot) = -g(\cdot, V)|_{N_t}$.
- The metric a_t on the space sections Q_t is defined by $a_t(w, v) = g(W, V)$.
- Given a 1-parameter family of sections $\sigma_t : Q \to N_t$ of the 1-parameter family of principal bundles $\pi_t : N_t \to Q_t$, the potential b_t reads, $b_t = -\sigma_t^* \omega_t$.

Lightlike lift I

Assume from now on that $M = T \times Q \times Y$ with $T \simeq Y \simeq \mathbb{R}$ is given the metric

$$\mathrm{d}s^2 = a_t - \mathrm{d}t \otimes (\mathrm{d}y - b_t) - (\mathrm{d}y - b_t) \otimes \mathrm{d}t - 2V\mathrm{d}t^2.$$

Every C^1 curve $(t, q(t))$ on $E = T \times Q$ is the projection of a lightlike curve on (M, g), $\gamma(t) = (t, q(t), y(t))$ where

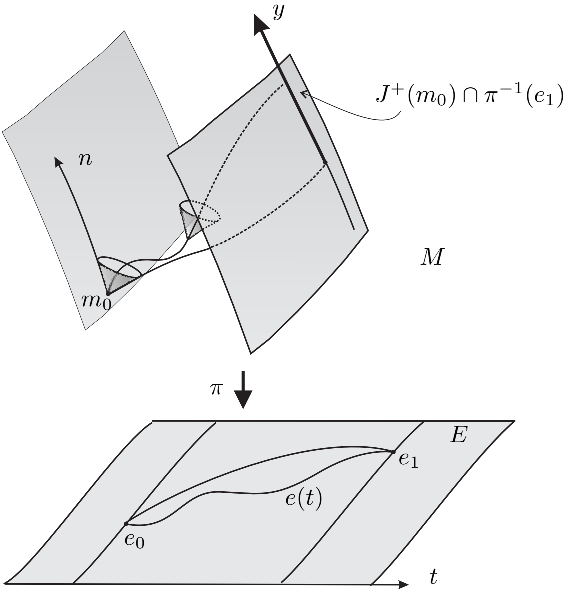
$$y(t) = y_0 + \int_{t_0}^{t} [\frac{1}{2} a_t(\dot{q}, \dot{q}) + b_t(\dot{q}) - V(t, q)]\mathrm{d}t = y_0 + \mathcal{S}_{e_0, e(t)}[q|_{[0,t]}].$$

this result follows from

$$g(\dot{\gamma}, \dot{\gamma}) = a_t(\dot{q}, \dot{q}) - 2(\dot{y} - b_t[\dot{q}]) - 2V = 2(L - \dot{y}),$$

Conversely, every lightlike curve on (M, g) with tangent vectors nowhere proportional to $n = \partial/\partial y$ projects on a C^1 curve on E.
Also, given a timelike curve γ we have necessarily

$$y(t) > y_0 + \mathcal{S}_{e_0, e(t)}[q|_{[0,t]}].$$



y
$J^+(m_0) \cap \pi^{-1}(e_1)$
n
M
m_0
π
E
e_1
$e(t)$
e_0
t

The light lift III

Proposition

Every geodesic on (M, g) not coincident with a flow line of n admits the function t as affine parameter and once so parametrized projects on a solution to the E-L equations. The light lift of a solution to the E-L equation is a lightlike geodesic.

