From causality to time and back

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based on

- Time functions as utilities Archive: 0909.0890
- K-causality coincides with stable causality - *Commun. Math. Phys.* 290 (2009) 239-248

Partially ordered sets

Let $\Delta = \{(p, p) : p \in M\}$

Preorder

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A preorder which is

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They can be regarded as relations on M i.e. as subsets of $M \times M$

$$I^+ = \{(p,q) \in M \times M : p \ll q\}, \qquad \text{chronology relation}$$
$$J^+ = \{(p,q) \in M \times M : p \le q\}, \qquad \text{causal relation}$$
$$E^+ = \{(p,q) \in M \times M : p \to q\} = J^+ \backslash I^+, \qquad \text{horismos relation}$$

I^+ and J^+ are transitive. I^+ is open but J^+ and E^+ are not necessarily closed.

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By definition $K^+ \subset J_S^+$; do they coincide?

No, but

- K-causality is equivalent to stable causality and in this case $K^+ = J_S^+$.

Time and stable causality

Time functions and temporal functions

Semi-time function: a *continuous* real function such that $p \ll q \Rightarrow t(p) < t(q)$.

Time function: a *continuous* real function such that $p < q \Rightarrow t(p) < t(q)$.

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Relation with stable causality

Hawking 1968	Temporal function $\Rightarrow$ stable causality
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Bernal and Sánchez 2004	Time function $\Rightarrow$ temporal function

So to prove "Time function $\Rightarrow$ stable causality" you pass thorough a smooth time function.

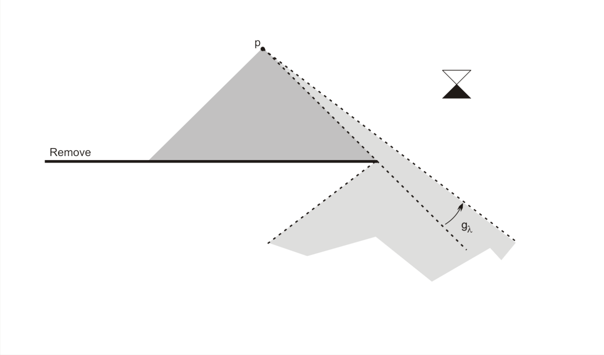
Hawking's averaging method

Geroch's time $\mu(I^-(x))$ is only lower semi-continuous.

- Stable causality $\Rightarrow$ time function.

Let $g_\lambda = (1-\frac{\lambda}{2}) + \frac{\lambda}{2}\tilde{g}$ with $\tilde{g} > g$, define

$$t(x) = \int_0^1 \mu(I^-_{(M,g_\lambda)}(x))d\lambda$$



Other route: prove directly

(i) The existence of a time function implies K-causality (skip smoothability),

(ii) K-causality implies the existence of a time function (skip Hawking's averaging technique).

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Utility theory comes into play...

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Utility theory

An individual has preferences (an apple over an orange) on an abstract space of alternatives A. These preferences give a preorder R. Write $x \sim_R y$ if $x \leq_R y$ and $y \leq_R x$, and $x <_R y$ if $x \leq_R y$ and not $y \leq_R x$.
Daniel Bernoulli (1728) introduced the concept of *utility*:

$$''x \sim_R y \Rightarrow u(x) = u(y)'' \text{ and } ''x <_R y \Rightarrow u(x) < u(y).''$$

to quantify preference and solve S. Petersburg paradox.
Let A be a topological space. The problem of establishing the existence of a *continuous* utility is formally similar to that of establishing the existence of a time function, but much older. Mathematicians tried every condition on R. Finally they reached (Levin's theorem) the conclusion that if R is closed then u exists!

Utilities in spacetime

Utilities for I^+

In a chronological spacetime the utilities of the relation I^+ are the semi-time functions.

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Given this correspondences Levin's and Peleg's theorems of utility theory lead to the following results

Theorem

A spacetime is K-causal if and only if it admits a time function. In this case, denoting with $\mathscr{A}$ the set of time functions we have that the partial order K^+ can be recovered from the time functions, that is

$$(x, y) \in K^+ \Leftrightarrow \forall t \in \mathscr{A},\ t(x) \le t(y).$$

Theorem

A chronological spacetime in which $\overline{J^+}$ is transitive admits a semi-time function.

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Considerations about time suggest to regard J_S^+ (or K^+) as more fundamental than J^+.