Time and Causality in General Relativity

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Lorentzian manifolds and light cones

Lorentzian manifolds

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Time orientation and spacetime

At every point there are two cones of timelike vectors. The Lorentzian manifold is time orientable if a continuous choice of one of the cones, termed future, can be made. If such a choice has been made the Lorentzian manifold is time oriented and is also called spacetime.

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They can be regarded as relations on M i.e. as subsets of $M \times M$

$$\begin{split} I^+ &= \{(p,q) \in M \times M : p \ll q\}, & \text{chronology relation} \\ J^+ &= \{(p,q) \in M \times M : p \leq q\}, & \text{causal relation} \\ E^+ &= \{(p,q) \in M \times M : p \to q\} = J^+ \backslash I^+, & \text{horismos relation} \end{split}$$

They are all transitive. I^+ is open but J^+ and E^+ are not necessarily closed.

Let
$$\Delta = \{(p, p) : p \in M\}$$

Preorder

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Total (linear) order

A partial order which is total.

In other words in a total ordering given two points you can decide which one comes before and which one after.

Abstract framework

General Philosophy

Forget about the metric and the conformal structure and work with relations on M.

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Strategies

■ Work with all I^+ , J^+ and E^+ and promote their relationships, such as

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to the status of axioms (Kronheimer and Penrose '67)

 \blacksquare Try to build everything from one single relation (e.g. Causal Set Theory '87), maybe $J^+.$

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Other candidates?

But none of I^+ , J^+ or E^+ are both closed and transitive while Seifert's is!

$$J_S^+ = \bigcap_{g'>g} J_g^+.$$

Here g' > g if the light cones of g are everywhere strictly larger than those of g.

Time and stable causality

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A continuous function $t:M\to\mathbb{R}$ such that if p< q then t(p)< t(q). (Example Minkowski)

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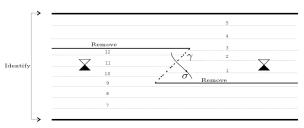
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This causal spacetime does not admit a time function (continuity fails) but it is not stably causal.



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- J_S^+ is a partial order iff (M,g) admits a time function,
- Under stable causality J_S^+ is the smallest closed and transitive relation which contains J^+ (i.e. it coincides with K^+),
- For every time function t defined the total preorder $T^+[t] = \{(p,q) \in M \times M : t(p) \le t(q)\}$ and denoted with A the set of time functions on spacetime

$$J_S^+ = \bigcap_{t \in A} T^+[t]$$

That is from 'time' one recovers J_S^+ not J^+ ! The two relations coincide under strong causality assumptions (causal simplicity).

From causality to time

Stable causality (antisymmetry of J_S^+) implies the existence of time.

This is the analog of Szpilrajn order extension principle: every partial order can be extended to a total order. (But here $T^+[t]$ is a total preorder and continuity comes into play!)

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Abstract approach

Time considerations suggest to regard J_S^+ (or K^+) as a natural candidate to build up an abstract causality framework.