It is based on

$$\begin{aligned}\mathcal{I}[\eta] &= \frac{1}{2}\int_{\lambda_0}^{\lambda_1} g(\eta', \eta')\,\mathrm{d}\lambda = \frac{1}{2}\int_{t_0}^{t_1} g(\dot\eta, \dot\eta)(t')\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \left[L(t, q(t), \dot q(t)) - \dot y\right](t')\,\mathrm{d}t.\end{aligned}$$

The least action

also called Hamilton's principal function is $S : E \times E \to [-\infty, +\infty]$ given by

$$
\begin{aligned}
&S(e_0, e_1) = \inf_{q \in C^1_{e_0,e_1}} \mathcal{S}_{e_0,e_1}[q], \qquad \text{for } \ t_0 < t_1, \\
&S(e_0, e_1) = 0, \qquad \text{for } \ t_0 = t_1 \text{ and } q_0 = q_1, \\
&S(e_0, e_1) = +\infty, \qquad \text{elsewhere.}
\end{aligned}
$$

Proposition

Let $x_0 = (e_0, y_0) \in M$, it holds

$$I^+(x_0) = \{x_1 : y_1 - y_0 > S(e_0, e_1) \text{ and } t_0 < t_1\},$$
$$J^+(x_0) \subset \{x_1 : y_1 - y_0 \geq S(e_0, e_1)\},$$
$$E^+(x_0) \subset r_{x_0} \cup \{x_1 : y_1 - y_0 = S(e_0, e_1)\}.$$

Analogous past versions hold.

A lemma

The statement

Let $x_1 \in J^+(x_0)$ and $t_0 < t_1$, thus in particular $y_1 - y_0 \geq S(e_0, e_1)$. Let $e(t) = (t, q(t))$ be a C^1 curve which is the projection of some C^1 causal curve connecting x_0 to x_1 then $y_1 - y_0 \geq \mathcal{S}_{e_0,e_1}[q]$. Among all the C^1 causal curves $x(t) = (t, q(t), y(t))$, connecting x_0 to x_1, which project on $e(t)$, the causal curve $\gamma(t) = (t, q(t), y(t))$ with

$$y(t) = y_0 + \mathcal{S}_{e_0,e(t)}[q|_{[t_0,t]}] + \frac{t - t_0}{t_1 - t_0}(y_1 - y_0 - \mathcal{S}_{e_0,e_1}[q]) \tag{1}$$

is the one and the only one that maximizes the Lorentzian length. The maximum is

$$l(\gamma) = \{2(y_1 - y_0 - \mathcal{S}_{e_0,e_1}[q])(t_1 - t_0)\}^{1/2}. \tag{2}$$

The proof

Let $\eta(t) = (t, q(t), w(t))$ be a C^1 causal curve connecting x_0 to x_1, then since it is causal by Eq. $-g(\dot\eta, \dot\eta) = 2(\dot w - L)$, $\dot w \geq L$ and integrating $y_1 - y_0 \geq \mathcal{S}_{e_0,e_1}[q]$. The curve γ is causal because (use Eq. $-g(\dot\gamma, \dot\gamma) = 2(\dot y - L)$)

$$-g(\dot\gamma, \dot\gamma) = \frac{2}{t_1 - t_0}(y_1 - y_0 - \mathcal{S}_{e_0,e_1}[q]) \geq 0,$$

taking the square root and integrating one gets

$$l(\gamma) = \{2(y_1 - y_0 - \mathcal{S}_{e_0,e_1}[q])(t_1 - t_0)\}^{1/2}.$$

If $\tilde\gamma = (t, q(t), \tilde y(t))$ is another C^1 timelike curve connecting x_0 to x_1 and projecting on $e(t)$

$$-g(\dot\gamma, \dot\gamma) = \frac{2}{t_1 - t_0}(y_1 - y_0 - \mathcal{S}_{e_0,e_1}[q]) = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1}[-g(\dot{\tilde\gamma}, \dot{\tilde\gamma})]\mathrm{d}t.$$

Using the Cauchy-Schwartz inequality $\int_{t_0}^{t_1}[-g(\dot{\tilde\gamma}, \dot{\tilde\gamma})]\mathrm{d}t \geq (t_1 - t_0)^{-1} l(\tilde\gamma)^2$, remplacing in the above equation, taking the square root and integrating $l(\gamma) \geq l(\tilde\gamma)$, thus γ is longer than $\tilde\gamma$. In order to prove the uniqueness note that the equality sign in $l(\gamma) \geq l(\tilde\gamma)$ holds iff it holds in the Cauchy-Schwarz inequality which is the case iff $g(\dot{\tilde\gamma}, \dot{\tilde\gamma}) = const.$, that is iff $\dot{\tilde y} - L = const.$ which integrated, once used the suitable boundary conditions, gives Eq. (1).

Corollary

Let $x_0, x_1 \in M$, $x_0 = (e_0, y_0)$, $x_1 = (e_1, y_1)$ then if $x_1 \in J^+(x_0)$,

$$d(x_0, x_1) = \sqrt{2[y_1 - y_0 - S(e_0, e_1)](t_1 - t_0)}. \tag{3}$$

In particular, $S(e_0, e_1) = -\infty$ iff $d(x_0, x_1) = +\infty$.

The triangle inequality

The function S is upper semi-continuous everywhere but on the diagonal of $E \times E$ and satisfies the triangle inequality: for every $e_0, e_1, e_2 \in E$

$$S(e_0, e_2) \leq S(e_0, e_1) + S(e_1, e_2),$$

with the convention that $(+\infty) + (-\infty) = +\infty$.

What is the relation between the reverse triangle inequality satisfied by d and the usual triangle inequality satisfied by S?

An abstract framework

Suppose on X you are given a function $s : X \to (-\infty, +\infty]$ such that $s(x, x) = 0$. On the cartesian product $X \times \mathbb{R}$ define the relation

$$(x, a) \leq (y, b) \quad \text{if} \quad b - a \geq s(x, y)$$

Assume futhermore that $\preceq$ is a total preorder on X and that $t : X \to \mathbb{R}$ is an utility function, i.e. $x \preceq y \Leftrightarrow t(x) \leq t(y)$. Finally, let $x \npreceq y \Rightarrow s(x, y) = +\infty$ be the compatibility condition of the total preorder with s. Define $d : (X \times \mathbb{R})^2 \to [0, +\infty]$ by

$$d((x, a), (y, b)) = \sqrt{2[b - a - s(x, y)](t(y) - t(x))} \tag{4}$$

if $(x, a) \leq (y, b)$ and 0 otherwise. Given $x_1, x_2, x_3 \in X$, the reverse triangle inequality

$$d((x_1, a_1), (x_2, a_2)) + d((x_2, a_2), (x_3, a_3)) \leq d((x_1, a_1), (x_3, a_3)) \tag{5}$$

holds for every triple $(x_1, a_1) \leq (x_2, a_2) \leq (x_3, a_3)$ if and only if for all $x, y, z \in X$, $s(x, z) \leq s(x, y) + s(y, z)$.

$$\begin{array}{c}\text{Global hyperbolicity}\\ \Downarrow \\ \text{Causal simplicity}\\ \Downarrow \\ \text{Causal continuity}\\ \Downarrow \\ \text{Stable causality}\\ \Downarrow \\ \text{Strong causality}\\ \Downarrow \\ \text{Distinction}\\ \Downarrow \\ \text{Causality}\\ \Downarrow \\ \text{Chronology}\end{array}$$

How does it change for our case? Can this properties be related with properties of the Lagrangian problem and in particular of the least action S?

Why should we expect a connection with the least action?

Because some causality properties can be expressed in terms of the Lorentzian distance d

Global hyperbolicity

A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor d is finite.

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A strongly causal spacetime is globally hyperbolic if and only if whatever the chosen conformal factor d is continuous.

Causal simplicity

A strongly causal spacetime is causally simple if and only if whatever the chosen conformal factor d is continuous wherever it vanishes.

and d and S are related...

The answer

Global hyperbolicity:

(i) $S(e_0, e_1)$, $t_0 < t_1$, is finite,

(ii) $\mathcal{S}_{e_0,e_1}$, $t_0 < t_1$, is coercive.

$$\Downarrow$$

Causal simplicity:

(a) $\mathcal{S}$ attains its infimum S wherever S is finite,

(b) S is lower semi-continuous.

Completeness

$$\Downarrow$$

Causal continuity:

$$\liminf_{e \to e_1} S(e_0, e) = \liminf_{e \to e_0} S(e, e_1),$$

and this quantity vanishes for $e_0 = e_1$.

$\Downarrow$

$$\boxed{\begin{array}{l}\text{Stable/Strong causality:}\\ \\ S \text{ is lower semi-continuous on the di-}\\ \text{agonal.}\end{array}}$$

$\Downarrow$

$$\boxed{\begin{array}{l}\text{Distinction:}\\ \\ \liminf\limits_{e\to\tilde{e}} S(\tilde{e},e) = \liminf\limits_{e\to\tilde{e}} S(e,\tilde{e}) = 0.\end{array}}$$

(M, g) is always causal.

Locally the Lorentzian distance $d(x_0, x)$ satisfies the eikonal equation

$$g(\nabla d, \nabla d) + 1 = 0,$$

while $S(e_0, e)$ satisfies the Hamilton-Jacobi equation. They are related because, using the relation between S and d

$$g(\nabla d,\nabla d) + 1 = \frac{2(t - t_0)^2}{d^2} \,[\, \frac{\partial S}{\partial t} + \frac{1}{2}\, a_t^{-1}(\mathrm{d}S - b_t, \mathrm{d}S - b_t) + V].$$

The function $u : E \to \mathbb{R}$ is a viscosity solution of the Hamilton-Jacobi equation if

- It is a *viscosity subsolution*: for every $(t,q) \in E$ there is a C^1 function $\varphi : E \to \mathbb{R}$ such that $u - \varphi$ has a local maximum at e and at e

$$\partial_t \varphi + H(t,q,D_q\varphi) \leq 0.$$

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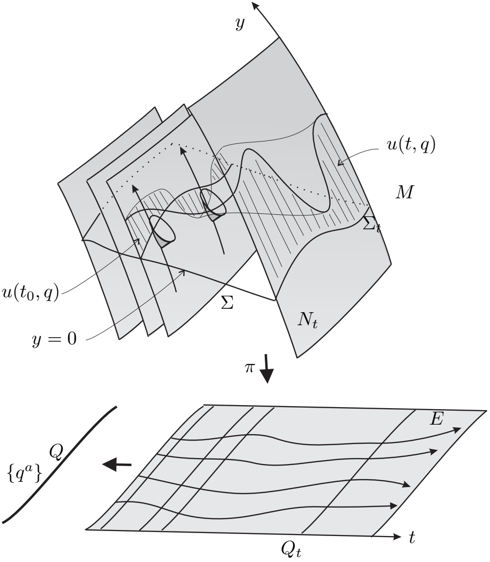
$$\partial_t \varphi + H(t,q,D_q\varphi) \geq 0.$$

Viscosity solutions and the causal future

The slices of the causal future of the initial condition give a viscosity solution.



Smoothness properties follow from theorems in Lorentzian geometry.

Lax-Oleinik semigroup

This solution is that of the Lax-Oleinik semigroup

$$u(t,q) = \inf_{q_0 \in Q, \alpha \in C^1} \{u(t_0,q_0) + \int_{t_0}^{t} L(t,\alpha,\dot{\alpha})\mathrm{d}t\}$$

Conclusions

- The study of spacetimes admitting a parallel null vector is tightly related with the study of Lagrangian mechanical systems.
- In this framework there is a simple relation between the Lorentzian distance and the least action.
- The causality properties of the spacetime are connected with lower semi-continuity properties of the least action.
- Tonelli's theorem on the existence of minimizers is basically the statement that global hyperbolicity implies causal simplicity.
- The Hamilton-Jacobi equation and the Lax-Oleinik semigroup are nothing but the causal relation in disguise.